

Research Article

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On the order of approximation by modified summation-integral-type operators based on two parameters

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Abstract: In this article, we study generalized family of positive linear operators based on two parameters, which are a broad family of many well-known linear positive operators, e.g., Baskakov-Durrmeyer, Baskakov-Szász, Szász-Beta, Lupaş-Beta, Lupaş-Szász, genuine Bernstein-Durrmeyer, and Păltănea. We first find direct estimates in terms of the second-order modulus of continuity, then we find an estimate of error in the Ditzian-Totik modulus of smoothness. Then we discuss the rate of approximation for functions in the Lipschitz class. Furthermore, we study the pointwise Grüss-Voronovskaja-type result and also establish the Grüss-Voronovskaja-type asymptotic formula in quantitative form.

Keywords: Ditzian-Totik modulus of smoothness, Peetre's K -functional, Grüss-Voronovskaja-type asymptotic formula

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1 Introduction

In approximation theory, linear positive operators (LPO) play a vital contribution to approximate functions of various classes. In this direction, several operators were constructed by various researchers, but here we mention the operators related to work, namely, Miheşan-Durrmeyer [1], summation-integral type [2,3] and references therein. The researchers are attempting to modify the order of approximation of their operators as well as study their local and global approximation results.

Suppose $\Pi_2[0, \infty)$ is the set of all functions ζ that are real-valued and defined on $[0, \infty)$ such that $|\zeta(y)| \leq M_\zeta(1 + y^2)$, where a constant M_ζ is dependent on ζ and $M_\zeta > 0$. In addition, let $C_2[0, \infty)$ denote the subspace of all continuous functions in $\Pi_2[0, \infty)$. Furthermore, let $C_2^*[0, \infty)$ be the space of all functions $\zeta \in C_2[0, \infty)$ for which $\lim_{y \rightarrow \infty} \frac{|\zeta(y)|}{1 + y^2}$ exists and is finite. The norm on space $\Pi_2[0, \infty)$ is

$$\|\zeta\|_2 = \sup_{y \in [0, \infty)} \frac{|\zeta(y)|}{1 + y^2}.$$

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It was shown (see [4]) that for every $\zeta \in C_2^*[0, \infty)$, there hold:

$$\lim_{\delta \rightarrow 0} \Omega(\zeta, \delta) = 0$$

and

$$\Omega(\zeta, l\delta) \leq 2(1+l)(1+\delta^2)\Omega(\zeta, \delta), \quad l > 0, \quad (1.1)$$

where $\Omega(\zeta, \delta)$ is the weighted modulus of continuity, given by

$$\Omega(\zeta, \delta) = \sup_{0 \leq h < \delta, y \in [0, \infty)} \frac{\zeta(y+h) - \zeta(y)}{(1+y^2)(1+h^2)}.$$

Using the above definition and in view of (1.1), we may write

$$|\zeta(w) - \zeta(u)| \leq (1+u^2)(1+(w-u)^2)\Omega(\zeta, |w-u|) \leq 2\left(1 + \frac{|w-u|}{\delta}\right)(1+\delta^2)\Omega(\zeta, \delta)(1+w^2)(1+(w-u)^2). \quad (1.2)$$

Recently, Gupta [5] studied the general class of family of LPO for $\zeta \in C_2[0, \infty)$ for parameters $\rho, \tau > 0$ as:

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho}(\zeta, y) = \sum_{v=1}^{\infty} Y_{n,v}^{\tau}(y) \int_0^{\infty} Y_{n,v-1}^{\sigma+1,\rho}(v) \zeta(v) dv + Y_{n,0}^{\tau}(y) \zeta(0), \quad y \geq 0, \quad (1.3)$$

where

$$Y_{n,v}^{\tau}(y) = \frac{(\tau)_v}{v!} \frac{\left(\frac{ny}{\tau}\right)^v}{\left(1 + \frac{ny}{\tau}\right)^{\tau+v}}, \quad (1.4)$$

$$Y_{n,v-1}^{\sigma+1,\rho}(v) = \frac{n}{\sigma \cdot B(v\rho, \sigma\rho + 1)} \cdot \frac{\left(\frac{nv}{\sigma}\right)^{v\rho-1}}{\left(1 + \frac{nv}{\sigma}\right)^{\sigma\rho+v\rho+1}},$$

and

$$B(v\rho, \sigma\rho + 1) = \frac{\Gamma(v\rho)\Gamma(\sigma\rho + 1)}{\Gamma(v\rho + \sigma\rho + 1)}$$

having rising factorial $(\tau)_v = \tau(\tau+1)\dots(\tau+v-1)$ and $(\tau)_0 = 1$ (also see [6]). We see that this sequence of LPO reproduces linear functions.

