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Research Article

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On Cauchy problem for pseudo-parabolic equation with Caputo-Fabrizio operator

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Abstract: In this article, we considered the pseudo-parabolic equation with Caputo-Fabrizio fractional derivative. This equation has many applications in different fields, such as science, technology, and so on. In this article, we gave the formula of mild solution, which is represented in the form of Fourier series by some operators . In the linear case, we investigated the continuity of the mild solution with respect to the fractional order. For the nonlinear case, we investigated the existence and uniqueness of a global solution. The main proof technique is based on the Banach fixed point theorem combined with some Sobolev embeddings. For more detailed, we obtained two other interesting results: the continuity of mild solution with respect to the derivative order and the convergence of solution as the coefficient k approaches to zero.

Keywords: Caputo-Fabrizio derivative operator, Burger equation, Banach fixed point theory, Sobolev embeddings

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1 Introduction

Recently, fractional differential equations have been extensively used in describing various mathematical models of physical processes and natural phenomena, for example, in mechanics, physics, and engineering sciences, etc. The number of works in this direction is quite abundant and attracts many interested mathematicians. We list here some works using fractional derivative models that closely related to this article, such as [1–19] and references therein.

Let $\mathbb{M}$ be a simply connected and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\mathbb{M}$. Let T be a positive real number. In this article, we are interested in considering the pseudo-parabolic equation as follows:

$$\begin{cases} {}_{\mathrm{CF}}D_t^{\beta}(Z + k\mathbb{L}Z) + \mathbb{L}^p Z = G(Z), & \text{in } \mathbb{M} \times (0, T], \\ Z(x, 0) = z_0(x), & \text{in } \mathbb{M}, \\ z(x, t) = 0, & \text{in } \partial\mathbb{M}, \end{cases} \tag{1.1}$$

where ${}_{\mathrm{CF}}D_t^{\beta}$ is the Caputo-Fabrizio derivative operator of order β, which is defined as (see [20,21]) follows:

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$$ {}_{\mathrm{CF}}D_t^{\beta} z(t) = \frac{Y(\beta)}{1-\beta}\int_0^t \mathbf{D}_\beta(t-r)\frac{\partial z(r)}{\partial r}\mathrm{d}r, \quad \text{for } t \geq 0, $$

where $\mathbf{D}_\beta(r) = \exp\left(-\frac{\beta}{1-\beta}r\right)$ and $Y(\beta)$ satisfies $Y(0) = Y(1) = 1$.

It is easy to observe that if $\beta = 1$, then problem (1.1) turns to the classical pseudo-parabolic equation. The pseudo-parabolic equation describes several important physical and biological phenomena, such as the analysis of unstable processes in [22] and population aggregation in [23]. We encourage the interested reader to the recent work by Tuan and Caraballo [24] on classical pseudo-parabolic equation, which is in the spirit of the Fourier series semigroup. Currently, several models with pseudo-parabolic equations, combined with fractional derivative, have attracted the interest of a number of mathematicians. Luc et al. in [25] derived the pseudo-parabolic equation with Caputo derivative as follows:

$$ \begin{cases} {}_{\mathrm{C}}D_t^{\beta}(Z + k\mathbb{L}Z) + \mathbb{L}^p Z = G(Z), & \text{in } \mathbb{M}\times(0,T], \\ Z(x,0) = z_0(x), & \text{in } \mathbb{M}, \\ Z(x,t) = 0, & \text{in } \partial\mathbb{M}, \end{cases} \tag{1.2} $$

where $0 < \beta < 1$, ${}_{\mathrm{C}}D_t^{\beta}$ is a Caputo fractional derivative operator of order β. The authors have obtained the existence of mild solutions in both aspects: local and global sense. The main method is to use the Banach contraction mapping theorem in Hilbert scales space. In [5], Tuan et al. studied problem (1.2) when the source function has a logarithmic form. The technique used in this article is quite interesting with many different embeddings in L^p and rather complicated evaluations. In [26], Can et al. focused on the nonlocal pseudo-parabolic equation with linear case. In [27], Tuan et al. considered the pseudo-parabolic equation associated with integral condition under nonlinear case. Shen et al. in [28] studied the fractional pseudo-parabolic equation with the Riemann-Liouville derivative. They obtained the global and local existence of weak solutions by using the Galerkin method.

