Research Article

Nadiyah Hussain Alharthi, Abdon Atangana*, and Badr S. Alkahtani

Analysis of Cauchy problem with fractal-fractional differential operators

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Abstract: Cauchy problems with fractal-fractional differential operators with a power law, exponential decay, and the generalized Mittag-Leffler kernels are considered in this work. We start with deriving some important inequalities, and then by using the linear growth and Lipchitz conditions, we derive the conditions under which these equations admit unique solutions. A numerical scheme was suggested for each case to derive a numerical solution to the equation. Some examples of fractal-fractional differential equations were presented, and their exact solutions were obtained and compared with the used numerical scheme. A nonlinear case was considered and solved, and numerical solutions were presented graphically for different values of fractional orders and fractal dimensions.

Keywords: fractal-fractional, power law, exponential decay, Mittag-Leffler function, numerical scheme, inequalities

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1 Introduction

Linear and nonlinear ordinary differential equations are important mathematical tools used to replicate behaviors observed in nature. To obtain these equations, some important mathematical formulas called derivatives are used. In the last decades, different definitions were suggested to help replicate different processes found in nature. One of the old definitions that were applied in classical mechanics is based on the rate of change. This definition is reported to have been introduced independently by Newton and Leibniz [1,2]. This concept together with its integral is the foundation of the nowadays differential and integral calculus [2–6]. Several important mathematical models that have changed our globe were built using this derivative [2–9]. Nevertheless, due to the complexity of nature, it has been reported in several instances that some models obtained using the classical differential operators do not always agree with the experimental data. This disagreement led many researchers to see the limitations of the derivative based on the rate of change. Several attempts have been done to capture different processes that the classical derivative cannot replicate [10–12], for example, the concept of fractional differential and integral calculus-based power law [6,10,12]. With the power-law kernel, two major fractional derivatives were suggested, namely, the Riemann-Liouville and the Caputo derivatives [6,10,12]. Indeed, several processes in nature follow the power-law behaviors, and such processes can be therefore replicated using a power-law-based

*** Corresponding author: Abdon Atangana,** Institute for Groundwater Studies, Faculty of Natural and Agricultural Science, University of the Free State, 9300 Bloemfontein, South Africa; Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, e-mail: AtanganaA@ufs.ac.za
Nadiyah Hussain Alharthi: Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University College of Science, P.O. Box 90950, 11623, Riyadh, Saudi Arabia, e-mail: nhalharthi@imamu.edu.sa
Badr S. Alkahtani: Department of Mathematics, College of Science, King Saud University, P.O. Box 1142, Riyadh 11989, Saudi Arabia

fractional derivative. However, there are plenty of natural processes that do not follow power-law behaviors rather they follow fading memory trends. A clear indication shows that the power-law-based fractional derivative cannot be used to model such processes, and therefore, a new definition was needed and introduced by Caputo and Fabrizio in 2015 [13]. This concept has with no doubt opened new doors of investigations within the framework of theory and application in the last years. Nevertheless, the fact that the associate integral of this derivative is the average of the function, and its classical integral led some researchers to ask some fundamental questions [14]. While this is not at all any weakness of this derivative, it only raises some issues about the fractional view of the derivative. To solve this problem, Atangana and Baleanu suggested a different definition with the general Mittag-Leffler kernel [14,15]. We have to note that, although several modifications have been suggested, the major strength of these three definitions is that their kernels appear naturally in many real-world problems. They were used to form very important distributions (Power-law distribution or the Pareto distribution, the Poisson distribution, and the generalized Mittag-Leffler distribution that was used to form super statistics) that are used in many statistical problems. On the other hand, a more general classical differential operator called fractal derivative was suggested to capture processes with local self-similar behaviors [16]. One of the properties of this derivative is that if the function is classically differentiable, then the fractal derivative is the product of the classical derivative and the power law function. By using this property, Atangana suggested a new concept called fractal-fractional derivative [17]. This concept was suggested for the power law, exponential decay, and generalized Mittag-Leffler functions [17]. Indeed, these three differential operators gave birth to three classes of Cauchy problems that will be analyzed in this work. The remainder of this essay is organized as follows: We derive some significant inequality in Section 2 that is related to fractal-fractional differential and integral operators. In Section 3, we provide a theoretical analysis of the power law-kernel fractal-fractional Cauchy problem, outlining the existence and uniqueness of the exact solution as well as the numerical approach. The same analysis is presented in Sections 4 and 5, with the generalized Mittag-Leffler and exponential decay as the kernels.

2 Some inequalities

We present here some inequalities associated with fractal-fractional differential operators.

Let $f(t)$ be a continuous bounded function admitting maximum and minimum M and M_1, respectively. We have that:

$$M \leq f(t) \leq M_1. \tag{1}$$

With ${}^{\mathrm{FFP}}_{0}D_t^{\alpha}$, we have:

$${}^{\mathrm{FFP}}_{0}D_t^{\alpha} M \leq {}^{\mathrm{FFP}}_{0}D_t^{\alpha} f(t) \leq {}^{\mathrm{FFP}}_{0}D_t^{\alpha} M_1, \tag{2}$$

where

$${}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t^{\beta}} \int_0^t f(\tau)(t-\tau)^{-\alpha} \mathrm{d}\tau, \tag{3}$$

where $0 < \alpha \leq 1$, $\beta > 0$ is a fractal dimension.

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t^{\beta}} = \lim_{t_1 \to t} \frac{f(t_1) - f(t)}{t_1^{\beta} - t^{\beta}}. \tag{4}$$

Thus,

$$\frac{Mt^{1-\beta}}{\beta\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{-\alpha} \mathrm{d}\tau \leq {}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta} f(t) \leq \frac{M_1 t^{1-\beta}}{\beta\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{-\alpha} \mathrm{d}\tau. \tag{5}$$

Noting that:

$$\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-\alpha}\mathrm{d}\tau = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{6}$$

Therefore:

$$\frac{Mt^{1-\beta-\alpha}}{\beta\Gamma(1-\alpha)} \leq {}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta}f(t) \leq \frac{M_1 t^{1-\beta-\alpha}}{\beta\Gamma(1-\alpha)}. \tag{7}$$

With ${}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta}$, we have:

$${}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta}M \leq {}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta}f(t) \leq {}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta}M_1. \tag{8}$$

Noting that,

$${}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta}f(t) = \frac{1}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t^{\beta}}\int_0^t f(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau. \tag{9}$$

Thus,

$$\frac{t^{1-\beta}}{(1-\alpha)\beta}M\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau \leq {}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta}f(t) \leq \frac{t^{1-\beta}}{(1-\alpha)\beta}M_1\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau. \tag{10}$$

We note that:

$$\int_0^t \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau = \frac{1-\alpha}{\alpha}\left[1-\exp\left[-\frac{\alpha}{1-\alpha}t\right]\right]. \tag{11}$$

