Research Article

Ovidiu Popescu*

Fixed-point results for convex orbital operators

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Abstract: The aim of this article is to introduce a new type of operator similar to those of A. Petruşel and G. Petruşel type (*Fixed point results for decreasing convex orbital operators*, J. Fixed Point Theory Appl. **23** (2021), no. 35) and prove some fixed-point theorems which generalize and complement several results in the theory of nonlinear operators.

Keywords: Hilbert space, graphic contraction, convex orbital Lipschitz operator, weakly Picard operator, fixed-point

MSC 2020: 47H10, 54H25

1 Introduction and preliminaries

We recall some important concepts for the fixed-point theory.

Definition 1.1. Let (X, d) be a metric space. Then, $T : X \to X$ is called a *Picard operator* if:
(i) $F_T = \{x^*\}$, where $F_T = \{x \in X : x = T(x)\}$ is the fixed-point set of T;
(ii) the sequence of iterates $(T^n(x))_{n\in\mathbf{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 1.2. Let (X, d) be a metric space. Then, $T : X \to X$ is called a *weakly Picard operator* if, for any $x \in X$, the sequence of iterates $(T^n(x))_{n\in\mathbf{N}}$ converges to a fixed-point of T.

In this case, the mapping $T^\infty : X \to F_T$, given by $T^\infty := \lim_{n\to\infty} T^n(x)$ is a set retraction on F_T.

Definition 1.3. [1] Let (X, d) be a metric space and $T : X \to X$ be an operator such that F_T is nonempty. Let $r : X \to F_T$ be a set retraction. We say that T satisfies a *retraction-displacement condition* if there exists $c > 0$ such that for every $x \in X$

$$d(x, r(x)) \le cd(x, T(x)).$$

Definition 1.4. Let (X, d) be a metric space, $T : X \to X$ be an operator such that F_T is nonempty, and $r : X \to F_T$ be a set retraction. Then:
(i) the fixed-point equation $x = T(x)$ is called *well posed* in the sense of Reich and Zaslavski (see [2,3]) if for each $x^* \in F_T$ and any sequence $(u_n)_{n\in\mathbf{N}}$ in $r^{-1}(x^*)$ for which $d(u_n, T(u_n)) \to 0$ as $n \to \infty$, we have that $u_n \to x^*$ as $n \to \infty$.
(ii) the operator T has *the Ostrowski property* (see [4,5]) if for each $x^* \in F_T$ and any sequence $(u_n)_{n\in\mathbf{N}}$ in $r^{-1}(x^*)$ for which $d(u_{n+1}, T(u_n)) \to 0$ as $n \to \infty$, we have that $u_n \to x^*$ as $n \to \infty$.

* **Corresponding author: Ovidiu Popescu,** Department of Mathematics and Computer Sciences, Transilvania University of Braşov, Braşov, Romania, e-mail: ovidiu.popescu@unitbv.ro

(iii) the fixed-point equation $x = T(x)$ is *Ulam-Hyers stable* (see [6,7]) if there exists $c > 0$ such that for every $\varepsilon > 0$ and every ε-fixed-point $y^* \in X$ of T (i.e., $d(y^*, T(y^*)) \leq \varepsilon$), there exists a fixed-point $x^* \in X$ of T such that $d(x^*, y^*) \leq c\varepsilon$.

Recently, Petruşel and Rus [8] proved the following important theorem :

Theorem 1.1. (Graphic contraction principle) *Let (X, d) be a complete metric space and $f : X \to X$ be a graphic k-contraction, i.e., there exists $k \in (0, 1)$ such that*

$$d(f(x), f^2(x)) \leq kd(x, f(x)),$$

for every $x \in X$.

If f has a closed graph, then:

(1) *the sequence of iterates $(f^n(x_0))_{n\in\mathbf{N}}$ converges in (X, d) to a fixed-point $x^*(x_0)$ of f;*
(2) $F_f = F_{f^n} \neq \emptyset$ *for all $n \in \mathbf{N}^*$;*
(3) *f is a weakly Picard operator;*
(4) $d(x, f^\infty(x)) \leq \frac{1}{1-k} d(x, f(x))$, *for every $x \in X$, i.e., f is a $\frac{1}{1-k}$-weakly Picard operator;*
(5) *the fixed-point equation $x = f(x)$ is well posed in the sense of Reich and Zaslavski;*
(6) *the fixed-point equation $x = f(x)$ is Ulam-Hyers stable*:
(7) *if $k < 1/3$, then $d(f(x), f^\infty(x)) \leq \frac{k}{1-2k} d(x, f^\infty(x))$, for every $x \in X$, i.e., f is a $\frac{k}{1-2k}$-quasicontraction;*
(8) *if $k < 1/3$, then f has the Ostrowski stability property.*

Very recently, Petruşel and Petruşel [9] gave the following notion of convex orbital β-Lipschitz operator.