Stancu [7] presented the modification of renowned Bernstein operators by using real parameters and discussed some approximation properties. Inspired from this modification, most recently, Alotaibi et al. [8], Milovanovic et al. [9], Mohiuddine et al. [10], and Mohiuddine and Özger [11] defined and discussed the Durrmeyer-Stancu, Stancu-type modification of Szász-Kantorovich, α -Baskakov-Kantorovich, and α -Bernstein-Kantorovich operators, respectively.

Motivated by the study of the above operators and their order of convergence, in the proceeding section, we will first define the Stancu kind modification of (1.3) and then moment estimates of our newly defined operators and their bound. An interesting property of LPO is to find the estimate of their differences using K -functional approach and in terms of appropriate modulus of continuity, so, in Section 3, we discuss the rate of convergence of our operators, i.e., estimation of error in terms of the usual modulus of continuity of second order as well as in terms of Ditzian-Totik modulus of smoothness using Peetre's K -functional approach. For more details, we refer to [12–22].

In [23], Grüss introduced an inequality, nowadays called Grüss inequality, in which he discusses the difference between the product of the integrals of two functions and the integral of the product of the same functions. Furthermore, Andrica and Badea [24] studied Grüss inequality by taking into account positive linear functionals. Acu et al. [25] extended this inequality for the Bernstein operators, convolution-type

operators, and Hermite-Fejer interpolation operators. Motivated by the study in this direction (cf. [25–28]), it is very interesting topic to study Grüss-Voronovskaja asymptotic result for general family of our newly aforementioned operators. In continuation, we derive the Grüss-Voronovskaja-type approximation theorem and also establish Grüss-Voronovskaja-type asymptotic result in quantitative form in the last section.

2 Construction of operators and estimates of moments

For real parameters a, b ($0 \leq a \leq b$), we define the Stancu generalization of operators (1.3) as follows:

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) = \sum_{v=1}^{\infty} Y_{n,v}^{\tau}(y) \int_0^{\infty} Y_{n,v-1}^{\sigma+1,\rho}(v) \zeta\left(\frac{nv+a}{n+b}\right) dv + Y_{n,0}^{\tau}(y) \zeta\left(\frac{a}{n+b}\right), \quad y \geq 0. \quad (2.1)$$

For specific values $a = b = 0$ in (2.1), we obtain

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,0,0}(\zeta, y) = \sum_{v=1}^{\infty} Y_{n,v}^{\tau}(y) \int_0^{\infty} Y_{n,v-1}^{\sigma+1,\rho}(v) \zeta(v) dv + Y_{n,0}^{\tau}(y) \zeta(0),$$

which is (1.3).

Equivalently, our operators $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)$ can also be written as follows:

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) = \sum_{v=0}^{\infty} Y_{n,v}^{\tau}(y) \mathfrak{Q}_{n,v}^{\sigma,\rho,a,b}(\zeta),$$

where $Y_{n,v}^{\tau}(y)$ is as given in (1.4) and

$$\mathfrak{Q}_{n,v}^{\sigma,\rho,a,b}(\zeta) = \begin{cases} \int_0^{\infty} Y_{n,v-1}^{\sigma+1,\rho}(v) \zeta\left(\frac{nv+a}{n+b}\right) dv, & 1 \leq v < \infty \\ \zeta\left(\frac{a}{n+b}\right), & v = 0. \end{cases}$$

We can obtain the following lemma with the help of Remark 4 of [5].

Lemma 1. *The r -th moment is defined as follows:*

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(e_r, y), \quad e_r = v^r, \quad r = 0, 1, 2, \dots$$

Using the definition of operators (2.1), we obtain

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(e_0, y) = 1, \quad \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(e_1, y) = \frac{ny+a}{n+b},$$

and

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(e_2, y) = \frac{1}{(n+b)^2} \left[a^2 + \frac{ny}{\sigma\rho-1} \left\{ (2a(\sigma\rho-1) + \sigma(1+\rho)) + \sigma\rho \left(1 + \frac{1}{\tau} \right) ny \right\} \right].$$

Remark 1. Let us define the central moments of r -th order by $\lambda_{n,\tau}^{\sigma,\rho,a,b}(y) = \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v-y)^r, y)$. Then, central moments of operators of order 1 and 2 are given as follows:

$$\lambda_{n,\tau,1}^{\sigma,\rho,a,b}(y) = \frac{a-by}{n+b}$$

and

$$\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) = \frac{1}{(n+b)^2} \left[a^2 + \frac{ny}{\sigma\rho-1} \left\{ (2a(\sigma\rho-1) + \sigma(1+\rho)) + \sigma\rho \left(1 + \frac{1}{\tau} \right) ny \right\} \right] + \frac{y((b-n)y-2a)}{(n+b)},$$

respectively.