To our knowledge, there have been quite a few investigations on diffusion equations with the presence of the Caputo-Fabrizio derivative. Let us refer some interesting papers that focus on the existence of the mild solution to some problems, e.g., [29,20,30,31,21]. In [20], Tuan and Zhou established the existence and uniqueness of the mild solution for diffusion equation (1.1) with $k = 0$. They also provided the existence of local mild solutions to the problem, and then a blow-up alternative is established. In [30], Tuan studied the Cahn-Hilliard equation with the Caputo-Fabrizio operator as follows:

$$ \begin{cases} {}_{\mathrm{CF}}D_t^{\beta} Z + \Delta^2 Z = \Delta(Z - Z^3), & \text{in } \mathbb{M}\times(0,T], \\ Z(x,0) = z_0(x), & \text{in } \mathbb{M}, \\ Z(x,t) = \Delta Z = 0, & \text{in } \partial\mathbb{M}, \end{cases} \tag{1.3} $$

and he proved the local existence result for problem (1.3). He first provided that the connections of the mild solution to problem (1.3) between the Cahn-Hilliard equation in the case $0 < \beta < 1$ and $\beta = 1$. The main key of the proof is the proficient use of some embeddings between L^p spaces and Hilbert scales spaces. Recently, the time-fractional integro-differential equation with the Caputo-Fabrizio type derivative has been considered in [31]. Let us refer the reader to some works in the spirit of mild solution with Caputo Fabrizio derivative, such as [29,32,21,36–39,41–50].

There are two advantages when we investigated the equation using this type of derivative. The most important thing is that we can avoid the singularity kernel, which frequently occurs in the case of the Caputo or Riemann-Liouville derivative. The second advantage is that the model with Caputo-Fabrizio is effectively used in physical models involving an exponential power multiplied to some components of the equations. To the best of our knowledge, the current article is the first study about pseudo-parabolic equations with Caputo-Fabrizio derivative.

The main results of this article are described as follows:

- In the linear case, we investigate the continuity of the mild solution respect to the derivative order. This research direction was inspired by an article by Dang et al. [33]. This article focuses on the question: “Does $Z_{\beta'}$ tends to Z_β in an appropriate sense as $\beta' \to \beta$?”

– For the nonlinear case, we prove the existence and uniqueness of global mild solution when the Lipschitz source function is global in Hilbert spaces $\mathbb{H}^p$. The main tool is to use Banach's theorem in suitable spaces. We also obtained continuous dependence of mild solution respect to derivative order for the nonlinear case.
– Another interesting contribution of this article is to show that the solution of the pseudo-parabolic equation converges to the solution of the corresponding parabolic problem. The ideas and methods are partially found from our recent paper by Tuan et al. [40] and Tuan [30]. To complete the proofs, we had to overcome many challenges by skillfully evaluating the upper bounds.

This article is organized as follows. Section 2 presents some preliminaries, the formula of the mild solution, and its representation in the operator form. Section 3 is dedicated to the results on continuity of fractional order for the linear problem. In Section 4, we showed the global existence of the mild solution in the nonlinear case. We also obtained the continuity results of fractional order in the case where the function G is globally Lipschitz function. In Section 5, we investigate the convergence of mild solutions of the problem (1.1) when $k \to 0^+$.

2 Preliminaries, mild solution, and solution operators

The Hilbert scale space $\mathbb{H}^m(\mathbb{M})$ is defined in the following line:

$$\mathbb{H}^m(\mathbb{M}) = \left\{\theta \in L^2(\mathbb{M}), \quad \sum_{n=1}^{\infty} \lambda_n^{2m} \left(\int_{\mathbb{M}} \theta(x)\psi_n(x)\mathrm{d}x \right)^2 < \infty \right\},$$

for some $m \geq 0$. The norm of $\mathbb{H}^m(\mathbb{M})$ is also given as follows:

$$\|\theta\|_{\mathbb{H}^m(\mathbb{M})} = \sqrt{\sum_{n=1}^{\infty} \lambda_n^{2m} \left(\int_{\mathbb{M}} \theta(x)\psi_n(x)\mathrm{d}x \right)^2}, \ \theta \in \mathbb{H}^m(\mathbb{M}).$$

This Hilbert scales space plays an important role in investigating the properties of regularity for mild solutions.

Lemma 2.1. (See [4]) *There hold that*
(*a*) $\mathbb{H}^{m/2}(\mathbb{M}) \hookrightarrow L^p(\mathbb{M})$, *for* $m = \frac{N}{2}$, $1 \leq p < \infty$, *or* $0 \leq m < \frac{N}{2}$, $1 \leq p \leq \frac{2N}{N-2m}$;
(*b*) $L^p(\mathbb{M}) \hookrightarrow \mathbb{H}^{m/2}(\mathbb{M})$, *for* $-\frac{N}{2} < m \leq 0$, $p \geq \frac{2N}{N-2m}$.