Then,

$$\frac{t^{1-\beta}M}{\beta^{\alpha}}\frac{\mathrm{d}}{\mathrm{d}t}\left[1-\exp\left[-\frac{\alpha}{1-\alpha}t\right]\right] \leq {}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta}f(t) \leq \frac{t^{1-\beta}M_1}{\beta^{\alpha}}\frac{\mathrm{d}}{\mathrm{d}t}\left[1-\exp\left[-\frac{\alpha}{1-\alpha}t\right]\right]. \tag{12}$$

With ${}^{\mathrm{FFM}}_{0}D_t^{\alpha,\beta}$ we have:

$$\frac{Mt^{2-\beta}}{(1-\alpha)\beta}E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}t^{\alpha}\right] \leq {}^{\mathrm{FFM}}_{0}D_t^{\alpha,\beta}f(t) \leq \frac{M_1 t^{2-\beta}}{(1-\alpha)\beta}E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}t^{\alpha}\right]. \tag{13}$$

We evaluate ${}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta}f(t)$ assuming $f'(t) > 0$:

$$\begin{aligned}
|{}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta}f(t)\mathrm{d}\tau|^2 &= \left|\frac{\beta t^{1-\beta}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau\right|^2 \\
&= \left|\beta t^{1-\beta}\left(+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0)+\frac{1}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau\right)\right|^2 \\
&< \frac{2\beta^2 t^{2-2\beta-2\alpha}}{(\Gamma(1-\alpha))^2}f^2(0) + 2\left|\frac{\beta t^{1-\beta}}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau\right|^2 < \frac{2\beta^2 t^{2(1-\alpha-\beta)}}{\Gamma(1-\alpha)^2}f^2(0) \\
&+ \frac{2\beta^2 t^{2-2\beta}}{\Gamma(1-\alpha)^2}\left|\int_0^t f'(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau\right|^2 < \frac{2\beta^2 t^{2(1-\alpha-\beta)}}{\Gamma(1-\alpha)^2}f^2(0) \\
&+ \frac{4\beta^2 t^{2-2\beta}}{\Gamma(1-\alpha)^2}\frac{t^{1-2\beta}}{(1-2\alpha)}(f(t)-f(0))^2\left(\frac{2\beta^2 t^{2(1-\alpha-\beta)}}{(\Gamma(1-\alpha))^2}+\frac{8\beta^2 t^{3-2\beta-2\alpha}}{\Gamma(1-\alpha)^2(1-2\alpha)}\right)f^2(0) \\
&+ \frac{8\beta^2 t^{3-2\beta-2\alpha}}{(1-2\alpha)\Gamma(1-\alpha)^2}|f(t)|^2,
\end{aligned} \tag{14}$$

$$\sup_{t>0}\left|{}_{0}^{\mathrm{FFP}}D_t^{\alpha,\beta}f(t)\right|^2 < \left(\frac{2\beta^2 t^{2(1-\alpha-\beta)}}{(\Gamma(1-\alpha))^2}+\frac{8\beta^2 t^{3-2\beta-2\alpha}}{\Gamma(1-\alpha)^2(1-2\alpha)}\right)f^2(0)\left\{1+\frac{\frac{8\beta^2 t^{3-2\beta-2\alpha}}{\Gamma(1-\alpha)^2(1-2\alpha)}\sup_{t\in Df}|f(t)|^2}{\left(\frac{2\beta^2 t^{2(1-\alpha-\beta)}}{(\Gamma(1-\alpha))^2}+\frac{8\beta^2 t^{3-2\beta-2\alpha}}{\Gamma(1-\alpha)^2(1-2\alpha)}\right)f'(0)}\right\} < K(1+\|f\|_\infty^2), \quad (15)$$

where

$$K=\left(\frac{2\beta^2 T^{2(1-\alpha-\beta)}}{\Gamma^2(1-\alpha)}+\frac{8\beta^2 T^{3-2\beta-2\alpha}}{\Gamma^2(1-\alpha)(1-2\alpha)}\right)f^2(0). \quad (16)$$

Under the condition that:

$$\frac{8\beta^2 T^{3-2\beta-2\alpha}}{2(1-2\alpha)\beta^2 T^{2(1-\alpha-\beta)}+8\beta^2 T^{3-2\beta-2\alpha}}<1. \quad (17)$$

We consider the case where the derivative is based on:

$${}_{0}^{\mathrm{CFR}}D_t^{\alpha,\beta}y(t)=\frac{1}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t^\beta}\int_0^t y(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau, \quad (18)$$

$$=\frac{1}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t^\beta}\int_0^t y(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau\frac{1}{t^{\beta-1}}, \quad (19)$$

$$\begin{aligned}
\left|{}_{0}^{\mathrm{CFR}}D_t^{\alpha}y(t)\right|^2 &= \left|{}_{0}^{\mathrm{CFR}}D_t^{\alpha}y(t)\frac{t^{1-\beta}}{\beta}\right|^2 = \frac{t^{2-2\beta}}{\beta^2}\left|{}_{0}^{\mathrm{CFR}}D_t^{\alpha}y(t)\right|^2 = \frac{t^{2-2\beta}}{\beta^2}\left|{}_{0}^{\mathrm{CFC}}D_t^{\alpha}y(t)+y(0)\frac{\exp}{1-\alpha}\left(-\frac{\alpha}{1-\alpha}t\right)\right|^2\\
&\le \frac{t^{2-2\beta}}{\beta^2}\left\{2\left|{}_{0}^{\mathrm{CFC}}D_t^{\alpha}y(t)\right|^2+2\left|\frac{y(0)}{1-\alpha}\exp\left(-\frac{\alpha}{1-\alpha}t\right)\right|^2\right\}\\
&\le \frac{t^{2-2\beta}}{\beta^2}\left\{\frac{2}{(1-\alpha)^2}\left|\int_0^t y'(\tau)\mathrm{d}\tau\right|^2+2\left|\frac{y(0)}{1-\alpha}\exp\left(-\frac{\alpha}{1-\alpha}t\right)\right|^2\right\}\\
&< \frac{t^{2-2\beta}}{\beta^2}\left\{\frac{2}{(1-\alpha)^2}\{y(t)-y(0)\}^2+2|y(0)|^2\right\} < \frac{4t^{2-2\beta}}{\beta^2(1-\alpha)^2}\|y\|_\infty^2\\
&\quad + |y(0)|^2\left\{2+\frac{2}{(1-\alpha)^2}\right\}\frac{t^{2-2\beta}}{\beta^2}.
\end{aligned} \quad (20)$$

3 Analysis of Cauchy problem with fractal-fractional with power law

We consider the following general Cauchy problem.

$$\begin{cases} {}_{0}^{\mathrm{FFP}}D_t^{\alpha,\beta}y(t)=f(t,y(t)) \text{ if } t>0\\ y(0)=y_0 \text{ if } t=0. \end{cases} \quad (21)$$

We define the following norm:

$$\|\varphi\|_\infty^2=\sup_{t\in D\varphi}|\varphi(t)|. \quad (22)$$

$f\in C^2[0,T]$, $0<\alpha\le 1$, $\beta>0$.

where α is a fractional order and β is a fractal dimension.

