DE GRUYTER Demonstratio Mathematica 2023; 56: 20220185

Research Article

Mădălina Păcurar*

Asymptotic stability of equilibria for difference equations via fixed points of enriched Prešić operators

https://doi.org/10.1515/dema-2022-0185
received August 16, 2022; accepted November 18, 2022

Abstract: We introduced a new general class of Prešić-type operators, by enriching the known class of Prešić contractions. We established conditions under which enriched Prešić operators possess a unique fixed point, proving the convergence of two different iterative methods to the fixed point. We also gave a data dependence result that was finally applied in proving the global asymptotic stability of the equilibrium of a certain k-th order difference equation.

Keywords: enriched Prešić operator, fixed point, global asymptotic stability, difference equation

MSC 2020: 54H25, 47H10

1 Introduction and preliminaries

A simple literature search reveals the constant interest of researchers for the so-called Prešić-type operators since the original article [1] was published, with an obvious increase in this interest over the past decade (see, for example, [2–6] for some of the most recent articles). This increasing number of dedicated papers is due to the numerous directions in which the initial contraction condition can be generalized, in various space settings and from different points of view, see, for example, [7–14] and references therein. But another reason is the interesting application area of the operators defined on product spaces, namely in studying the solutions of some particular types of difference equations, see, for example, [15–17].

We recall that in a metric space (X, d), for k a positive integer, and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$ such that $\sum_{i=1}^{k}\alpha_i = \alpha < 1$, a mapping $f : X^k \to X$ is called a Prešić operator if

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \le \sum_{i=1}^{k}\alpha_i d(x_{i-1}, x_i) \tag{1}$$

for all $x_0, \ldots, x_k \in X$. It is obvious that for $k = 1$, this reduces to a Banach contraction condition.

As we have to refer to the original result of Prešić in order to prove our main result, we shall recall it the way it was formulated in [18], where we added some information regarding the rate of convergence of the iterative method.

Theorem 1.1. *Let (X, d) be a complete metric space, k a positive integer, and $f : X^k \to X$ a Prešić operator. Then,*
(1) *f has a unique fixed point x^*;*
(2) *the sequence $\{y_n\}_{n\ge0}$*

*** Corresponding author: Mădălina Păcurar,** Faculty of Economics and Business Administration, Babeș-Bolyai University, T. Mihali 58-60, 400591 Cluj-Napoca, Romania, e-mail: madalina.pacurar@econ.ubbcluj.ro

$$y_{n+1} = f(y_n, y_n, \ldots, y_n),\ n \geq 0,$$

converges to x^*;

(3) *the sequence* $\{x_n\}_{n\geq 0}$ *with* $x_0, \ldots, x_{k-1} \in X$ *and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}),\ n \geq k,$$

also converges to x^*, *with a rate estimated by*

$$d(x_{n+1}, x^*) \leq \alpha d(x_n, x^*) + M \cdot \theta^n,\ n \geq 0,$$

where $M > 0$ *and* $\theta \in (0, 1)$ *are constant.*

The new general class of operators that we shall introduce in the following extends on the one hand the Prešic operators, but on the other hand the class of enriched Banach contractions introduced in [19], where more details about the technique of enriching an existing class of operators can be found. Several papers on different classes of enriched operators have appeared recently (see, for example, [20–27]).

We recall that in a linear normed space $(X, \|\cdot\|)$, a mapping $T : X \to X$ is said to be an enriched (Banach) contraction if there exist $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x - y\|,\ \forall x, y \in X.$$

As shown in [19], such operators have a unique fixed point, that can be obtained by means of an appropriate Krasnoselski iteration. This iteration is in fact the Picard iteration of the averaged mapping $T_\lambda : X \to X$, $T_\lambda = (1-\lambda)I + \lambda T$, with $\lambda \in (0, 1]$, corresponding to the initial operator $T : X \to X$. Among the properties of the averaged mapping T_λ, we mention the fact that it has the same set of fixed points as the initial operator T.

The article is rather concise and contains three more sections.

In Section 2, we introduce a nice generalization of the averaged mapping, corresponding to the case of operators defined on product spaces, and the class of enriched Prešić-type contractions. In the same section, we state the first result of the article, establishing conditions under which enriched Prešić operators have a unique fixed point and also proving the convergence of two iterative methods to this fixed point.