Definition 1.5. Let $(X, \|\cdot\|)$ be a normed space and Y be a nonempty and convex subset of X. Let $T : Y \to Y$ be an operator and $\lambda \in (0, 1]$. We say that T is a *convex orbital β-Lipschitz operator* if $\beta > 0$ and for any $x \in Y$

$$\|T(x) - T((1-\lambda)x + \lambda T(x))\| \leq \beta\lambda\|x - T(x)\|.$$

They proved that this class of operators includes the Banach contractions, Kannan contractions, Ćirić-Reich-Rus contractions, Berinde contractions, nonexpansive operators, enriched (β, θ) contractions, and Lipschitz operators [10–16]. From now on, we will use the symbol Tx instead of $T(x)$. The main results in [9] are the following theorems:

Theorem 1.2. *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, Y be a nonempty closed and convex subset of X, and $T : Y \to Y$ be an operator with closed graph. We suppose that*:
(*i*) *T is a convex orbital β-Lipschitz;*
(*ii*) *T is decreasing, i.e.,* $\mathrm{Re}(\langle Tu - Tv, u - v\rangle) \leq 0$, *for every $u, v \in Y$.*

Then, there exists $\lambda \in (0, 1)$ such that, for every $x_0 \in Y$, the sequence $(x_n)_{n\in N} \subset Y$, defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ n \in N,$$

converges to the unique fixed-point $x^* \in Y$ of T.

Theorem 1.3. *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, Y be a nonempty closed and convex subset of X, and $T : Y \to Y$ be an operator satisfying all the assumptions in* Theorem 1.2. *If $x^* \in Y$ is the unique fixed-point of T, then the following conclusions hold*:
(*a*) *T satisfies the following retraction-displacement condition*

$$\|x - x^*\| \leq (1 + k)\|x - Tx\|,$$

for every $x \in Y$, where $k := \frac{\beta}{\sqrt{1+\beta^2}}$;
(*b*) *the fixed-point equation $x = Tx$ is Ulam-Hyers stable;*
(*c*) *the fixed-point equation $x = Tx$ is well posed.*

Theorem 1.4. *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, Y be a nonempty closed and convex subset of X, and $T : Y \to Y$ be an operator satisfying all the assumptions in* Theorem 1.2. *If $\beta < \sqrt{\frac{5-2\sqrt{5}}{5}}$, then T is a quasicontraction.*

In this article we will clarify and complement the notion of convex orbital β-Lipschitz, by introducing some new classes of convex orbital operators. Then similar results as Theorems 1.2, 1.3, and 1.4 for these new types of operators are proved.

2 Main results

Definition 2.1. [9] Let $(X, \|\cdot\|)$ be a normed space and Y be a nonempty and convex subset of X. Let $T : Y \to Y$ be an operator. We say that T is a *convex orbital β-Lipschitz operator* if there exists $\beta > 0$ such that for any $\lambda \in (0, 1]$, and for any $x \in Y$,

$$\|Tx - TT_\lambda x\| \leq \beta\lambda\|x - Tx\|,$$

where $T_\lambda x := (1-\lambda)x + \lambda Tx$.

Definition 2.2. Let $(X, \|\cdot\|)$ be a normed space and Y be a nonempty and convex subset of X. Let $T : Y \to Y$ be an operator. We say that T is a *weak convex orbital Lipschitz operator* if for any $\lambda \in (0, 1]$ there exists $\beta > 0$ such that for any $x \in Y$,

$$\|Tx - TT_\lambda x\| \leq \beta\lambda\|x - Tx\|.$$

Definition 2.3. Let $(X, \|\cdot\|)$ be a normed space and Y be a nonempty and convex subset of X. Let $T : Y \to Y$ be an operator. We say that T is a *convex orbital (λ, β)-Lipschitz operator* if there exists $\lambda \in (0, 1]$ and $\beta > 0$ such that for any $x \in Y$

$$\|Tx - TT_\lambda x\| \leq \beta\lambda\|x - Tx\|.$$

Obviously, every convex orbital β-Lipschitz operator is a weak convex orbital Lipschitz operator and every weak convex orbital Lipschitz operator is a convex orbital (λ, β)-Lipschitz operator.