It is assumed that $f(t,y(t))$ verifies the following criteria:

1. $\forall t\in[0,T]\quad |f(t,y(t))|^2<\underline{R}(1+|y|^2)$,
2. $\forall t\in[0,T]$, $y_1,y_2\in C^2[0,T]\,|f(t,y_1)-f(t,y_2)|^2<\bar{R}|y_1-y_2|^2$.

Note that,

$$\begin{cases} {}^{\mathrm{FFP}}_{\ 0}D_t^{\alpha,\beta}y(t) = f(t, y(t)) \quad t > 0 \\ y(0) = y_0 \text{ if } t = 0. \end{cases} \tag{23}$$

Can be transformed to:

$$\begin{cases} {}^{\mathrm{RL}}_{\ 0}D_t^{\alpha,\beta}y(t) = \beta t^{\beta-1}f(t, y(t)), \quad t > 0 \\ y(0) = y_0 \text{ if } t = 0. \end{cases} \tag{24}$$

$$\begin{cases} y(t) = \dfrac{\beta}{\Gamma(1-\alpha)}\displaystyle\int_0^t \tau^{\beta-1}f(\tau, y(\tau))(t-\tau)^{-\alpha}\mathrm{d}t, \quad t > 0 \\ y(0) = y_0. \end{cases} \tag{25}$$

To achieve the existence and uniqueness of exacts condition, we defined the following mapping and show that this mapping verified conditions described earlier:

$$\Phi(y(t)) = \frac{\beta}{\Gamma(1-\alpha)}\int_0^t \tau^{\beta-1}f(\tau, y(\tau))(t-\tau)^{-\alpha}\mathrm{d}\tau, \tag{26}$$

$$|\Phi(y(t))|^2 = \frac{\beta^2}{(\Gamma(1-\alpha))^2}\left|\int_0^t \tau^{\beta-1}f(\tau, y(\tau))(t-\tau)^{-\alpha}\mathrm{d}\tau\right|^2 < \frac{2\beta^2}{\Gamma^2(1-\alpha)}\int_0^t \tau^{2\beta-2}(t-\tau)^{-2\alpha}\mathrm{d}\tau\int_0^t |f(\tau, y(\tau))|^2\mathrm{d}\tau. \tag{27}$$

We evaluate first,

$$\int_0^1 \tau^{2\beta-2}(t-\tau)^{-2\alpha}\mathrm{d}\tau. \tag{28}$$

By the following change of variable:

$$tz = \tau. \tag{29}$$

Such that:

$$\int_0^t (tz)^{2\beta-2}(t-zt)^{-2\alpha}t\mathrm{d}z = t^{2\beta-2\alpha-1}\int_0^t z^{2\beta-2}(1-z)^{-2\alpha}\mathrm{d}z = t^{2\beta-2\alpha-1}B(2\beta-1, \quad 1-2\alpha), \tag{30}$$

where

$$B(z, z_1) = \int_0^t t^{z-1}(1-t)^{z_1-1}\mathrm{d}\tau, \tag{31}$$

where $\mathrm{Re}(z)$, $\mathrm{Re}(z_1) > 0$, in our case,

$$2\beta - 1 > 0 \text{ and } 1 - 2\alpha > 0, \tag{32}$$

$$\beta > \frac{1}{2} \text{ and } 1 > 2\alpha \geq \alpha < \frac{1}{2}. \tag{33}$$

Thus,

$$\begin{aligned} |\Phi(y(t))|^2 &< \frac{2\beta^2}{\Gamma(1-\alpha)^2}t^{2\beta-2\alpha-1}B(2\beta-1, \quad 1-2\alpha)\int_0^t |f(\tau, y(\tau))|^2\mathrm{d}\tau \\ &< \frac{2\beta^2}{\Gamma(1-\alpha)^2}t^{2\beta-2\alpha-1}B(2\beta-1, \quad 1-2\alpha)K\int_0^t (1+|y(t)|^2)\mathrm{d}\tau \\ &< \frac{2\beta^2}{\Gamma(1-\alpha)^2}t^{2\beta-2\alpha-1}B(2\beta-1, \quad 1-2\alpha)Kt(1+\|y\|_\infty^2), \end{aligned} \tag{34}$$

$$|\Phi(y(t))|^2 < \frac{2\beta^2 T^{2\beta-2\alpha}}{\Gamma(1-\alpha)^2} B(2\beta-1,1-2\alpha)K(1+\|y\|_\infty^2) < K_1(1+\|y\|_\infty^2), \tag{35}$$

where

$$K_1 = \frac{2\beta^2 T^{2\beta-2\alpha}}{(\Gamma(1-\alpha))^2} B(2\beta-1,1-2\alpha)K. \tag{36}$$

This shows that the mapping Φ satisfies the linear growth condition. We now proceed with the Lipschitz condition.

$$|\Phi_1 y_1 - \Phi_2 y_2|^2 = \left| \frac{\beta}{\Gamma(1-\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{-\alpha} f(\tau, y_1(\tau))\mathrm{d}\tau - \frac{\beta}{\Gamma(1-\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{-\alpha} f(\tau, y_2(\tau))\mathrm{d}\tau \right|^2. \tag{37}$$

The linearity of the fractal-fractional integral yields:

$$|\Phi_1 y_1 - \Phi_2 y_2|^2 = \left| \frac{\beta}{\Gamma(1-\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{-\alpha} f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))\mathrm{d}\tau \right|^2. \tag{38}$$

Thanks to the Cauchy inequality, we have:

$$\begin{aligned} |\Phi_1 y_1 - \Phi_2 y_2|^2 &< \frac{2\beta^2}{(\Gamma(1-\alpha))^2} \int_0^t \tau^{2\beta-2}(t-\tau)^{-2\alpha}\mathrm{d}\tau \int_0^t |f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))|^2 \mathrm{d}\tau \\ &< \frac{2\beta^2}{(\Gamma(1-\alpha))^2} t^{2\beta-2\alpha-1} B(2\beta-1,\quad 1-2\alpha) \int_0^t |f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))|^2 \mathrm{d}\tau. \end{aligned} \tag{39}$$

Using the Lipschitz condition of $f(t, y(t))$ with respect to the second parameter yields:

$$|\Phi_1 y_1 - \Phi_2 y_2|^2 < \frac{2\beta^2 t^{2\beta-2\alpha-1}}{(\Gamma(1-\alpha))^2} B(2\beta-1,\quad 1-2\alpha)\bar{K} t \|y_1 - y_2\|_\infty^2. \tag{40}$$

Therefore,

$$|\Phi_1 y_1 - \Phi_2 y_2|^2 < \frac{2\beta^2 T^{2\beta-2\alpha}}{(\Gamma(1-\alpha))^2} B(2\beta-1,\quad 1-2\alpha)\bar{K} \|y_1 - y_2\|_\infty^2 < \bar{K} \|y_1 - y_2\|_\infty^2, \tag{41}$$

where

$$\bar{K}_1 = \frac{2\beta^2 T^{2\beta-2\alpha}}{(\Gamma(1-\alpha))^2} B(2\beta-1,\quad 1-2\alpha)\bar{K}. \tag{42}$$

Under the condition of the linear growth, we can conclude that our equation is a unique solution.