In Section 3 we prove a data dependence result for enriched Prešić operators.

Based on this theorem, we prove in the last section a result concerning the asymptotic stability of equilibria for a particular class of difference equations.

Similar research could be carried out for other classes of operators, in various frameworks, or in other directions, having in view the results in articles like [8,9, 21,28–31].

2 Fixed points of enriched Prešić operators

As the averaged mapping plays an important role in constructing the enriched Banach contractions, see [19], we start our approach by introducing an analogue of the averaged mapping for the case of operators defined on product spaces.

Definition 2.1. Let $(X, +, \cdot)$ be a linear vector space, k a positive integer, and $T : X^k \to X$ an operator. For $\lambda_0, \lambda_1, \ldots, \lambda_k \geq 0$, with $\sum_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$, the operator $T_\lambda : X^k \to X$

$$T_\lambda(x_0, x_1, \ldots, x_{k-1}) = \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{k-1} x_{k-1} + \lambda_k T(x_0, x_1, \ldots, x_{k-1}) \tag{2}$$

will be called the averaged mapping corresponding to T.

Remark 2.1. One can easily see that, for $k = 1$, the previous definition reduces to $T_\lambda(x_0) = \lambda_0 x_0 + \lambda_1 T(x_0)$, for $x_0 \in X$, where $\lambda_0 + \lambda_1 = 1$, that is, the averaged mapping $T_\lambda : X \to X$ mentioned in the previous section.

Remark 2.2. As in the case of the averaged mapping corresponding to an operator defined on X, it is not difficult to show that $x^* \in X$ is a fixed point of $T^k : X \to X$ if and only if it is a fixed point of the corresponding $T_\lambda : X^k \to X$, for some $\lambda_i \geq 0$, $i = 0, 1, \dots, k$, with $\sum_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$.

Indeed, supposing $x^* \in X$ such that $T_\lambda(x^*, x^*, \dots, x^*) = x^*$, it follows that

$$\lambda_0 x^* + \lambda_1 x^* + \dots + \lambda_{k-1} x^* + \lambda_k T(x^*, x^*, \dots, x^*) = x^*,$$

so

$$(1 - \lambda_k) x^* + \lambda_k T(x^*, x^*, \dots, x^*) = x^*.$$

Since $\lambda_k \neq 0$, it follows immediately that $T(x^*, x^*, \dots, x^*) = x^*$. The inverse is obvious.

Using the averaged mapping defined earlier, we can now define a new general class of Prešić-type operators:

Definition 2.2. Let $(X, \|\cdot\|)$ be a linear normed space and k a positive integer. A mapping $T : X^k \to X$ is said to be an enriched Prešić operator if there exist $b_i \geq 0$, $i = 0, 1, \dots, k-1$, and $\theta_i \geq 0$, $i = 0, 1, \dots, k-1$, with $\sum_{i=0}^{k-1} (\theta_i - b_i) < 1$ such that:

$$\left\| \sum_{i=0}^{k-1} b_i (x_i - x_{i+1}) + T(x_0, x_1, \dots, x_{k-1}) - T(x_1, x_2, \dots, x_k) \right\| \leq \sum_{i=0}^{k-1} \theta_i \|x_i - x_{i+1}\|$$

for all $x_0, x_1, \dots, x_k \in X$.

Remark 2.3. For $k = 1$, this reduces to the definition of an enriched Banach contraction, see [19].

Remark 2.4. If $b_0 = b_1 = \dots = b_{k-1} = 0$ in the previous definition, then we obtain the definition of a Prešić operator, see (1).

Next, we prove that the enriched Prešić operators possess a unique fixed point, which can be obtained by means of some appropriate iterative methods.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space, k a positive integer, and $T : X^k \to X$ an enriched Prešić operator with constants b_i, θ_i, $i = 0, 1, \dots, k-1$. Then,*
(1) *T has a unique fixed point $x^* \in X$ such that $T(x^*, x^*, \dots, x^*)$;*
(2) *There exists $a \in (0, 1]$ such that the iterative method $\{y_n\}_{n\geq 0}$ given by*