Remark 2. We have

$$|\lambda_{n,\tau,1}^{\sigma,\rho,a,b}(y)| \leq \frac{|a| + |b|y}{n+b}.$$

3 Direct results

We start this section by discussing the pointwise convergence theorem for (2.1).

Theorem 1. Let $\zeta \in C_2[0, \infty)$. Then, for each $y \in [0, \infty)$, we have

$$\lim_{n \rightarrow \infty} \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) = \zeta(y). \quad (3.1)$$

Proof. From Lemma 1, it is evident that

$$\lim_{n \rightarrow \infty} \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(e_i, y) = y^m, \quad m = 0, 1, 2.$$

Hence, (3.1) follows by applying the universal Korovkin theorem (see [29]). $\square$

Let $C_B[0, \infty)$ denote the class of bounded and continuous functions on $[0, \infty)$, normed by

$$\|\xi\| = \sup_{y \in [0, \infty)} |\xi(y)|.$$

For $\xi \in \bar{C}_B[0, \infty) := \{\xi \in C_B[0, \infty) : \xi \text{ is uniformly continuous on } [0, \infty)\}$ and $\delta > 0$, the modulus of continuity of order j is given as follows:

$$\omega_j(\xi, \delta) = \sup_{0 < \eta \leq \delta} \sup_{y \in [0, \infty)} |\Delta_\eta^j \xi(y)|,$$

where Δ_η^j is the j th order forward difference of step length η . For $j = 1$, we call it the usual modulus of continuity which is denoted by $\omega(\xi, \delta)$.

Furthermore, let $\kappa^2 = \{p \in \bar{C}_B[0, \infty) : p', p'' \in \bar{C}_B[0, \infty)\}$. For $\xi \in \bar{C}_B[0, \infty)$, the κ -functional is defined as

$$\kappa_2(\xi, \delta) = \inf_{p \in \kappa^2} \{\|\xi - p\| + \delta \|p''\|\},$$

where $\delta > 0$.

By Theorem 2.4 of [30], there is an absolute constant $C > 0$ such that

$$\kappa_2(\xi, \delta) \leq C\omega_2(\xi, \sqrt{\delta}),$$

where $\omega_2(\xi, \delta)$ is the modulus of continuity of the second order of ξ .

Let us define

$$\lambda_{n,\tau,i}^{\sigma,\rho,a,b,*}(y) = \sup_{y \in [0, \infty)} |\lambda_{n,\tau,i}^{\sigma,\rho,a,b}(y)| \quad (i = 1, 2).$$

We are now ready to estimate the error of approximation by (2.1) in terms of first- and second-order moduli of continuity.

Theorem 2. *If $g \in \bar{C}_B[0, \infty)$ and $y \in [0, \infty)$, then*

$$\|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(g) - g\| \leq C\omega_2\left(g, \frac{\sqrt{\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2}}{2}\right) + \omega(g, \lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))$$

for any constant $C > 0$.

Proof. Define the auxiliary operators $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}$ as

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(g, y) = \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(g, y) - g(s) + g(y), \quad (3.2)$$

where $s = \frac{ny+a}{n+b}$, $y \in [0, \infty)$.

By Lemma 1, we see that the operator $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}$ is linear and preserves the linear functions. Hence,

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(u - y, y) = 0. \quad (3.3)$$

Let $\zeta \in \mathfrak{K}^2$. From Taylor's expansion,

$$\zeta(u) = \zeta(y) + \zeta'(y)(u - y) + \int_y^u (u - w)\zeta''(w)dw, \quad u \in [0, \infty).$$

Using (3.3), we have

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(\zeta, y) = \zeta(y) + \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}\left(\int_y^u (u - w)\zeta''(w)dw, y\right).$$

Hence, by (3.2), one has

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(\zeta, y) - \zeta(y) = \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}\left(\int_y^u (u - w)\zeta''(w)dw, y\right) - \int_y^s (s - w)\zeta''(w)dw.$$

Now,

$$\begin{aligned} |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(\zeta, y) - \zeta(y)| &\leq \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}\left(\left|\int_y^u (u - w)\zeta''(w)dw\right|, y\right) + \left|\int_y^s (s - w)\zeta''(w)dw\right| \\ &\leq [\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}((u - y)^2; y) + (s - y)^2]\|\zeta''\| \\ &\leq (\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2)\|\zeta''\|. \end{aligned}$$

For $\zeta \in \mathfrak{K}^2$ and $g \in \bar{C}_B[0, \infty)$, we have

$$\begin{aligned} |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(g, y) - g(y)| &\leq |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(g - \zeta, y)| + |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(\zeta, y) - \zeta(y)| + |g(y) - \zeta(y)| + |g(s) - g(x)| \\ &\leq 4\|g - \zeta\| + (\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2)\|\zeta''\| + \omega(g, \lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y)). \end{aligned}$$