Example 2.1. (See Example 2.7 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be an L-Lipschitz operator, i.e., $L > 0$ and for each $x, y \in Y$

$$\|Tx - Ty\| \leq L\|x - y\|.$$

Then, T is a convex orbital L-Lipschitz operator. Indeed, if we choose $y := T_\lambda x$ in the aforementioned inequality, we have that

$$\|Tx - TT_\lambda x\| \leq L\|x - T_\lambda x\| = L\lambda\|x - Tx\|,$$

for any $\lambda \in (0, 1]$ and for every $x \in Y$.

Example 2.2. (See Example 2.1 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be an α-contraction, i.e., $\alpha \in (0, 1)$ and for each $x, y \in Y$

$$\|Tx - Ty\| \leq \alpha\|x - y\|.$$

Since every α-contraction is a Lipschitz operator with $L := \alpha$, then T is a convex orbital α-Lipschitz operator.

Example 2.3. (See Example 2.5 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be a nonexpansive operator, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Since every nonexpansive operator is a Lipschitz operator with $L := 1$, then T is a convex orbital 1-Lipschitz operator.

Example 2.4. (See Example 2.6 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be an enriched (b, θ)-contraction, i.e., there exist $b \geq 0$, $\theta \in [0, b + 1)$ such that for each $x, y \in Y$

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\|.$$

Then, T is a convex orbital $(b + \theta)$-Lipschitz operator. Indeed, if we choose $y := T_\lambda x$ in the aforementioned relation, we obtain that

$$\|b\lambda(x - Tx) + Tx - TT_\lambda x\| \leq \theta\lambda\|x - Tx\|,$$

from which we obtain

$$\|Tx - T_\lambda x\| - b\lambda\|x - Tx\| \leq \theta\lambda\|x - Tx\|,$$

for any $\lambda \in (0, 1]$ and for every $x \in Y$. Hence, we obtain that

$$\|Tx - TT_\lambda x\| \leq (b + \theta)\lambda\|x - Tx\|.$$

Example 2.5. (See Example 2.2 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be a Kannan γ-contraction, i.e., $\gamma \in [0, 1/2)$, and for each $x, y \in Y$,

$$\|Tx - Ty\| \leq \gamma[\|x - Tx\| + \|y - Ty\|].$$

Then, T is a weak convex orbital Lipschitz operator. Indeed, if we insert in the aforementioned inequality $y := T_\lambda x$, then we obtain for $\lambda \in (0, 1]$ and $x \in Y$ that

$$\|Tx - TT_\lambda x\| \leq \gamma[\|x - Tx\| + \|(1 - \lambda)x + \lambda Tx - TT_\lambda x\|] \leq \gamma[\|x - Tx\| + (1 - \lambda)\|x - Tx\| + \|Tx - TT_\lambda x\|].$$

Hence, we obtain

$$\|Tx - TT_\lambda x\| \leq \frac{\gamma(2 - \gamma)}{1 - \gamma}\|x - Tx\|.$$

Therefore, T is a weak convex orbital Lipschitz operator $\left(\beta = \frac{\gamma(2-\gamma)}{\lambda(1-\gamma)}\right)$. Obviously, $\lim_{\lambda \to 0}\beta(\lambda) = \infty$, so T is not a convex orbital β-Lipschitz operator.

Example 2.6. (See Example 2.3 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be a Ćirić-Reich-Rus (α, γ)-contraction, i.e., $\alpha, \gamma \in \mathbf{R}_+$ with $\alpha + 2\gamma < 1$, and for each $x, y \in Y$,

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \gamma[\|x - Tx\| + \|y - Ty\|].$$

Then, T is a weak convex orbital Lipschitz operator. Indeed, if we insert in the aforementioned inequality $y := T_\lambda x$, we obtain that

$$\|Tx - TT_\lambda x\| \leq \alpha\|x - T_\lambda x\| + \gamma[\|x - Tx\| + \|T_\lambda x - TT_\lambda x\|].$$

This implies

$$\|Tx - TT_\lambda x\| \leq \alpha\lambda\|x - Tx\| + \gamma[\|x - Tx\| + (1 - \lambda)\|x - Tx\| + \|Tx - TT_\lambda x\|].$$

Hence,

$$\|Tx - TT_\lambda x\| \leq \frac{\alpha\lambda + \gamma(2 - \lambda)}{1 - \gamma}\|x - Tx\|.$$

Therefore, T is a weak convex orbital Lipschitz operator $\left(\beta = \frac{\alpha\lambda + \gamma(2-\lambda)}{\lambda(1-\gamma)}\right)$. Obviously, $\lim_{\lambda \to 0}\beta(\lambda) = \infty$, so T is not a convex orbital β-Lipschitz operator.