$$y_n = (1 - a) y_{n-1} + a T(y_{n-1}, y_{n-1}, \dots, y_{n-1}), \; n \geq 1,$$

converges to the unique fixed point x^*, starting from any initial point $y_0 \in X$.
(3) *There exist $\lambda_0, \lambda_1, \dots, \lambda_k \geq 0$ with $\sum_{i=0}^{k} \lambda_i = 1$ and $\lambda_k \neq 0$ such that the iterative method $\{x_n\}_{n\geq 0}$ given by*

$$x_n = \lambda_0 x_{n-k} + \lambda_1 x_{n-k+1} + \dots + \lambda_{k-1} x_{n-1} + \lambda_k T(x_{n-k}, x_{n-k+1}, \dots, x_{n-1})$$

or simply

$$x_n = T_\lambda(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), \quad n \geq 1,$$

converges to x^*, for any initial points $x_0, x_1, \dots, x_{k-1} \in X$.

Proof. If $\sum_{i=0}^{k-1} b_i = 0$ (which happens only if $b_0 = b_1 = \dots = b_{k-1} = 0$), then T is a Prešić operator and all the conclusions of Theorem 1.1 follow.

Let us consider that $\sum_{i=0}^{k-1} b_i > 0$. Denoting $b = \sum_{i=0}^{k-1} b_i$, it follows that $b > 0$.

Now, we take $\lambda_k = \frac{1}{b+1} > 0$ (k is a fixed positive integer) and

$$\lambda_i = \lambda_k b_i, \quad \text{for } i = 0, 1, \ldots, k-1.$$

This way, we have $\lambda_0, \lambda_1, \ldots, \lambda_k$ with the property that

$$\sum_{i=0}^{k} \lambda_i = \sum_{i=0}^{k-1} \lambda_i + \lambda_k = \sum_{i=0}^{k-1} \lambda_k b_i + \lambda_k = \lambda_k(b+1) = 1. \tag{3}$$

Since T is an enriched Prešić operator, by replacing the constants b_i, $i = 0, 1, \ldots, k-1$ in the definition relation (2), we obtain

$$\left\| \sum_{i=0}^{k-1} \frac{\lambda_i}{\lambda_k}(x_i - x_{i+1}) + T(x_0, x_1, \ldots, x_{k-1}) - T(x_1, x_2, \ldots, x_k) \right\| \le \sum_{i=0}^{k-1} \theta_i \|x_i - x_{i+1}\|,$$

for any $x_0, x_1, \ldots, x_k \in X$. By multiplying with $\lambda_k > 0$, this yields

$$\left\| \sum_{i=0}^{k-1} \lambda_i(x_i - x_{i+1}) + \lambda_k T(x_0, x_1, \ldots, x_{k-1}) - \lambda_k T(x_1, x_2, \ldots, x_k) \right\| \le \sum_{i=0}^{k-1} \theta_i \lambda_k \|x_i - x_{i+1}\|.$$

Now, by denoting $\alpha_i = \lambda_k \theta_i$, for $i = 0, 1, \ldots, k-1$, this can be written as

$$\left\| \sum_{i=0}^{k-1} \lambda_i x_i + \lambda_k T(x_0, x_1, \ldots, x_{k-1}) - \sum_{i=0}^{k-1} \lambda_i x_{i+1} - \lambda_k T(x_1, x_2, \ldots, x_k) \right\| \le \sum_{i=0}^{k-1} \alpha_i \|x_i - x_{i+1}\|,$$

so having in view the definition of the averaged operator T_λ, this means that

$$\|T_\lambda(x_0, x_1, \ldots, x_{k-1}) - T_\lambda(x_1, x_2, \ldots, x_k)\| \le \sum_{i=0}^{k-1} \alpha_i \|x_i - x_{i+1}\| \tag{4}$$

for any $x_0, x_1, \ldots, x_k \in X$. As T is assumed to be an enriched Prešić operator, according to the definition $\sum_{i=0}^{k-1}(\theta_i - b_i) < 1$, so

$$\sum_{i=0}^{k-1} \theta_i < \sum_{i=0}^{k-1} b_i + 1 = b + 1.$$

It follows that

$$\sum_{i=0}^{k-1} \alpha_i = \sum_{i=0}^{k-1} \lambda_k \theta_i = \lambda_k \sum_{i=0}^{k-1} \theta_i < \lambda_k(b+1) = 1,$$

and thus by (4), T_λ is a Prešić operator.

By Theorem 1.1 and having in view Remark 2.2, we have the following conclusions:

(1) T_λ has a unique fixed point $x^* \in X$, which is in the same time the unique fixed point of T.