Taking $\inf_{\zeta \in \mathfrak{K}^2}$, we obtain

$$\|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(g) - g\| \leq 4\kappa_2\left(g, \frac{(\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2)}{4}\right) + \omega(g, \lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y)).$$

In view of $\kappa_2(g, \delta) \leq \omega_2(g, \sqrt{\delta})$, we obtain

$$\|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(g) - g\| \leq 4\omega_2\left(g, \frac{\sqrt{\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2}}{2}\right) + \omega(g, \lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y)),$$

which completes the proof. $\square$

For the choice of $a = b = 0$ in the last Theorem 2, we obtain the following corollary:

Corollary 1. *If $g \in \tilde{C}_B[0, \infty)$ and $y \in [0, \infty)$, then*

$$\|\mathfrak{L}_{n,\tau}^{\sigma,\rho,0,0,*}(g) - g\| \leq C\omega_2\left(g, \frac{\sqrt{\lambda_{n,\tau,2}^{\sigma,\rho,0,0,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,0,0,*}(y))^2}}{2}\right) + \omega(g, \lambda_{n,\tau,1}^{\sigma,\rho,0,0,*}(y))$$

for any constant $C > 0$. Note that the operators $\mathfrak{L}_{n,\tau}^{\sigma,\rho,0,0,*}$ were discussed in [5].

For $\zeta \in \tilde{C}_B[0, \infty)$ and $\delta > 0$, the Ditzian-Totik modulus of smoothness of second order is defined as follows:

$$\omega_\varphi^2(\zeta, \delta) = \sup_{0 < \eta \leq \delta y \pm \eta \varphi(y) \in [0, \infty)} \sup_{\eta \varphi(y)} |\zeta(y + \eta \varphi(y)) - 2\zeta(y) + \zeta(y - \eta \varphi(y))|,$$

where $\varphi(y) = y(1 - y)$.

The corresponding κ -functional is given as follows:

$$\kappa_{2,\varphi}(\zeta, \delta^2) = \inf_{g \in \mathfrak{K}_\infty^2(\varphi)} \{\|\zeta - g\| + \delta^2 \|\varphi^2 g''\|\},$$

where $\mathfrak{K}_\infty^2(\varphi) = \{g \in C_B[0, \infty) : g' \in AC_{\text{loc}}[0, \infty), \varphi^2 g'' \in C_B[0, \infty)\}$ and φ is an admissible step-function on $[0, \infty)$.

By Theorem 2.11 of [31], there is an absolute constant $C > 0$ such that

$$C^{-1}\omega_\varphi^2(\zeta, \delta) \leq \kappa_{2,\varphi}(\zeta, \delta^2) \leq C\omega_\varphi^2(\zeta, \delta).$$

Now, we estimate an error in terms of weighted Ditzian-Totik modulus of smoothness using κ -functional approach.

Theorem 3. *Let $\zeta \in \tilde{C}_B[0, \infty)$ and $y \in (0, \infty)$. Then,*

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(\zeta, y) - \zeta(y)| \leq C\omega_{2,\varphi}\left(\zeta, \frac{\sqrt{\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2}}{\varphi(y)}\right) + \omega_\psi\left(\zeta, \frac{1}{n+b}\right),$$

where $\varphi(y) = y(1 - y)$ and $\psi(x) = |a| + |b|y$.

Proof. Let the operator $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}$ be defined as in (3.2). For $p \in \mathfrak{K}_\infty^2(\varphi)$ and by Taylor's series expansion, we obtain

$$p(v) = p(y) + (v - y)p'(y) + \int_y^v (v - u)p''(u)du. \quad (3.4)$$

Applying $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}$ on both sides of (3.4) and in view of (3.2), we obtain

$$\begin{aligned}
|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(p, y) - p(y)| &\leq \left(\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*} \left| \int_y^v |v-u| |p''(u)| du \right|, y \right) + \left| \int_y^s |s-u| |p''(u)| du \right| \\
&\leq \frac{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}((v-y)^2, y)}{\varphi^2(y)} \|\varphi^2 p'\| + \|\varphi^2 p'\| \left| \int_y^s \frac{|s-u|}{\varphi^2(y)} du \right| \\
&\leq \frac{\|\varphi^2 p'\|}{\varphi^2(y)} (\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b}(y))^2),
\end{aligned} \tag{3.5}$$

since $\frac{|v-u|}{\varphi^2(u)} \leq \frac{|v-y|}{\varphi^2(y)}$ for u between v and y .