Example 2.7. (See Example 2.4 of [9]) Let $(X, \|.\|)$ be a normed space, Y be a nonempty and convex subset of X, and $T : Y \to Y$ be a Berinde (α, L)-contraction, i.e., $\alpha, L \in \mathbf{R}_+$ with $\alpha < 1$, and for each $x, y \in Y$,

$$\|Tx - Ty\| \le \alpha\|x - y\| + L\|y - Tx\|.$$

Then, T is a weak convex orbital Lipschitz operator. Indeed, if we insert in the aforementioned inequality $y := T_\lambda x$, we obtain that

$$\|Tx - TT_\lambda x\| \le \alpha\|x - Tx\| + L\|T_\lambda x - Tx\| \le \alpha\|x - Tx\| + L(1 - \lambda)\|x - Tx\|,$$

where we obtain for every $\lambda \in (0, 1]$ and each $x \in Y$

$$\|Tx - TT_\lambda x\| \le (\alpha\lambda + L(1 - \lambda))\|x - Tx\|.$$

Therefore, T is a weak convex orbital Lipschitz operator $\left(\beta = \frac{\alpha\lambda + L(1-\lambda)}{\lambda}\right)$. Obviously, $\lim_{\lambda\to 0}\beta(\lambda) = \infty$, so T is not a convex orbital β-Lipschitz operator.

The following examples show that there exist convex orbital (λ, β)-Lipschitz operators, which are not weak convex orbital Lipschitz operators.

Example 2.8. Let $X = Y = \mathbf{R}$, $T : \mathbf{R} \to \mathbf{R}$ be a mapping defined by $Tx := -x$ if $x \ne 0$ and $Tx := 1$ if $x = 0$. We have $T_1x = Tx$, $TT_1x = T^2x = x$ if $x \ne 0$ and $TT_1x = T^2x = -1$ if $x = 0$. Then, we obtain that $Tx - TT_1x = -2x$ if $x \ne 0$, $Tx - TT_1x = 2$ if $x = 0$, $x - Tx = 2x$ if $x \ne 0$, and $x - Tx = -1$ if $x = 0$. It is easy to see that the inequality $\|Tx - TT_1x\| \le 2\|x - Tx\|$ holds for every x, so T is a convex orbital $(1, 2)$-Lipschitz operator. Moreover, $T_{1/2}x = (x + Tx)/2 = 0$ if $x \ne 0$ and $T_{1/2}x = 1/2$ if $x = 0$. Thus, $TT_{1/2}x = 1$ if $x \ne 0$, $TT_{1/2}x = -1/2$ if $x = 0$, $Tx - TT_{1/2}x = -x - 1$ if $x \ne 0$, and $Tx - TT_{1/2}x = 3/2$ if $x = 0$. For $x \ne 0$, the inequality $\|Tx - TT_{1/2}x\| \le \beta/2\|x - Tx\|$ is equivalent to $|x + 1| \le \beta|x|$. For $x \to 0$, we obtain a contradiction. Then, T is not a weak convex orbital Lipschitz operator.