We now present a numerical solution to the Cauchy problem with the fractal-fractional derivative with power law.

$$ {}^{\mathrm{FFP}}_{\ \ 0}D_t^{\alpha,\beta} y(t) = f(t, y(t))\ t > 0, \tag{43}$$

$$y(0) = y_0. \tag{44}$$

We consider the following: $0 < t_1 < t_2 < t_3 < \ldots < t_{n+1} = T$.

We have that $\forall t \in (0, T)$

$$ {}^{\mathrm{FFP}}_{\ \ 0}D_t^{\alpha,\beta} y(t) = f(t, y(t)). \tag{45}$$

Can be converted to:

$$y(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1} f(\tau, y(\tau))\mathrm{d}\tau. \tag{46}$$

But then at $t = t_{n+1}$, we have:

$$y(t_{n+1}) = \frac{\beta}{\Gamma(\alpha)}\int_0^{t_{n+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}f(\tau, y(\tau))\mathrm{d}\tau, \tag{47}$$

$$=\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}f(\tau, y(\tau))\mathrm{d}\tau. \tag{48}$$

Within $[t_j, t_{j+1}]$, we approximate:

$$f(\tau, y(\tau)) \approx P_j(\tau) = f(t_j, y(t_j)) + \frac{(t-t_j)}{h}(f(t_{j+1}, y(t_{j+1})) - f(t_j, y(t_j))). \tag{49}$$

Replacing the aforementioned into the general equation yields:

$$y(t_{n+1}) \approx \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}\left[f(t_j, y(t_j)) + \frac{\tau-t_j}{h}(f(t_{j+1}, y(t_{j+1})) - f(t_j, y(t_j)))\right]\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\mathrm{d}\tau, \tag{50}$$

$$\begin{aligned}&= \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left[f(t_j, y(t_j))\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\mathrm{d}\tau\right] + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\Bigg[[f(t_{j+1}, y(t_{j+1}))\\ &- f(t_j, y(t_j))]\Bigg]\frac{(\tau-t_j)}{h}(t_{n+1}-\tau)^{\alpha-1}\tau^{\beta-1}d\tau + \frac{\beta}{\Gamma(\alpha)}\int_{t_n}^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}\tau^{\beta-1}\frac{(\tau-t_n)}{h}[f(t_{n+1}, y^p(t_{n+1}))\\ &- f(t_n, y(t_n))]\mathrm{d}\tau.\end{aligned} \tag{51}$$

$$\begin{aligned}y_{n+1} = {} & \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}f(t_j, y(t_j))\int_{t_j}^{t_{j+1}}\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\mathrm{d}\tau + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n-1}\frac{(f(t_{j+1}, y(t_{j+1})) - f(t_j, y(t_j)))}{h}\int_{t_j}^{t_{j+1}}(\tau\\ &- t_j)(t_{n+1}-\tau)^{\alpha-1}\tau^{\beta-1}\mathrm{d}\tau + \frac{\beta}{\Gamma(\alpha)}\int_{t_n}^{t_{n+1}}(t_{n+1}-\tau)^{\alpha}(f(t_{n+1}, y^p(t_{n+1})) - f(t_n, y(t_n)))\frac{(\tau-t_n)\tau^{\beta}}{h}\mathrm{d}\tau.\end{aligned} \tag{52}$$

We shall put,

$$\int_{t_j}^{t_{j+1}}\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\mathrm{d}\tau = \varnothing_{n,j}^{\alpha,\beta}. \tag{53}$$

We consider the fractal-fractional with exponential decay kernel.

$${}^{\mathrm{FCP}}_{\ \ 0}D_t^{\alpha,\beta}y(t) = f(t, y(t))\ t > 0, \tag{54}$$

$$y(0) \text{ if } t = 0. \tag{55}$$

We convert the system into the following:

$$\begin{cases}{}^{\mathrm{CF}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}f(t, y(t))\ t > 0\\ y(0) = y_0\ \text{ if } t = 0,\end{cases} \tag{56}$$

$$\begin{cases}y(t) = (1-\alpha)\beta t^{\beta-1}f(t, y(t)) + \alpha\beta\displaystyle\int_0^t \tau^{\beta-1}f(\tau, y(\tau))\mathrm{d}\tau \quad \text{if} t > 0\\ y(0) = y_0.\end{cases} \tag{57}$$

At $t = t_{n+1}$,

$$\begin{cases} y(t_{n+1}) = (1-\alpha)\beta t_{t_{n+1}}^{\beta-1} f(t_{n+1}, y(t_{n+1})) + \alpha\beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} f(\tau, y(\tau))\mathrm{d}\tau \\ y(t_n) = (1-\alpha)\beta t_n^{\beta-1} f(t_n, y(t_n)) + \alpha\beta \int_0^{t_n} \tau^{\beta-1} f(\tau, y(\tau))\mathrm{d}\tau. \end{cases} \tag{58}$$

The difference produces:

$$y(t_{n+1}) - y(t_n) = (1-\alpha)\beta(t_{n+1}^{\beta-1} f(t_{n+1}, y(t_{n+1})) - t_n^{\beta-1} f(t_n, y(t_n))) + \alpha\beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} f(\tau, y(\tau))\mathrm{d}\tau. \tag{59}$$

Within $[t_n, t_{n+1}]$, we approximate $f(\tau, y(\tau))\mathrm{d}\tau$ as follows:

$$f(\tau, y(\tau)) \approx P_n(\tau) = \frac{(t_{j+1} - \tau)}{t_{j+1} - t_j} f(t_j, y(t_j)) + \frac{\tau - t_j}{t_{j+1} - t_j} f(t_{j+1}, y(t_{j+1})). \tag{60}$$