Now, for $\zeta \in \tilde{C}_B[0, \infty)$ and any $p \in \mathfrak{X}_{\infty}^2(\varphi)$, we write

$$\begin{aligned}
|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| &\leq |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(\zeta - p, y)| + |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b,*}(p, y) - p(y)| + |\zeta(y) - p(y)| + |\zeta(s) - \zeta(y)| \\
&\leq 4\|\zeta - p\| + \frac{\|\varphi^2 p'\|}{\varphi^2(y)} (\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b}(y))^2) + |\zeta(s) - \zeta(y)|.
\end{aligned}$$

Letting $\inf_{p \in \mathfrak{X}_{\infty}^2(\varphi)}$, we obtain

$$\begin{aligned}
|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| &\leq 4\kappa_{2,\varphi} \left(\zeta, \frac{\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2}{4\varphi^2(y)} \right) + \left| \zeta \left(y + \psi(y) \frac{\lambda_{n,\tau,1}^{\sigma,\rho,a,b}(y)}{\psi(y)} \right) - \zeta(y) \right| \\
&\leq 4\kappa_{2,\varphi} \left(\zeta, \frac{\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2}{4\varphi^2(y)} \right) + \omega_{\psi} \left(\zeta, \frac{1}{\psi(y)} |\lambda_{n,\tau,1}^{\sigma,\rho,a,b}(y)| \right) \\
&\leq 4\kappa_{2,\varphi} \left(\zeta, \frac{\lambda_{n,\tau,2}^{\sigma,\rho,a,b,*}(y) + (\lambda_{n,\tau,1}^{\sigma,\rho,a,b,*}(y))^2}{4\varphi^2(y)} \right) + \omega_{\psi} \left(\zeta, \frac{1}{n+b} \right),
\end{aligned}$$

in view of Remark 2.

Using the equivalence between $\kappa_{2,\varphi}(\zeta, \delta^2)$ and $\omega_{\varphi}^2(\zeta, \delta)$, the theorem follows. $\square$

We consider the Lipschitz-type space (see [32]) for parameters $c, d > 0$ to investigate the approximation of functions as follows:

$$\text{Lip}_M^{(c,d)}(\varrho) = \left\{ \zeta \in C[0, \infty) : |\zeta(v) - \zeta(y)| \leq M \frac{|v-y|^{\varrho}}{(v+cy^2+dy)^{\frac{\varrho}{2}}}, \quad v, y \in [0, \infty) \right\},$$

where a constant $M > 0$ and $0 < \varrho \leq 1$.

The proceeding result gives the degree of approximation for $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta)$ for $\zeta \in \text{Lip}_M^{(c,d)}(\varrho)$.

Theorem 4. Let $0 < \varrho \leq 1$ and $\zeta \in \text{Lip}_M^{(a,b)}(\varrho)$. Then,

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq M \left(\frac{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}{cy^2 + dy} \right)^{\varrho/2} \quad (\forall y > 0).$$

Proof. We first consider to obtain our result for $\varrho = 1$. Then, for $\zeta \in \text{Lip}_M^{(c,d)}(\varrho)$ and $y > 0$, we have

$$\begin{aligned}
|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| &\leq \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|\zeta(v) - \zeta(y)|, y) \\
&\leq M \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b} \left(\frac{|v-y|}{(v+cy^2+dy)^{1/2}}, y \right) \\
&\leq \frac{M}{(v+cy^2+dy)^{1/2}} \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|v-y|, y),
\end{aligned}$$

since

$$\frac{1}{v + cy^2 + dy} \leq \frac{1}{cy^2 + dy}.$$

Using Cauchy's Schwarz inequality together with Lemma 1, we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \frac{M}{(cy^2 + dy)^{1/2}} (\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v - y)^2, y))^{1/2} \leq M \left(\frac{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}{cy^2 + dy} \right)^{1/2}.$$

Thus, result is true for $\varrho = 1$.

Now, let us prove the result for the case $0 < \varrho < 1$. So,

$$\begin{aligned} |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| &\leq \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|\zeta(v) - \zeta(y)|, y) \\ &\leq M \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b} \left(\frac{|v - y|^\varrho}{(v + cy^2 + dy)^{\varrho/2}}, y \right) \\ &\leq \frac{M}{(cy^2 + dy)^{\varrho/2}} \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|v - y|^\varrho, y). \end{aligned}$$

In view of Lemma 1 and Hölder's inequality with $p = \frac{1}{\varrho}$ and $q = \frac{1}{1-\varrho}$, we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|\zeta(v) - \zeta(y)|, y) \leq \frac{M}{(cy^2 + dy)^{\varrho/2}} (\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|v - y|, y))^\varrho.$$

It follows from the Cauchy's Schwarz inequality and Lemma 1 that

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq M \left(\frac{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(x)}{cy^2 + dy} \right)^{\varrho/2},$$

which completes the proof. $\square$

In the last result of this section, we consider the Lipschitz-type maximal function of order ϱ (see [33]) as

$$\tilde{\omega}_\varrho(\zeta, y) = \sup_{v \neq y, v \in [0, \infty)} \frac{|\zeta(v) - \zeta(y)|}{|v - y|^\varrho}, \quad y \in [0, \infty) \text{ and } \varrho \in (0, 1] \quad (3.6)$$

to study the local direct estimate for (2.1).