Example 2.9. Let $X = Y = \mathbf{R}$, $T : \mathbf{R} \to \mathbf{R}$ be a mapping defined by $Tx := 1 + \frac{1}{x-1}$ if $x > 1$, $Tx := 2$ if $x \in [-1, 1]$, and $Tx := 1 + \frac{1}{-x-1}$ if $x < -1$. For $x > 1$, we have $T_1x = Tx$, $TT_1x = T^2x = x$, $Tx - TT_1x = 1 + \frac{1}{x-1} - x$, and $x - Tx = x - 1 - \frac{1}{x-1}$. Then, $\|Tx - TT_1x\| \le \|x - Tx\|$. If $x \in [-1, 1]$, we have $TT_1x = T^2x = 2$, $Tx - TT_1x = 2 - x$ and $x - Tx = x - 2$, by where we obtain $\|Tx - TT_1x\| \le \|x - Tx\|$. For $x < -1$, we have $TT_1x = T^2x = -x$, $Tx - TT_1x = 1 - \frac{1}{1+x} + x = \frac{x^2+2x}{1+x}$ and $x - Tx = x - 1 + \frac{1}{1+x} = \frac{x^2}{1+x}$. Since $1 + 2/x \in (-1, 1)$, we obtain $|x^2 + 2x| \le |x^2|$, by where $\|Tx - TT_1x\| \le \|x - Tx\|$. Therefore, T is a convex orbital $(1, 1)$-Lipschitz operator. Since $T_{1/2}\frac{1}{n} = 1 + \frac{1}{2n}$, $TT_{1/2}\frac{1}{n} = 1 + 2n$, $T\frac{1}{n} - TT_{1/2}\frac{1}{n} = 1 - 2n$, and $\frac{1}{n} - T\frac{1}{n} = \frac{1}{n} - 1$, the inequality $\left\|T\frac{1}{n} - TT_{1/2}\frac{1}{n}\right\| \le (1/2)\beta\left\|\frac{1}{n} - T\frac{1}{n}\right\|$ is not satisfied for n sufficiently large. Hence, T is not a weak convex orbital Lipschitz operator. Also, it is easy to see that T has a closed graph.

In the following example, we present a weak convex orbital Lipschitz operator with a closed graph, which is not a convex orbital β- Lipschitz operator.

Example 2.10. Let $X = Y = \mathbf{R}$, $T : \mathbf{R} \to \mathbf{R}$ be a mapping defined by $Tx := 1/x$ if $x \ne 0$ and $Tx := 1$ if $x = 0$. Obviously, T has a closed graph. For $x \ne 0$, we have $T_\lambda x = (1 - \lambda)x + \frac{\lambda}{x} > 0$, $TT_\lambda x = \frac{x}{(1-\lambda)x^2+\lambda}$, $Tx - TT_\lambda x = \frac{\lambda(1-x^2)}{x[(1-\lambda)x^2+\lambda]}$, and $x - Tx = \frac{x^2-1}{x}$. Taking $\beta := 1/\lambda$, we have $\frac{1}{(1-\lambda)x^2+\lambda} \le \beta$, so $\frac{\lambda|1-x^2|}{x[(1-\lambda)x^2+\lambda]} \le \lambda\beta\frac{|x^2-1|}{|x|}$. Then, we obtain $\|Tx - TT_\lambda x\| \le \lambda\beta\|x - Tx\|$. If $x = 0$, we have $T_\lambda x = \lambda$, $TT_\lambda x = \frac{1}{\lambda}$, $Tx - TT_\lambda x = 1 - \frac{1}{\lambda}$, and $x - Tx = -1$. Taking $\beta = \frac{1-\lambda}{\lambda^2}$, we obtain $\left|1 - \frac{1}{\lambda}\right| \le \lambda\beta$, so $\|Tx - TT_\lambda x\| \le \lambda\beta\|x - Tx\|$. Therefore, for any $\lambda \in (0, 1]$, there exists $\beta := \max\left\{\frac{1}{\lambda}, \frac{1-\lambda}{\lambda^2}\right\}$ such that $\|Tx - TT_\lambda x\| \le \lambda\beta\|x - Tx\|$, i.e., T is a weak convex orbital Lipschitz operator. Now, if we take $x = \lambda = 1/n$ with $n \ge 2$, we have $Tx - TT_\lambda x = \frac{n^3-n}{n^2+n-1}$, $x - Tx = \frac{1-n^2}{n}$. Then, the inequality $\|Tx - TT_\lambda x\| \le \lambda\beta\|x - Tx\|$ is equivalent to the inequality $\frac{n^3}{n^2+n-1} \le \beta$, which is not satisfied for n sufficiently large. Hence, T is not a convex orbital β- Lipschitz operator.

Now, we give similar results of Theorems 1.2, 1.3, and 1.4 (which hold for convex orbital β-Lipschitz operators) for convex orbital (λ, β)-Lipschitz operators.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and Y be a nonempty closed and convex subset of X. Let $T : Y \to Y$ be a convex orbital (λ, β)-Lipschitz operator with closed graph, where $\beta < 1$. Then, for every $x_0 \in Y$, the sequence $(x_n)_{n\in N} \subset Y$, defined by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \in N,$$

converges to a fixed-point $x^*(x_0)$ of T.