We replace the approximate function to obtain:

$$y_{n+1} = y_n + \beta(t_{n+1}^{\beta-1} f(t_{n+1}, y_{n+1}^p) - t_n f(t_n, y_n))(1-\alpha) + \alpha\beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} \left\{ \frac{t_{j+1} - T}{h} f(t_j, y_j) + \frac{\tau - t_j}{h} f(t_{j+1}, y_{j+1}) \right\} \mathrm{d}\tau, \tag{61}$$

$$\begin{aligned} y_{n+1} = {} & y_n + \beta(1-\alpha)(t_{n+1}^{\beta-1} f(t_{n+1}, y_{n+1}^p) - t_{t_n}^{\beta-1} f(t_n, y_n)) + \alpha\beta \frac{f(t_n, y_n)}{h} \int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{j+1} - \tau)\mathrm{d}\tau \\ & + \alpha\beta \frac{f(t_{n+1}, y_{n+1})}{h} \int_{t_j}^{t_{j+1}} \tau^{\beta-1}(\tau - t_n)\mathrm{d}\tau, \end{aligned} \tag{62}$$

$$\int_{t_n}^{t_{n+1}} (\tau^{\beta-1} t_{n+1} - \tau^{\beta-1})\mathrm{d}\tau = \frac{\tau^\beta t_{n+1}}{\beta} - \frac{T^{\beta+1}}{\beta+1} \Bigg|_{t_n}^{t_{n+1}}, \tag{63}$$

$$= \frac{t_{n+1}^{\beta+1}}{\beta} - \frac{t^{\beta+1}}{\beta+1} - \frac{t_n^\beta t_{n+1}}{\beta} + \frac{t_n^{\beta+1}}{\beta+1}, \tag{64}$$

$$= (\Delta t)^{\beta+1} \left\{ \frac{(n+1)^{\beta+1}}{\beta} - \frac{(n+1)^{\beta+1}}{\beta+1} - \frac{n^\beta(n+1)}{\beta} + \frac{n^{\beta+1}}{\beta+1} \right\}, \tag{65}$$

$$\int_{t_n}^{t_{n+1}} (\tau^{\beta-1}\tau - \tau^{\beta-1} t_n)\mathrm{d}\tau = \frac{\tau^{\beta+1}}{\beta+1} - \frac{T^\beta t_n}{\beta} \Bigg|_{t_n}^{t_{n+1}}, \tag{66}$$

$$= \frac{t_{n+1}^{\beta+1}}{\beta+1} - \frac{t_{n+1}^\beta t_n}{\beta} - \frac{t_n^{\beta+1}}{\beta+1} + \frac{t_n^{\beta+1}}{\beta}, \tag{67}$$

$$= (\Delta t)^{\beta+1} \left\{ \frac{(n+1)^{\beta+1}}{\beta+1} - \frac{n(n+1)^\beta}{\beta} - \frac{n^{\beta+1}}{\beta+1} + \frac{n^{\beta+1}}{\beta+1} \right\}, \tag{68}$$

$$\begin{aligned} y_{n+1} = {} & y_n + \beta(1-\alpha)(\Delta t)^{\beta-1}\{(n+1)^{\beta+1} f(t_{n+1}, y_{n+1}^p) - n^{\beta+1} f(t_n, y_n)\} \\ & + \beta\alpha f(t_n, y_n)(\Delta t)^{\beta+1} \left\{ \frac{(n+1)^{\beta+1}}{\beta} - \frac{(n+1)^{\beta+1}}{\beta+1} - \frac{n^\beta(n+1)}{\beta} + \frac{n^{\beta+1}}{\beta+1} \right\} \\ & + \alpha\beta\,\Delta t^{\beta+1} \left\{ \frac{(n+1)^{\beta+1}}{\beta+1} - \frac{n(n+1)^\beta}{\beta} - \frac{n^{\beta+1}}{\beta+1} + \frac{n^{\beta+1}}{\beta+1} \right\} f(t_{n+1}, y_{n+1}^p), \end{aligned} \tag{69}$$

where y_{n+1}^{p} is the predictor, and here, the use of the predictor angle rule is applied:

$$\beta\int_{t_n}^{t_{n+1}}\tau^{\beta-1}f(\tau,y(\tau))\mathrm{d}\tau\approx\beta f(t_n,y_n)\int_{t_n}^{t_{n+1}}\tau^{\beta-1}\mathrm{d}\tau, \tag{70}$$

$$\approx\beta f(t_n,y_n)\left\{\frac{t_{n+1}^{\beta}}{\beta}-\frac{t_n^{\beta}}{\beta}\right\}, \tag{71}$$

$$\approx\beta f(t_n,y_n)\frac{\Delta t^{\beta}}{\beta}((n+1)^{\beta}-n^{\beta}), \tag{72}$$

$$y_{n+1}^{p}\approx f(t_n,y_n)\Delta t^{\beta}\{(n+1)^{\beta}-n^{\beta}\}. \tag{73}$$

Therefore, the scheme is completed and given as follows:

$$\begin{aligned}y_{n+1}&=y_n+\beta(1-\alpha)(\Delta t)^{\beta-1}\{(n+1)^{\beta}f(t_{n+1},y_{n+1}^{p})-n^{\beta}f(t_n,y_n)\}\\&+\beta\alpha f(t_n,y_n)(\Delta t)^{\beta-1}\left\{\frac{(n+1)^{\beta+1}}{\beta}-\frac{(n+1)^{\beta+1}}{\beta+1}-\frac{n^{\beta}(n+1)}{\beta}+\frac{n^{\beta+1}}{\beta+1}\right\}\\&+\alpha\beta(\Delta t)^{\beta+1}\left\{\frac{(n+1)^{\beta+1}}{\beta+1}-\frac{n(n+1)^{\beta}}{\beta}-\frac{n^{\beta+1}}{\beta+1}+\frac{n^{\beta+1}}{\beta+1}\right\}f(t_{n+1},y_{n+1}^{p}),\end{aligned} \tag{74}$$

$$y_{n+1}^{p}=f(t_n,y_n)\Delta t^{\beta}\{(n+1)^{\beta}-n^{\beta}\}. \tag{75}$$

We carry on our analysis by considering the following Cauchy problem:

$$\begin{cases}{}^{\mathrm{FFM}}_{\ 0}D_t^{\alpha,\beta}y(t)=f(t,y(t)),\ t>0\\ y(0)=y_0t=0,\end{cases} \tag{76}$$

$${}^{\mathrm{ABR}}_{\ 0}D_t^{\alpha}y(t)=\beta t^{\beta-1}f(t,y(t)),\ t>0, \tag{77}$$

$$y(0)=y_0\quad\text{if}\quad t=0, \tag{78}$$

$$\begin{cases}y(t)=(1-\alpha)\beta t^{\beta-1}f(t,y(t))+\dfrac{\beta}{\Gamma(\alpha)}\displaystyle\int_0^t\tau^{\beta-1}f(\tau,y(\tau))(t-\tau)^{\alpha-1}\mathrm{d}\tau,\ t>0\\ y(0)=y_0\quad\text{if}\quad t=0.\end{cases} \tag{79}$$