Theorem 5. Let $0 < \varrho \leq 1$ and $\zeta \in C_B[0, \infty)$. Then,

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq C \tilde{\omega}_\varrho(\zeta, y) (\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y))^{\frac{\varrho}{2}} \quad (\forall y \in [0, \infty)).$$

Proof. From (3.6), we have

$$|\zeta(v) - \zeta(y)| \leq \tilde{\omega}_\varrho(\zeta, y) |v - y|^\varrho$$

and

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|\zeta(v) - \zeta(y)|, y) \leq \tilde{\omega}_\varrho(\zeta, y) \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|v - y|^\varrho, y).$$

It follows by using Hölder's inequality with

$$p_1 = \frac{2}{\varrho}, \quad \frac{1}{p_2} = 1 - \frac{1}{p_1}$$

and Lemma 1 that

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \tilde{\omega}_\varrho(\zeta, y) (\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v - y)^2, y))^{\varrho/2} \leq \tilde{\omega}_\varrho(\zeta, y) (\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y))^{\frac{\varrho}{2}}.$$

Consequently, the proof follows. $\square$

4 Grüss-Voronovskaja-type asymptotic result

We establish the Grüss-Voronovskaja-type theorem for (2.1). First, we derive a Grüss-type approximation theorem and then prove the Grüss-Voronovskaja-type asymptotic result.

The weighted modulus of continuity is given as follows:

$$\Omega(\zeta, \delta) = \sup_{0 < \xi < \delta, y \in [0, \infty)} \frac{\zeta(y + \xi) - \zeta(y)}{(1 + \xi^2)(1 + y^2)}$$

for the functions $\zeta \in C_2^*[0, \infty)$.

Theorem 6. If ζ, p, ζ^2, p^2 , and $\xi g \in C_2^*[0, \infty)$ then for fixed $y \geq 0$, there holds:

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta p, y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y)| \leq \wp_\zeta(x) \wp_p(y),$$

where

$$\wp_\zeta(y) = \sqrt{32(1 + y^2)\Omega(\zeta^2, (\lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y))^{1/4}) + 32(1 + C)\|\zeta\|_2(1 + y^2)^2\Omega(\zeta, (\lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y))^{1/4})},$$

$\wp_p(y)$ is the analogue of $\wp_\zeta(y)$, and C is a constant.

Proof. Define $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, p, y) = \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta p, x) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y)$. Using Cauchy-Schwarz inequality, one obtains

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, p, y)| \leq \sqrt{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, \zeta, y)} \sqrt{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, p, y)}.$$

In view of (1.2), we reach to

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq 2(1 + y^2)(1 + \delta^2)\Omega(\zeta, \delta) \times \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}\left(\left(1 + \frac{|z - y|}{\delta}\right)(1 + (z - y)^2), y\right). \quad (4.1)$$

Let us define $\wp(y, z, \delta) := \left(1 + \frac{|z - y|}{\delta}\right)(1 + (z - y)^2)$. So,

$$\wp(y, z, \delta) \leq \begin{cases} 2(1 + \delta^2), & |z - y| < \delta \\ 2(1 + \delta^2) \frac{(z - y)^4}{\delta^4}, & |z - y| \geq \delta. \end{cases}$$

Now, combining both cases for all $y, z \geq 0$, we obtain

$$\wp(y, z, \delta) \leq 2(1 + \delta^2) \left[1 + \frac{(z - y)^4}{\delta^4} \right]. \quad (4.2)$$

Combining (4.1)–(4.2), for $0 < \delta < 1$, we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq 16(1 + y^2) \left(1 + \frac{1}{\delta^4} \lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y) \right) \Omega(\zeta, \delta). \quad (4.3)$$

We can write

$$\begin{aligned} \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, \zeta, y) &= \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta^2, y) - \zeta^2(y) + \zeta^2(y) - (\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y))^2 \\ &= \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta^2, y) - \zeta^2(y) + (\zeta(y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y))(\zeta(y) + \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)). \end{aligned}$$

Now,

$$\frac{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)}{1 + y^2} \leq \frac{\|\zeta\|_2 \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(1 + t^2, y)}{1 + y^2} \leq \frac{\|\zeta\|_2 C(1 + y^2)}{1 + y^2} = C\|\zeta\|_2.$$