Proof. Let the operator $T_\lambda : Y \to Y$ defined by

$$T_\lambda := (1-\lambda)x + \lambda Tx,\ x \in Y.$$

It is easy to see that $F_T = F_{T_\lambda}$ and T_λ has a closed graph. For every $x, y \in Y$, we have

$$\|T_\lambda x - T_\lambda y\| = \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\| \le (1-\lambda)\|x-y\| + \lambda\|Tx - Ty\|.$$

Taking $y := T_\lambda x$, we obtain

$$\begin{aligned}\|T_\lambda x - T_\lambda^2 x\| &\le (1-\lambda)\|x - T_\lambda x\| + \lambda\|Tx - TT_\lambda x\| \\ &\le (1-\lambda)\|x - T_\lambda x\| + \beta\lambda^2\|x - Tx\| \\ &= (1-\lambda)\|x - T_\lambda x\| + \beta\lambda\|x - T_\lambda x\| \\ &= (1-\lambda+\beta\lambda)\|x - T_\lambda x\|.\end{aligned}$$

Since $\beta < 1$, if we denote $k := 1 - \lambda + \beta\lambda$, then $k < 1$ and

$$\|T_\lambda x - T_\lambda^2 x\| \le k\|x - T_\lambda x\|,$$

for every $x \in Y$. This shows that $T_\lambda : Y \to Y$ is a graphic k-contraction. Hence, by the graphic contraction principle, T_λ is a weakly Picard operator. Since $F_T = F_{T_\lambda}$, we have $F_T \ne \emptyset$ and the sequence $(T_\lambda^n x_0)_{n\in N}$ converges to $T_\lambda^\infty x_0 := x^*(x_0) \in F_T$, for every $x_0 \in Y$. □

The following theorem is our main result.

Theorem 2.2. *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, Y be a nonempty closed and convex subset of X, and $T : Y \to Y$ be an operator with a closed graph. We suppose that:*

(*i*) *T is a convex orbital (λ, β)-Lipschitz operator with $\beta \ge 1$;*

(*ii*) $\mathrm{Re}(\langle Tu - Tv, u - v\rangle) \le \mu\,\|u - v\|^2$, *for every $u, v \in Y$, where $\mu < \frac{2-\lambda(1+\beta^2)}{2(1-\lambda)}$.*

Then, for every $x_0 \in Y$, the sequence $(x_n)_{n\in N} \subset Y$, defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \in N,$$

converges to the unique fixed-point $x^* \in Y$ of T.

Proof. Consider the operator $T_\lambda : Y \to Y$ defined by

$$T_\lambda := (1-\lambda)x + \lambda Tx,\ x \in Y.$$

Obviously, $F_T = F_{T_\lambda}$ and T_λ has a closed graph. By using (ii), for every $x, u \in Y$, we have:

$$\begin{aligned}\|T_\lambda x - T_\lambda u\|^2 &= \|(1-\lambda)(x-u) + \lambda(Tx - Tu)\|^2 \\ &\le (1-\lambda)^2\|x-u\|^2 + \lambda^2\,\|Tx - Tu\|^2 + 2\lambda(1-\lambda)\,\mathrm{Re}(\langle Tx - Tu, x - u\rangle) \\ &\le (1-\lambda)^2\|x-u\|^2 + \lambda^2\,\|Tx - Tu\|^2 + 2\lambda(1-\lambda)\mu\,\|x-u\|^2.\end{aligned}$$

Taking $u := T_\lambda x$ in the aforementioned inequality, we obtain

$$\begin{aligned}\|T_\lambda x - T_\lambda^2 x\|^2 &\le [(1-\lambda)^2 + 2\lambda(1-\lambda)\mu]\|x - T_\lambda x\|^2 + \lambda^2\,\|Tx - TT_\lambda x\|^2\\ &= [(1-\lambda)^2 + 2\lambda(1-\lambda)\mu]\|x - T_\lambda x\|^2 + \lambda^4\beta^2\,\|x - Tx\|^2\\ &= [(1-\lambda)^2 + 2\lambda(1-\lambda)\mu]\|x - T_\lambda x\|^2 + \lambda^2\beta^2\,\|x - T_\lambda x\|^2\\ &= [(1-\lambda)^2 + 2\lambda(1-\lambda)\mu + \lambda^2\beta^2]\|x - T_\lambda x\|^2.\end{aligned}$$

If we denote by $k := \sqrt{(1-\lambda)^2 + 2\lambda(1-\lambda)\mu + \lambda^2\beta^2}$, we have by (ii) that $k < 1$ and

$$\|T_\lambda x - T_\lambda^2 x\| \le k\|x - T_\lambda x\|,$$

for every $x \in Y$. Thus, by graphic contraction principle, T_λ is a weakly Picard operator and the sequence $(T_\lambda^n x_0)_{n\in N}$ converges to $T_\lambda^\infty x_0 := x^*(x_0) \in F_T$, for every $x_0 \in Y$.