(2) x^* can be obtained starting from any $y_0 \in X$ by means of the iteration

$$y_n = T_\lambda(y_{n-1}, y_{n-1}, \ldots, y_{n-1}), \; n \ge 1,$$

that is, as the limit of the sequence

$$y_n = \lambda_0 y_{n-1} + \lambda_1 y_{n-1} + \cdots + \lambda_{k-1} y_{n-1} + \lambda_k T(y_{n-1}, y_{n-1}, \ldots, y_{n-1}), \; n \ge 1,$$

equivalent to

$$y_n = (1 - \lambda_k) y_{n-1} + \lambda_k T(y_{n-1}, y_{n-1}, \ldots, y_{n-1}), \; n \ge 1,$$

or simply

$$y_n = (1 - a) y_{n-1} + a T(y_{n-1}, y_{n-1}, \ldots, y_{n-1}), \; n \ge 1,$$

with $a \in (0, 1]$.

(3) As practice shows, k-step iterations converge faster than one-step methods. This is possible also in the case of enriched Prešić contractions, as by Theorem 1.1 applied for T_λ, it follows that the unique fixed point x^* can be obtained also starting from any $x_0, x_1, \dots, x_{k-1} \in X$ by means of the iteration

$$x_n = T_\lambda(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}),\ n \geq 1,$$

that is, as the limit of the sequence

$$x_n = \lambda_0 x_{n-k} + \lambda_1 x_{n-k+1} + \dots + \lambda_{k-1} x_{n-1} + \lambda_k T(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}).$$ □

3 Data dependence of the fixed point

First, we note a simple but useful property of the averaged mappings corresponding to operators defined on product spaces.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a linear normed space, k a positive integer, and $T, U : X^k \to X$ two operators. Let $\lambda_0, \lambda_1, \dots, \lambda_k \geq 0$ be fixed, such that $\lambda_k > 0$ and $\sum_{i=0}^{k} \lambda_i = 1$.*

If there exists $\eta > 0$ such that $\|T(x, x, \dots, x) - U(x, x \dots, x)\| \leq \eta$, for any $x \in X$, then

$$\|T_\lambda(x, x, \dots, x) - U_\lambda(x, x, \dots, x)\| \leq \lambda_k \eta,$$

for any $x \in X$.

Proof. Suppose there is $\eta > 0$ such that $\|T(x, x, \dots, x) - U(x, x \dots, x)\| \leq \eta$, for any $x \in X$. Let $x \in X$. Then,

$$\begin{aligned} \|T_\lambda(x, x, \dots, x) - U_\lambda(x, x, \dots, x)\| &= \|\lambda_0 x + \lambda_1 x + \dots + \lambda_{k-1} x + \lambda_k T(x, x, \dots, x) - \lambda_0 x - \lambda_1 x - \dots - \lambda_{k-1} x \\ &\quad - \lambda_k U(x, x, \dots, x)\| \\ &= \lambda_k \|T(x, x, \dots, x) - U(x, x, \dots, x)\| \\ &\leq \lambda_k \eta. \end{aligned}$$ □

Based on this lemma, the following data dependence result can be proved.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space, k a positive integer, and $T : X^k \to X$ an enriched Prešić operator with constants b_i, θ_i, $i = 0, 1, \dots, k-1$. Let $U : X^k \to X$ be an operator satisfying the following conditions:*
(i) *U has at least a fixed point $x_U^* \in X$;*
(ii) *there exists $\eta > 0$ such that for any $x \in X$,*

$$\|T(x, x, \dots, x) - U(x, x, \dots, x)\| \leq \eta.$$

Then,

$$\|x_T^* - x_U^*\| \leq \frac{\lambda_k \eta}{1 - \alpha},$$

where Fix(T) $= \{x_{\mathrm{T}}^*\}$, $\lambda_k = \frac{1}{\sum_{i=0}^{k-1} b_i + 1}$, *and* $\alpha = \lambda_k \sum_{i=0}^{k-1} \theta_i$.

Proof. For $\sum_{i=0}^{k-1} b_i = 0$, this reduces to the data dependence of Prešić operators, which has been studied in [18] in a metric space setting. So, further on we study the case $\sum_{i=0}^{k-1} b_i > 0$.