At $t=t_{n+1}$,

$$\begin{cases}y(t_{n+1})=(1-\alpha)\beta t_{t_{n+1}}^{\beta-1}f(t_{n+1},y(t_{n+1}))\\ \qquad+\dfrac{\beta}{\Gamma(\alpha)}\displaystyle\int_0^{t_{n+1}}\tau^{\beta-1}f(\tau,y(\tau))(t_{n+1}-\tau)^{\alpha-1}\mathrm{d}\tau\\ y(0)=y_0,\end{cases} \tag{80}$$

$$\begin{cases}y(t_{n+1})=(1-\alpha)\beta t_{t_{n+1}}^{\beta-1}f(t_{n+1},y(t_{n+1}))\\ \qquad+\dfrac{\beta}{\Gamma(\alpha)}\displaystyle\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}f(\tau,y(\tau))(t_{n+1}-\tau)^{\alpha-1}\mathrm{d}\tau\\ y(0)=y_{0.}\end{cases} \tag{81}$$

Within $[t_j,t_{j+1}]$, we approximate $f(\tau,y(\tau))$ as follows:

$$f(\tau,y(\tau))\approx\frac{t_{j+1}-\tau}{h}f(t_j,y_j)+\frac{\tau-t_{j+1}}{h}f(t_{j+1},y_{j+1}). \tag{82}$$

$$y_{n+1}=(1-\alpha)\beta t_{t_{n+1}}^{\beta-1}f(t_{n+1},y_{n+1}^{p})+\frac{\alpha\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left\{\frac{t_{j+1}-\tau}{h}f(t_j,y_j)+\frac{\tau-t_j}{h}f(t_{j+1},y_{j+1})\right\}(t_{n+1}-\tau)^{\alpha-1}\tau^{\beta-1}\mathrm{d}\tau. \tag{83}$$

The following are the examples of some fractal-fractional equations:

$$ {}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta} y(t) = t^{\gamma}, \tag{84} $$

$$ {}^{\mathrm{FFE}}_{0}D_t^{\alpha,\beta} y(t) = t^{\gamma}, \tag{85} $$

$$ {}^{\mathrm{FFM}}_{0}D_t^{\alpha,\beta} y(t) = t^{\gamma}. \tag{86} $$

1. We start with the first one,

$$ {}^{\mathrm{FFP}}_{0}D_t^{\alpha,\beta} y(t) = t^{\gamma} \Rightarrow {}^{\mathrm{RL}}_{0}D_t^{\alpha} y(t) = \beta t^{\gamma+\beta-1}. \tag{87} $$

By applying the RL integral, we obtain:

$$ y(t) = \frac{\beta}{\Gamma(\alpha)} \int_t^0 t^{\gamma+\beta-1}(t-\tau)^{\alpha-1}\mathrm{d}\tau, \tag{88} $$

$$ = \frac{\beta}{\Gamma(\alpha)} t^{\Upsilon+\beta+\alpha-1} B(\Upsilon+\beta,\alpha). \tag{89} $$

Thus, the exact solution for this equation is:

$$ y(t) = \frac{\beta}{\Gamma(\alpha)} t^{\Upsilon+\beta+\alpha-1} B(\Upsilon+\beta,\alpha). \tag{90} $$

We present some numerical simulation for different values of fractional orders in Figure 1.

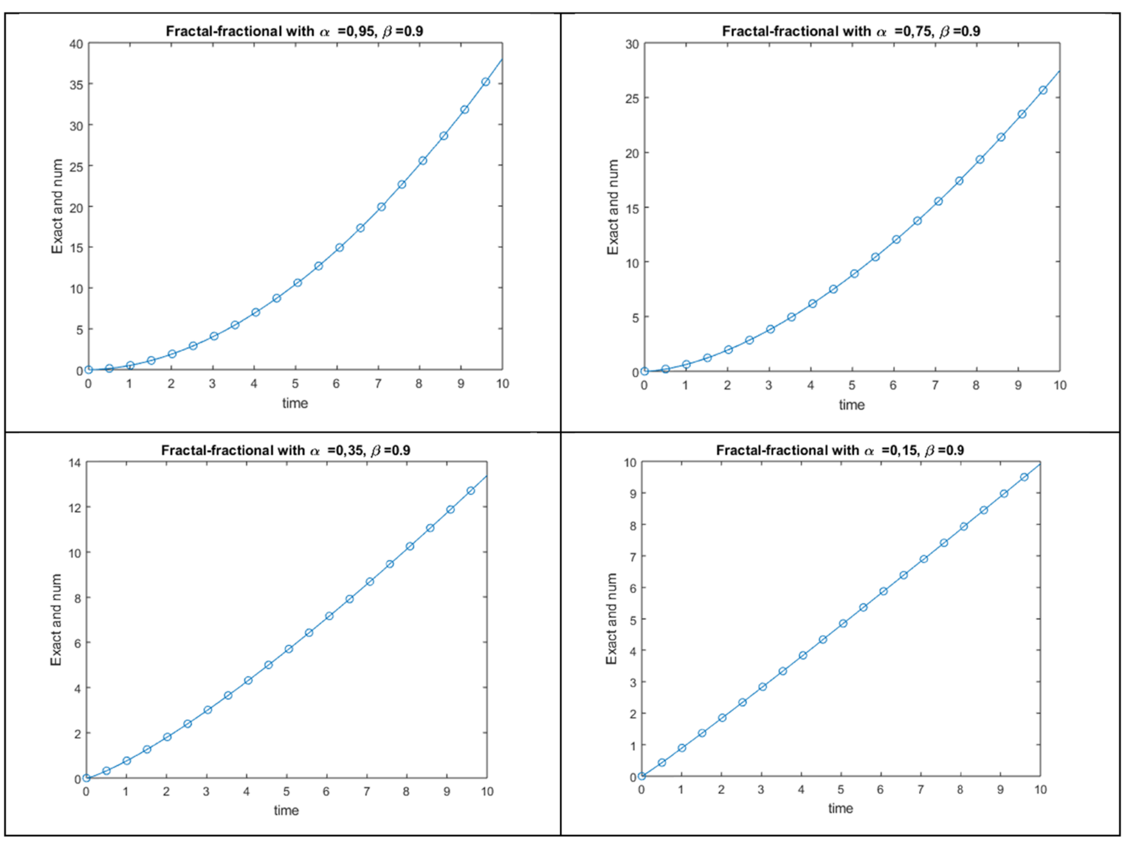

Figure 1: Numerical solution with the power-law case.