Now, let us suppose that there exist $x^*, y^* \in F_T$ with $x^* \neq y^*$. Then, we have $x^* = Tx^* = T_\lambda x^*$ and $y^* = Ty^* = T_\lambda y^*$. Taking $u := x^*$ and $v := y^*$ in (ii), we obtain

$$\mathrm{Re}(\langle Tx^* - Ty^*, x^* - y^*\rangle) \le \mu\,\|x^* - y^*\|^2.$$

Hence,

$$\|x^* - y^*\|^2 \le \mu\,\|x^* - y^*\|^2.$$

Since $\beta \ge 1$, we have $2 - \lambda(1+\beta)^2 \le 2(1-\lambda)$, and hence, $\mu < 1$. Therefore, $\|x^* - y^*\| = 0$, which is a contradiction. Thus, $F_T = F_{T_\lambda} = \{x^*\}$ and T_λ is a Picard operator. □

We will illustrate the aforementioned theorem by the following example:

Example 2.11. Let $T : R^2 \to R^2$ be a mapping defined by

$$T(x,y) := \frac{3}{4}(x - y, x + y).$$

Then:
(a) T is a convex orbital $(1/2, 3\sqrt{2}/2)$-Lipschitz operator;
(b) T satisfies (ii) of Theorem 2.2 with $\mu = 3/4$;
(c) T is continuous on R^2;
(d) T is not decreasing on R^2;
(e) $F_T = \{(0,0)\}$ and $\|T_{1/2}^n(x,y)\| = (\sqrt{58}/8)^n\|(x,y)\| \to 0$ as $n \to \infty$.

(a) For $(x,y) \in R^2$, we have:

$$T_{1/2}(x,y) = (1/2)(x,y) + (1/2)T(x,y) = (1/8)(7x - 3y, 3x + 7y),$$

$$TT_{1/2}(x,y) = (3/16)(2x - 5y, 5x + 2y).$$

Hence, we obtain:

$$T(x,y) - TT_{1/2}(x,y) = (3/16)(2x + y, -x + 2y).$$

Since $(x,y) - T(x,y) = (1/4)(x + 3y, x - 3y)$, we obtain:

$$\|T(x,y) - TT_{1/2}(x,y)\| = (3\sqrt{5}/16)\sqrt{x^2 + y^2}$$

and

$$\|(x,y) - T(x,y)\| = (\sqrt{10}/4)\sqrt{x^2 + y^2}.$$

Therefore, we have

$$\|T(x,y) - TT_{1/2}(x,y)\| = (1/2)(3\sqrt{2}/4)\|(x,y) - T(x,y)\|.$$

Thus, T is a convex orbital $(1/2, 3\sqrt{2}/4)$-Lipschitz operator.

(b) If $(x_1, y_1), (x_2, y_2) \in R^2$, then we have:

$$T(x_1, y_1) - T(x_2, y_2) = (3/4)(x_1 - x_2 - (y_1 - y_2), x_1 - x_2 + y_1 - y_2).$$

Hence,

$$\begin{aligned}&\mathrm{Re}(\langle T(x_1, y_1) - T(x_2, y_2), (x_1, y_1) - (x_2, y_2)\rangle)\\ &= (3/4)[(x_1 - x_2) - (y_1 - y_2)](x_1 - x_2) + (3/4)[x_1 - x_2 + y_1 - y_2](y_1 - y_2)\\ &= (3/4)[(x_1, -x_2)^2 + (y_1 - y_2)^2] = (3/4)\|(x_1, y_1) - (x_2, y_2)\|^2.\end{aligned}$$

Therefore, T satisfies (ii) of Theorem 2.2 with $\mu = 3/4$. We note that $\mu < \frac{2-\lambda(1+\beta^2)}{2(1-\lambda)} = 15/16$.

(c) It is obvious.

(d) For $(x_1, y_1) = (2, 0)$ and $(x_2, y_2) = (1, 0)$, we have

$$\mathrm{Re}(\langle T(x_1, y_1) - T(x_2, y_2), (x_1, y_1) - (x_2, y_2)\rangle) = 3/4 > 0,$$

hence T is not decreasing.