So, we have

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, \zeta, y)| \leq |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta^2, y) - \zeta^2(y)| + |\zeta(y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)|(\|\zeta\|_2 + M\|\zeta\|_2)(1 + y^2).$$

Furthermore, using (4.3), we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, \zeta, y)| \leq 16(1+y^2) \left(1 + \frac{1}{\delta^4} \lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y)\right) \Omega(\zeta^2, \delta) + 16(1+C) \|\zeta\|_2 (1+y^2)^2 \left(1 + \frac{1}{\delta^4} \lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y)\right) \Omega(\zeta, \delta).$$

Choosing $\delta = (\lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y))^{\frac{1}{4}}$, we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, \zeta, y)| \leq 32(1+y^2) \Omega\left(\zeta^2, (\lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y))^{\frac{1}{4}}\right) + 32(1+C) \|\zeta\|_2 (1+y^2)^2 \Omega\left(\zeta, (\lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y))^{\frac{1}{4}}\right).$$

We find similar estimate for $|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, p, y)|$. $\square$

Next, we discuss the quantitative Voronovskaja-type result with a view of (2.1) for functions belonging to $C_2^*[0, \infty)$.

Theorem 7. If ζ, ζ' , and $\zeta'' \in C_2^*[0, \infty)$, then for $y \in [0, \infty)$, there holds

$$\left| \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y) - \frac{1}{2} \zeta''(y) \mu_{n,\tau,2}^{\sigma,\rho,a,b}(y) \right| \leq 16(1+y^2) \Omega\left(\zeta'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right) \lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y).$$

Proof. We obtain our result by proceeding along the similar lines of the proof of Theorem 2 of [26], hence the details are omitted. $\square$

Finally, we prove the following Grüss-Voronovskaja-type quantitative result for $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}$.

Theorem 8. Let $\zeta, p, \zeta p, \zeta', p', (\zeta p)', \zeta'', p'',$ and $(\zeta p)'' \in C_2^*[0, \infty)$, then at any point $y \in [0, \infty)$, we have

$$\begin{aligned} & n |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta p, y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y) - \lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) \zeta'(y) p'(y)| \\ & \leq 16(1+y^2) n \lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) \left\{ \Omega\left(\zeta p'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right) + \|\zeta\|_2 (1+y^2) \Omega\left(p'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right) \right. \\ & \quad \left. + \|p\|_2 (1+y^2) \Omega\left(\zeta'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right) \right\} + n \hbar_n(\zeta) \hbar_n(p), \end{aligned}$$

where

$$\hbar_n(\zeta) = \frac{1}{2} \|\zeta''\|_2 (1+y^2) \left(2 \lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) + \frac{2y}{1+y^2} |\lambda_{n,\tau,3}^{\sigma,\rho,a,b}(y)| + \frac{1}{1+y^2} \lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y) \right)$$

and $\hbar_n(p)$ is the analogue of $\hbar_n(\zeta)$.

Proof. By Taylor's series expansion, we have

$$\begin{aligned} & \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta p, y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v-y)^2, y) \zeta'(y) p'(y) \\ & = \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta p, y) - \zeta(y) p(y) - \frac{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v-y)^2, y)}{2} (\zeta(y) p(y))'' \\ & \quad - \zeta(y) \left[\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y) - p(y) - \frac{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v-y)^2, y)}{2} p''(y) \right] \\ & \quad - p(y) \left[\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y) - \frac{\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((v-y)^2, y)}{2} \zeta''(y) \right] \\ & \quad + (p(y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y)) \cdot (\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)) = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4. \end{aligned}$$

Using Theorem 7, we obtain

$$\begin{aligned} |\Gamma_1| &\leq 16(1+y^2)\Omega\left((\zeta p)'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y), \\ |\Gamma_2| &\leq 16|\zeta(y)|(1+y^2)\Omega\left(p'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y), \\ |\Gamma_3| &\leq 16|p(y)|(1+y^2)\Omega\left(\zeta'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y). \end{aligned}$$

Next, since $\zeta \in C_2^*[0, \infty)$, we can write

$$\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y) = \frac{1}{2}\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta''(\xi)(v-y)^2, y),$$

and hence, we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \frac{1}{2}\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(|\zeta''(\xi)|(v-y)^2, y) \leq \|\zeta''\|_2 \frac{1}{2}\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((1+\xi^2)(v-y)^2, y),$$

where $v < \xi < y$.