Similarly to the proof of Theorem 2.1, we denote $b = \sum_{i=0}^{k-1} b_i$. Then, we take $\lambda_k = \frac{1}{b+1} > 0$ and $\lambda_i = \lambda_k b_i$, for $i = 0, 1, \ldots, k-1$, thus having that $\sum_{i=0}^{k} \lambda_i = 1$.

For these values of $\lambda_0, \lambda_1, \ldots, \lambda_k$, we consider the averaged mappings $T_\lambda, U_\lambda : X^k \to X$ corresponding to the operators T and U, respectively.

According to Remark 2.2, x_T^* and x_U^* are fixed points for T_λ, and U_λ respectively, as well. Then,

$$\begin{aligned}\|x_U^* - x_T^*\| &= \|U_\lambda(x_U^*, x_U^*, \ldots, x_U^*) - T_\lambda(x_T^*, x_T^*, \ldots, x_T^*)\| \\ &\leq \|U_\lambda(x_U^*, x_U^*, \ldots, x_U^*) - T_\lambda(x_U^*, x_U^*, \ldots, x_U^*)\| + \|T_\lambda(x_U^*, x_U^*, \ldots, x_U^*) - T_\lambda(x_T^*, x_T^*, \ldots, x_T^*)\|.\end{aligned}$$

By Lemma 3.1, this becomes

$$\begin{aligned}\|x_U^* - x_T^*\| &\leq \lambda_k \eta + \|T_\lambda(x_U^*, x_U^*, \ldots, x_U^*) - T_\lambda(x_T^*, x_T^*, \ldots, x_T^*)\| \\ &\leq \lambda_k \eta + \|T_\lambda(x_U^*, x_U^*, \ldots, x_U^*) - T_\lambda(x_U^*, x_U^*, \ldots, x_U^*, x_T^*)\| + \\ &\quad + \|T_\lambda(x_U^*, x_U^*, \ldots, x_U^*, x_T^*) - T_\lambda(x_U^*, x_U^*, \ldots, x_U^*, x_T^*, x_T^*)\| + \\ &\quad + \cdots + \|T_\lambda(x_U^*, x_T^*, \ldots, x_T^*, x_T^*) - T_\lambda(x_T^*, \ldots, x_T^*)\|.\end{aligned}$$

Since T is an enriched Prešić operator, it follows that T_λ is a Prešić operator with constants $\alpha_i = \lambda_k \theta_i$, $i = 0, 1, \ldots, k-1$ (see the proof of Theorem 2.1).

Then,

$$\|x_U^* - x_T^*\| \leq \lambda_k \eta + \alpha_{k-1}\|x_U^* - x_T^*\| + \alpha_{k-2}\|x_U^* - x_T^*\| + \cdots + \alpha_0\|x_U^* - x_T^*\| = \lambda_k \eta + \alpha\|x_U^* - x_T^*\|,$$

where $\alpha = \sum_{i=0}^{k-1} \alpha_i < 1$.

The conclusion follows immediately. □

4 Stability for discrete population models. A discussion

In many articles on Prešić-type operators, various nonlinear difference equations of order k which model population dynamics are traditionally mentioned. An important step for any researcher developing new iterative methods, including operators on product spaces, is to have a closer look at this wide field of (potential) applications, understand the language in which it is described, see what kinds of problems are formulated there, and which are the most used techniques. The literature is very generous in this direction and the approaches are very diverse.

While the first-order difference equations appear generally in connection with discrete single-species models, there are also delayed-recruitment models, which involve higher-order difference equations, and age-class models, which lead to systems of difference equations, some of which can be written in an equivalent form as higher-order difference equations.

There is a standard approach of models that involve a k-th order difference equation (see, for example, [32–34]), that is, an equation of the form

$$x_{n+k} = f(x_{n+k-1}, \ldots, x_n). \tag{5}$$

The equilibrium is x_∞ such that $f(x_\infty, x_\infty, \ldots, x_\infty) = x_\infty$, which is actually a fixed point of f. The behavior of the solutions x_n near an equilibrium x_∞ is of great importance, once x_∞ is determined. An equilibrium is asymptotically stable if a small change in the initial size x_0 of a solution has a small effect on the behavior of the solutions when $n \to \infty$ or, in other words, if every solution with x_0 close enough to x_∞ remains close to x_∞ and tends to x_∞ as $n \to \infty$. Close enough is dictated actually by the second-order term, which is to be neglected when Taylor's theorem is applied.