(e) It is easy to see that $F_T = \{(0, 0)\}$ and $\|T_{1/2}(x, y)\| = (\sqrt{58}/8)\|(x, y)\|$. This implies that $\|T^n_{1/2}(x, y)\| = (\sqrt{58}/8)^n\|(x, y)\| \to 0$ as $n \to \infty$.

By the previous theorems, we obtain some additional properties of the fixed-point equation $x = Tx$.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and Y be a nonempty closed and convex subset of X. Let $T : Y \to Y$ be a convex orbital (λ, β)-Lipschitz operator with a closed graph, where $\beta < 1$. Then, the following conclusions hold*:

(*a*) *T satisfies the following retraction-displacement condition*

$$\|x - x^*(x)\| \le \frac{1}{1-\beta}\|x - Tx\|,$$

for every $x \in Y$;

(*b*) *the fixed-point equation $x = Tx$ is Ulam-Hyers stable*;

(*c*) *if $\beta < 1/3$ and $\lambda > \frac{2}{3(1-\beta)}$, then T has the Ostrowski stability property.*

Proof. (a) By the proof of Theorem 2.1, the operator $T_\lambda : Y \to Y$, given by $T_\lambda x := (1-\lambda)x + \lambda Tx$ is weakly Picard. By graphic contraction principle, we obtain

$$\|x - x^*(x)\| \le \frac{1}{1-k}\|x - T_\lambda x\|,$$

for every $x \in Y$, where $(T^n_\lambda x)_{n\in N}$ converges to $x^*(x)$ and $k = 1 - \lambda + \lambda\beta$. Since $\|x - T_\lambda x\| = \lambda\|x - Tx\|$, we obtain that

$$\|x - x^*(x)\| \le \frac{\lambda}{1-k}\|x - Tx\| = \frac{1}{1-\beta}\|x - Tx\|,$$

for every $x \in Y$. This proves that T satisfies the (c, r)-retraction-displacement condition, where $c := \frac{1}{1-\beta}$ and $r : Y \to F_T$ is given by $r(x) := x^*(x)$, $x \in Y$.

(b) Let $\varepsilon > 0$ and $y \in Y$ such that $\|y - Ty\| \le \varepsilon$. Then, we have

$$\|y - x^*(y)\| \le \frac{1}{1-\beta}\|y - Ty\| \le \frac{\varepsilon}{1-\beta}.$$

(c) By the graphic contraction principle, we know that T has the Ostrowski stability property if $k < 1/3$. This means that $1 - \lambda + \lambda\beta < 1/3$, i.e., $\lambda > \frac{2}{3(1-\beta)}$. Since $\beta < 1/3$, we have $\frac{2}{3(1-\beta)} < 1$, by where there exists $\lambda \le 1$ such that $\lambda > \frac{2}{3(1-\beta)}$. Also, in this case, T is a $\frac{k}{1-2k}$-quasicontraction. □

Theorem 2.4. *Let $(X, \langle\cdot\rangle)$ be a Hilbert space, Y be a nonempty closed and convex subset of X, and $T : Y \to Y$ be an operator satisfying all the conditions in Theorem 2.2. If x^* is the unique fixed-point of T, then the following conclusions hold*:

(*a*) *T satisfies the retraction-displacement condition*

$$\|x - x^*\| \leq \frac{\lambda}{1-k}\|x - Tx\|,$$

for every $x \in Y$, where $k := \sqrt{(1-\lambda)^2 + 2\lambda(1-\lambda)\mu + \lambda^2\beta^2}$;

(*b*) *the fixed-point equation $x = Tx$ is Ulam-Hyers stable*;

(*c*) *the fixed-point equation $x = Tx$ is well posed.*

Proof.

(a) By graphic contraction principle and the proof of Theorem 2.2, we have

$$\|x - x^*\| \leq \frac{1}{1-k}\|x - T_\lambda x\| = \frac{\lambda}{1-k}\|x - Tx\|,$$

for every $x \in Y$.

(b) Similarly with (b) from Theorem 2.3.

(c) Let $(u_n)_{n\in N}$ be a sequence in Y such that $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$. Then, by (a), we have that

$$\|u_n - x^*\| \leq \frac{\lambda}{1-k}\|u_n - Tu_n\| \to 0$$

as $n \to \infty$. Hence, $u_n \to x^*$ as $n \to \infty$. □

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