If ξ lies between v and y , then we obtain $1 + \xi^2 \leq 1 + y^2$. So, in this case, we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \frac{\|\zeta''\|_2(1+y^2)}{2}\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y).$$

Moreover, if ξ lies between y and v , then we obtain $1 + \xi^2 \leq 1 + v^2$. So, we obtain

$$\begin{aligned} |\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| &\leq \frac{\|\zeta''\|_2}{2}\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}((1+v^2)(v-y)^2, y) \\ &= \frac{\|\zeta''\|_2}{2}((1+y^2)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) + 2y\lambda_{n,\tau,3}^{\sigma,\rho,a,b}(y) + \lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y)). \end{aligned}$$

Therefore, by combining the two cases of ξ , we obtain

$$|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) - \zeta(y)| \leq \frac{\|\zeta''\|_2(1+y^2)}{2}\left\{2\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) + \frac{2y}{1+y^2}\lambda_{n,\tau,3}^{\sigma,\rho,a,b}(y) + \frac{1}{1+y^2}\lambda_{n,\tau,4}^{\sigma,\rho,a,b}(y)\right\} := \hbar_n(\zeta).$$

Analogously, we determine $|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y) - p(y)| \leq \hbar_n(p)$. Thus, we reach to

$$\begin{aligned} n|\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta p, y) - \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(p, y) - \lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)\zeta'(y)p'(y)| \\ \leq 16(1+y^2)n\Omega\left((\zeta p)'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) \\ + 16\|\zeta\|_2(1+y^2)^2n\Omega\left(p'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) \\ + 16\|p\|_2(1+y^2)^2n\Omega\left(\zeta'', \left(\frac{\lambda_{n,\tau,6}^{\sigma,\rho,a,b}(y)}{\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y)}\right)^{1/4}\right)\lambda_{n,\tau,2}^{\sigma,\rho,a,b}(y) + n\hbar_n(\zeta)\hbar_n(p). \end{aligned}$$

Hence, the result of the theorem is established. $\square$

5 Conclusion

In our discussion, we defined the Stancu-type modification of $\mathfrak{L}_{n,\tau}^{\sigma,\rho}(\zeta, y)$ with the help of parameters by

$$\begin{aligned} \mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y) = & \sum_{v=1}^{\infty} \frac{(\tau)_v}{v!} \frac{\left(\frac{ny}{\tau}\right)^v}{\left(1 + \frac{ny}{\tau}\right)^{\tau+v}} \int_0^{\infty} \frac{n\Gamma(v\rho + \sigma\rho + 1)}{\sigma\Gamma(v\rho)\Gamma(\sigma\rho + 1)} \cdot \frac{\left(\frac{nv}{\sigma}\right)^{v\rho-1}}{\left(1 + \frac{nv}{\sigma}\right)^{\sigma\rho+v\rho+1}} \\ & \times \zeta\left(\frac{nv+a}{n+b}\right) dv + \frac{1}{\left(1 + \frac{ny}{\tau}\right)^{\tau}} \zeta\left(\frac{a}{n+b}\right). \end{aligned} \quad (5.1)$$

We discussed several approximation results for (5.1), namely, pointwise convergence, degree of approximation, error estimations by means of suitable moduli of continuity as well as moduli of smoothness, Grüss-Voronovskaja-type results, etc.

By taking different values and limiting conditions of parameters τ and σ in (5.1), we obtain various linear positive operators, which were studied by several authors. In Table 1, we see that our operators $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)$ reduced to several previously studied operators.

Table 1: The operators $\mathfrak{L}_{n,\tau}^{\sigma,\rho,a,b}(\zeta, y)$ for some specific values

For the choice of	(5.1) reduces to	Studied in
$a = b = 0$	The operators $\mathfrak{L}_{n,\tau}^{\sigma,\rho}(\zeta, y)$	[5]
$a = b = 0, \rho = 1, \tau = \sigma \rightarrow \infty$	Phillips operators	[34] (also see [35])
$a = b = 0, \rho = 1, \tau = \sigma = n$	Baskakov-Durrmeyer type operators	[36]
$a = b = 0, \rho = 1, \tau \neq \sigma, \tau = n, \sigma \rightarrow \infty$	Baskakov-Szász type operators	[37]
$a = b = 0, \rho = 1, \tau \neq \sigma, \sigma = n, \tau \rightarrow \infty$	Szász-Beta type operators	[38]
$a = b = 0, \rho = 1, \tau \neq \sigma, \tau = ny, \sigma = n$	Lupaş-Beta integral operators	[39]
$a = b = 0, \rho = 1, \tau \neq \sigma, \tau = ny, \sigma \rightarrow \infty$	Lupaş-Szász operators	[40]
$a = b = 0, \rho = 1, \tau = \sigma = -n$	Genuine Bernstein-Durrmeyer operator	[41] (also see [42])
$a = b = 0, \rho > 0, \tau = \sigma \rightarrow \infty$	Păltănea operators	[43]

It is worth noting to the reader that one can further modify these operators to improve the order of approximation by taking their linear and iterative combination and studying their approximation properties. Moreover, the q -variant of these operators may also be constructed and studied.

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