1. With the exponential kernel, we have:

$$ {}_{0}^{\mathrm{CF}}D_{t}^{\alpha}y(t) = \beta t^{\beta+\gamma-1}, \tag{91} $$

$$ y(t) = (1-\alpha)\beta t^{\beta+\gamma-1} + \alpha\beta \int_{0}^{t} \tau^{\beta+\gamma-1}\mathrm{d}\tau, \tag{92} $$

$$ = (1-\alpha)\beta t^{\beta+\gamma-1} + \frac{\alpha\beta t^{\beta+\gamma}}{(\alpha+\gamma)}. \tag{93} $$

Therefore, the exact solution will be:

$$ y(t) = (1-\alpha)\beta t^{\beta+\gamma-1} + \frac{\alpha\beta t^{\beta+\gamma}}{(\beta+\gamma)}. \tag{94} $$

By using the same routine, we obtain the case with the Mittag-Leffler kernel as follows:

$$ y(t) = (1-\alpha)\beta t^{\beta+\gamma-1} + \frac{\beta\alpha}{\Gamma(\alpha)} t^{\gamma+\beta+\alpha-1} B(\gamma+\beta,\alpha). \tag{95} $$

We present some graphical representation for different values of fractional order in Figure 2.

Example 2. Nonlinear equation

We consider a system of nonlinear differential equation, where the nonlinear part is given as follows:

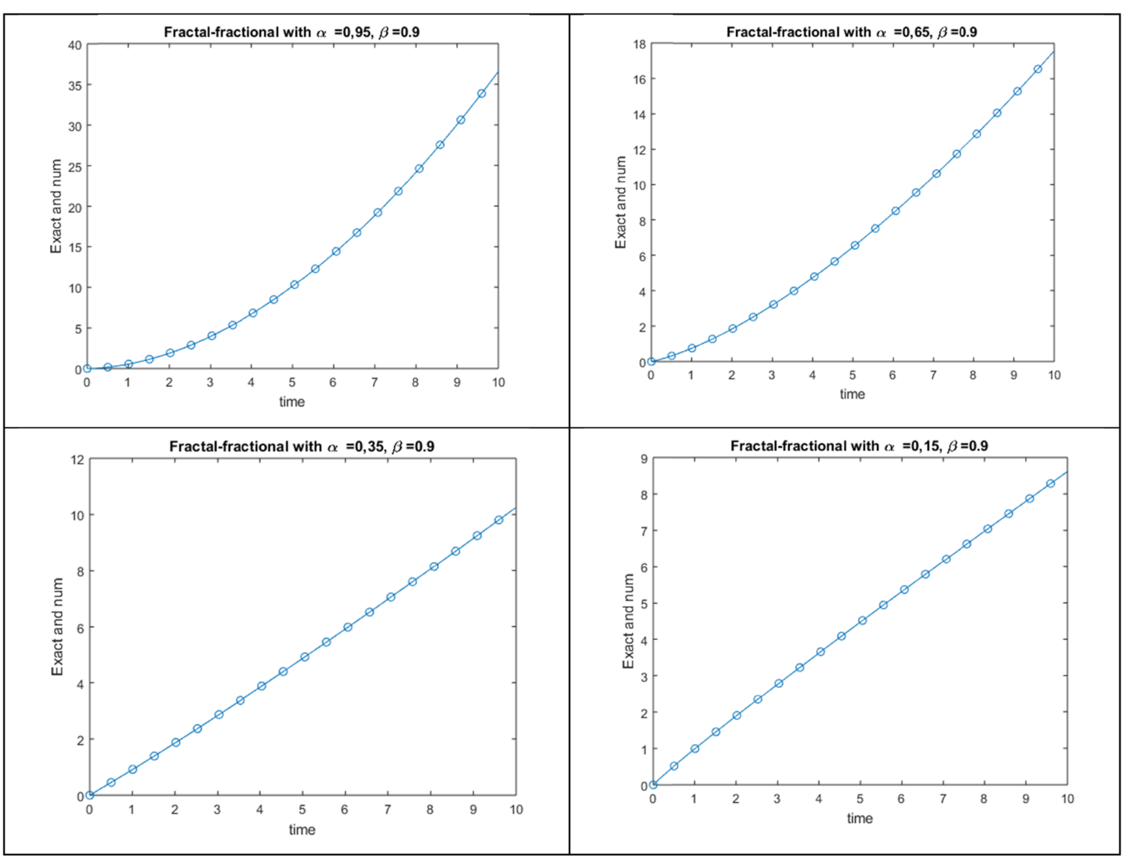

Figure 2: Numerical solution with the generalized Mittag-Leffler case.

$$f_1(\tau, x(\tau), y(\tau)) = \cos(x(\tau)) - \tau^5 y(\tau), \tag{96}$$

$$f_2(\tau, x(\tau), y(\tau)) = 2x(\tau) + \sin(x(\tau))y(\tau). \tag{97}$$

Indeed, the both functions f_1 and f_2 satisfy the linear growth and Lipschitz conditions. This is easily seen if we define the following norm:

$$X = (x, y), \tag{98}$$

$$\|X\|_\infty = \max\{\sup_{t\epsilon Dxny}|x|, \quad \sup_{t\epsilon Dxny}|y|\}, \tag{99}$$

where $Dxny$ is the intersection of the domain of x and y.

By using the numerical scheme presented earlier, we can now solve the aforementioned problem numerically.

We next consider the case with exponential decay.

$$\begin{cases} {}^{\mathrm{FFP}}_{\ \ 0}D_t^{\alpha,\beta}y(t) = \cos(x(t)) - t^5 y(t) \\ {}^{\mathrm{FFE}}_{\ \ 0}D_t^{\alpha,\beta}y(t) = 2x(t) + \sin(x(t))y(t). \end{cases} \tag{100}$$

We consider now the following well-known nonlinear differential equation that we convert into a fractal-fractional case.

We consider the following system:

$$x'(t) = \cos(x(t)) - t^5 y(t), \tag{101}$$

$$y'(t) = 2x(t) + \sin(x(t))y(t). \tag{102}$$

First, we consider the case with the power law.

$$\begin{cases} {}^{\mathrm{FFP}}_{\ \ 0}D_t^{\alpha,\beta}y(t) = \cos(x(t)) - t^5 y(t) \\ {}^{\mathrm{FFE}}_{\ \ 0}D_t^{\alpha,\beta}y(t) = 2x(t) + \sin(x(t))y(t), \end{cases} \tag{103}$$

$$\begin{cases} {}^{\mathrm{RL}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}(\cos(x(t)) - t^5 y(t)) \\ {}^{\mathrm{RL}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}(2x(t) + \sin(x(t))y(t)), \end{cases} \tag{104}$$

$$\begin{cases} x(t) = \dfrac{\beta}{\Gamma(\alpha)}\displaystyle\int_0^t \tau^{\beta-1}(\cos(x) - \tau^5 y(\tau))(t-\tau)^{\alpha-1}\mathrm{d}\tau, \\ y(t) = \dfrac{\beta}{\Gamma(\alpha)}\displaystyle\int_0^t \tau^{\beta-1}(2x(t) + \sin(x(t))y(t))(t-\tau)^{\alpha-1}\mathrm{d}\tau. \end{cases} \tag{105}$$

We present a numerical simulation of the aforementioned model using the suggested numerical scheme in Figure 3.