Furthermore, linearization theorems state that an equilibrium is asymptotically stable if and only if all solutions of the corresponding linearization tend to zero, which basically means that $x_n \to \infty$ as $n \to 0$, if one starts close enough to x_∞.

Now, looking at all this from the position of fixed point theory, we recall a result from [35], see also [18]. The equation (5) has a global asymptotically stable equilibrium (that is, the solutions are stable for any x_0, not only when starting close to the equilibrium) if and only if the operator $A_f : X^k \to X^k$, where $A_f(u_1, u_2, \dots, u_k) = (u_2, u_3, \dots, u_k, f(u_1, u_2, \dots, u_k))$ is a Picard operator.

Having in view the results proved in this article, the following is obvious:

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space, k a positive integer, and $T : X^k \to X$ an enriched Prešić operator with constants b_i, θ_i, $i = 0, 1, \dots, k-1$. Then, the equation*

$$x_n = \lambda_0 x_{n-k} + \lambda_1 x_{n-k+1} + \cdots + \lambda_{k-1} x_{n-1} + \lambda_k T(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}),$$

where $x_0, x_1, \dots, x_k \in X$, *has a global asymptotically stable equilibrium.*

Although fixed point techniques seem more powerful, at least from a theoretical point of view, as they enable determining the equilibrium of such an equation-offering error estimates, data dependence results, etc., they still do not seem to be very popular in the practical approaches, at least not in the literature on mathematical biology. As the computational instruments have evolved over the past years, it would be interesting and probably rewarding to re-evaluate what fixed point results could offer regarding methods and techniques currently used in the study of various population models. This remains an open problem.

Funding information: This study was supported by the Babeș-Bolyai University 2022 Research Fund.

Conflict of interest: The author states that there is no conflict of interests.

References

[1] S. B. Prešić, *Sur une classe dainéquations aux différences finite et sur la convergence de certaines suites*, Publ. Inst. Math. (Beograd) (N.S.) **5** (1965), no. 19, 75–78.
[2] M. Alansari and M. U. Ali, *Interpolative Prešić type contractions and related results*, J. Funct. Spaces. (2022), Art. ID 6911475, 10 pp.
[3] D. Alecsa, *Common fixed points for Prešić operators via simulation functions*, J. Nonlin. Conv. Analysis **20** (2019), no. 3, 363–377.
[4] F. S. Alshammari, K. P. Reshma, R. Rajagopalan, and R. George, *Generalised Prešić type operators in modular metric space and an application to integral equations of Caratheodory type functions*, J. Math. **2021** (2021), Art. ID 7915448, 20 pp.
[5] I. Altun, M. Qasim and M. Olgun, *A new result of Prešić type theorems with applications to second order boundary value problems*, Filomat **35** (2021), no. 7, 2257–2266.
[6] S. Batul, D. Sagheer, H. Aydi, A. Mukheimer, and S. S. Aiadi, *Best proximity point results for Preăić type nonself operators in b-metric spaces*, AIMS Math. **7** (2022), no. 6, 10711–10730.
[7] M. Păcurar, *Approximating common fixed points of Prešić-Kannan type operators by a multi-step iterative method*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **17** (2009), no. 1, 153–168.
[8] M. Păcurar, *A multi-step iterative method for approximating common fixed points of Prešić-Rus type operators on metric spaces*, Stud. Univ. Babeş-Bolyai Math. **55** (2010), no. 1, 149–162.
[9] M. Păcurar, *A multi-step iterative method for approximating fixed points of Prešić-Kannan operators*, Acta Math. Univ. Comenian. (N.S.) **79** (2010), no. 1, 77–88.
[10] V. Berinde and M. Păcurar, *Two elementary applications of some Prešić type fixed point theorems*, Creat. Math. Inform. **20** (2011), no. 1, 32–42.
[11] M. Păcurar, *Fixed points of almost Prešić operators by a k-step iterative method*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **57** (2011), suppl. 1, 199–210.
[12] M. Păcurar, *Common fixed points for almost Prešić type operators*, Carpathian J. Math. **28** (2012), no. 1, 117–126.
[13] M. Abbas, M. Berzig, T. Nazir, and E. Karapınar, *Iterative approximation of fixed points for Prešić type F-contraction operators*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78** (2016), no. 2, 147–160.
[14] H. Aydi, E. Karapınar, and C. Vetro, *Meir-Keeler type contractions for tripled fixed points*, Acta Math. Sci. Ser. B (Engl. Ed.) **32** (2012), no. 6, 2119–2130.
[15] Y.-Z. Chen, *A Prešić type contractive condition and its applications*, Nonlinear Analysis **71** (2009), no. 12, e2012–e2017 pp.

[16] S. Stević, *Asymptotic behavior of a class of nonlinear difference equations*, Discrete Dyn. Nat. Soc. **2006** (2006), Art. ID 47156, 10 pp.

[17] M. De la Sen and E. Karapinar, *Best proximity points of generalized semicyclic impulsive self-mappings: applications to impulsive differential and difference equations*, Abstr. Appl. Anal. (2013), Art. ID 505487, 16 pp.

[18] M. Păcurar, *Iterative Approximation of Fixed Points*, Risoprint, Cluj-Napoca, 2009.

[19] V. Berinde and M. Păcurar, *Approximating fixed points of enriched contractions in Banach spaces*, J. Fixed Point Theory Appl. **22** (2020), Paper no. 38, 10 pp.

[20] V. Berinde, *Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition*, Carpathian J. Math. **36** (2020), no. 1, 27–34.

[21] V. Berinde and M. Păcurar, *Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces*, Carpathian J. Math. **37** (2021), no. 2, 173–184.

[22] T. Kesahorm and W. Sintunavarat, *On novel common fixed point results for enriched nonexpansive semigroups*, Thai J. Math. **18** (2020), no. 3, 1549–1563.

[23] A. Marchiş, *Common fixed point theorems for enriched Jungck contractions in Banach spaces*, J. Fixed Point Theory Appl.. **23** (2021), no. 4, Paper no. 76, 13 pp.

[24] S. Suantai, D. Chumpungam and P. Sarnmeta, *Existence of fixed points of weak enriched nonexpansive mappings in Banach spaces*, Carpathian J. Math. **37** (2021), no. 2, 287–294.

[25] S. Panja, M. Saha and R. Bisht, *Existence of common fixed points of non-linear semigroups of enriched Kannan contractive mappings*, Carpathian J. Math. **38** (2022), no. 1, 169–178.

[26] R. Shukla and R. Panicker, *Approximating fixed points of enriched nonexpansive mappings in geodesic spaces*, J. Funct. Spaces **2022** (2022), Art. ID 6161839, 8 pp.

[27] M. Abbas, R. Anjum, and H. Iqbal, *Generalized enriched cyclic contractions with application to generalized iterated function system*, Chaos Solitons Fractals. **154** (2022), Paper no. 111591, 9 pp.

[28] A. Petruşel and I.A. Rus, *Stability of Picard operators under operator perturbations*, An. Univ. Vest Timiş. Ser. Mat.-Inform. **56** (2018), no. 2, 3–12.

[29] J. Brzdęk, E. Karapınar, and A. Petruşel, *A fixed point theorem and the Ulam stability in generalized dq-metric spaces*, J. Math. Anal. Appl. **467** (2018), no. 1, 501–520.

[30] E. Karapınar and A. Fulga, *An admissible hybrid contraction with an Ulam type stability*, Demonstr. Math. **52** (2019), no. 1, 428–436.

[31] A. Salim, S. Abbas, M. Benchohra, and E. Karapinar, *Global stability results for Volterra-Hadamard random partial fractional integral equations*, Rend. Circ. Mat. Palermo, series 2 **2022** (2022), 1–13.

[32] F. Brauer and C. Castillo-Chavez, Mathematical models in population biology and epidemiology, 2nd edition, *Texts in Applied Mathematics*, vol. 40, Springer, New York, 2012.

[33] S. Elaydi, An introduction to difference equations, 3rd edition, *Undergraduate Texts in Mathematics*, Springer, New York, 2005.

[34] A. R. Lucas, *A fixed point theorem for a general epidemic model*, J. Math. Anal. Appl. **404** (2013), no. 1, 135–149.

[35] I. A. Rus, An abstract point of view in the nonlinear difference equations, *Conf. on An., Functional Equations, App. and Convexity*, Cluj-Napoca, October 15–16, 1999, pp. 272–276