We next consider the same model where the fractional derivative is with the exponential decay kernel.

$$\begin{cases} {}^{\mathrm{CF}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}(\cos(x(t)) - t^5 y(t)), \\ {}^{\mathrm{CF}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}(2x(t) + \sin(x(t))y(t)). \end{cases} \tag{106}$$

$$\begin{cases} x(t) = (1-\alpha)\beta t^{\beta-1}(\cos(x(t)) - t^5 y(t)) + \beta\alpha\displaystyle\int_0^t \tau^{\beta-1}f_1(x(\tau), y(\tau))\mathrm{d}\tau, \\ y(t) = (1-\alpha)\beta t^{\beta-1}(2x(t) + \sin(x(t))y(t)) + \beta\alpha\displaystyle\int_0^t \tau^{\beta-1}f_2(x(\tau), y(\tau))\mathrm{d}\tau. \end{cases} \tag{107}$$

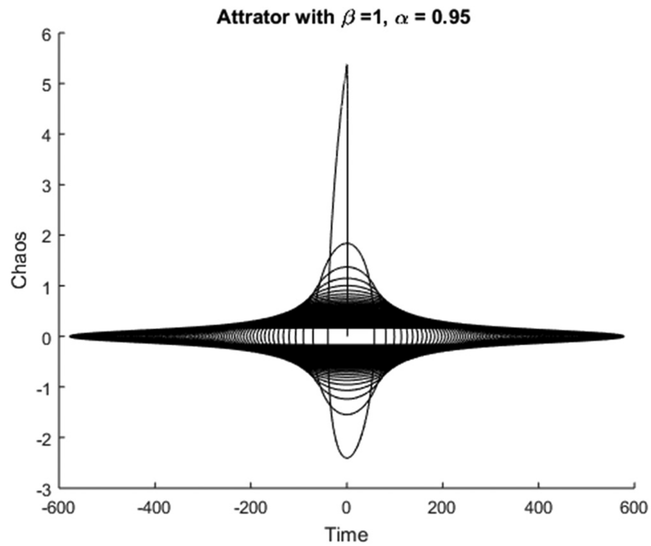

Figure 3: Chaos with the power-law case.

By using the numerical scheme presented in this work, we perform numerical simulation of the aforementioned model. The numerical simulation is presented in Figure 4.

We consider the same model by replacing the classical differential operator by the Fractal-fractional derivative with the generalized Mittag-Leffler function:

$$\begin{cases} {}^{\mathrm{ABC}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}(\cos(x(t)) - t^5 y(t)), \\ {}^{\mathrm{ABC}}_{\ 0}D_t^{\alpha}y(t) = \beta t^{\beta-1}(2x(t) + \sin(x(t))y(t)). \end{cases}$$

The same routine can be applied to the case with the Mittag-Leffler kernel to obtain:

$$\begin{cases} x(t) = (1-\alpha)\beta t^{\beta-1}(\cos(x(t)) - t^5 y(t)) + \dfrac{\beta\alpha}{\Gamma(\alpha)}\displaystyle\int_0^t \tau^{\beta-1} f_1(x(\tau), y(\tau))(t-\tau)^{\alpha-1}\mathrm{d}\tau, \\ y(t) = (1-\alpha)\beta t^{\beta-1}(2x(t) + \sin(x(t))y(t)) + \dfrac{\alpha\beta}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f_2(x(\tau), y(\tau))\tau^{\beta-1}\mathrm{d}\tau. \end{cases} \tag{108}$$

Numerical simulations are shown in Figure 5.

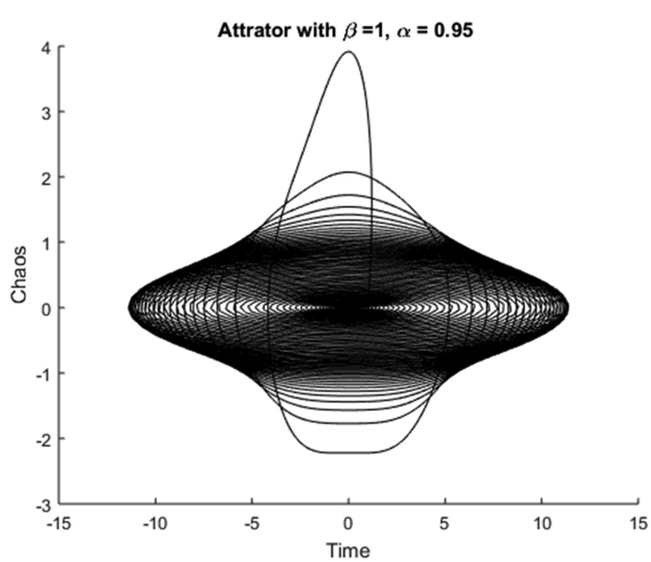

Figure 4: Chaos with exponential decay case.

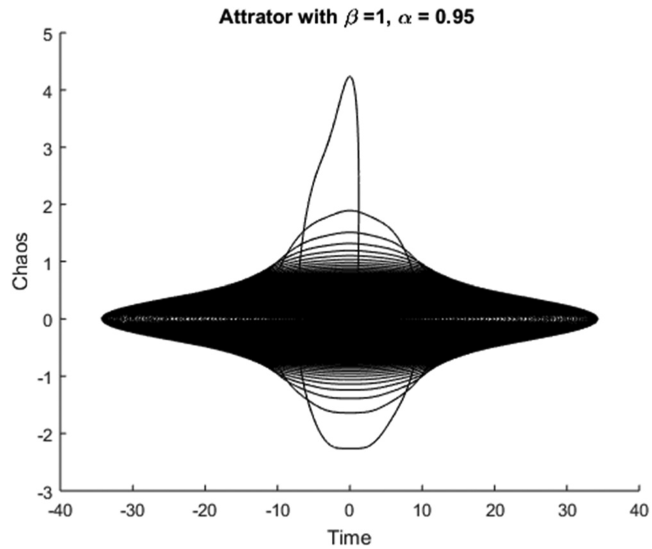

Figure 5: Chaos with the generalized Mittag-Leffler case.

4 Conclusion

Fractal-fractional differential equations have attracted the attention of several authors in the last past years due to their wider applicability. Several theoretical foundations have been laid down and more are still to be developed. In this work, a few additional results are presented. We considered three classes of Cauchy problems with fractal-fractional derivatives, including the class with power law kernel, exponential decay kernel, and the generalized Mittag-Leffler kernel. For each one of these classes, we presented a detailed investigation of the existence and uniqueness of the solution, then its numerical solutions with some simulations.

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