Definition 2.2. (See [35]) Let $\mathbf{Y}_{a,d}((0, T]; B)$ denote the weighted space of all functions $v \in C((0, T]; B)$ such that

$$\|f\|_{\mathbf{Y}_{a,d}((0,T];\, B)} := \sup_{t \in (0,T]} t^a e^{-\mathrm{d}t} \|f(t, \cdot)\|_B < \infty. \tag{2.4}$$

The space $\mathbf{Y}_{a,d}((0, T]; B)$ is used in proving the global solution of the nonlinear problem. In the following, we introduce the formula of mild solution to our problem. Let us assume that

$$Z(x, t) = \sum_{n=1}^{\infty} \left(\int_{\mathbb{M}} Z(x, t)\psi_n(x)\mathrm{d}x \right) \psi_n(x).$$

Then we obtain the following differential equation:

$$ {}_{\mathrm{CF}}D_t^{\beta}\left(\int_{\mathbb{M}} Z(x,t)\psi_n(x)\mathrm{d}x\right) + \frac{\lambda_n^p}{1+k\lambda_n}\left(\int_{\mathbb{M}} Z(x,t)\psi_n(x)\mathrm{d}x\right) = \frac{1}{1+k\lambda_n}\left(\int_{\mathbb{M}} G(x,t)\psi_n(x)\mathrm{d}x\right). \tag{2.5} $$

By solving the aforementioned differential equation, the mild solution is given by

$$ Z(t) = \mathbb{Q}_{\beta,p}(t)z_0 + \int_0^t \mathbb{J}_{\beta,p}(t-v)G(v)\mathrm{d}v, \tag{2.6} $$

where $\mathbb{Q}_{\beta,p}(t)$ and $\mathbb{J}_{\beta,p}(t)$ are defined via the Fourier series as follows:

$$ \mathbb{Q}_{\beta,p}(t)f(x) := \sum_{n=1}^{\infty} \frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p} \exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x), \tag{2.7} $$

$$ \mathbb{J}_{\beta,p}(t)f(x) := \sum_{n=1}^{\infty} \frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} \exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x), \tag{2.8} $$

for any $f \in L^2(\mathbb{M})$. Here, f_n is Fourier coefficient of f, which is defined by

$$ f_n = \int_{\mathbb{M}} f(x)\psi_n(x)\mathrm{d}x. \tag{2.9} $$

The following lemma is introduced to play an important role in investigating some properties of solutions.

Lemma 2.3. *Let m be any real number and $\theta \in \mathbb{H}^{m-p}(\mathbb{M})$. Then we obtain*

$$ \|\mathbb{Q}_{\beta,p}(t)\theta\|_{\mathbb{H}^m(\mathbb{M})} \le \frac{1}{1-\beta}\|\theta\|_{\mathbb{H}^{m-p}(\mathbb{M})} \tag{2.10} $$

and

$$ \|\mathbb{J}_{\beta,p}(t)\theta\|_{\mathbb{H}^m(\mathbb{M})} \le \frac{2\beta}{1-\beta}\|\theta\|_{\mathbb{H}^{m-p}(\mathbb{M})}. \tag{2.11} $$

Proof. In view of Parseval's equality and noting that $e^{-z} \le 1$ for $z \ge 0$, we obtain that

$$ \begin{aligned} \|\mathbb{Q}_{\beta,p}(t)\theta\|^2_{\mathbb{H}^m(\mathbb{M})} &= \sum_{n=1}^{\infty}\left(\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)^2 \lambda_n^{2m} \exp\left(\frac{-2\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)\theta_n^2 \\ &\le \sum_{n=1}^{\infty}\left(\frac{1}{(1-\beta)\lambda_n^p}\right)^2 \lambda_n^{2m} \exp\left(\frac{-2\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)\theta_n^2 \\ &= \frac{1}{(1-\beta)^2}\sum_{n=1}^{\infty}\lambda_n^{2m-2p}\theta_n^2 = \frac{1}{(1-\beta)^2}\|\theta\|^2_{\mathbb{H}^{m-p}(\mathbb{M})}, \end{aligned} $$

where θ_n is given by $\theta_n = \int_{\mathbb{M}} \theta(x)\psi_n(x)\mathrm{d}x$. By a similar explanation as mentioned earlier, we find that

$$ \|\mathbb{J}_{\beta,p}(t)\theta\|^2_{\mathbb{H}^m(\mathbb{M})} = \sum_{n=1}^{\infty}\left(\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\right)^2 \lambda_n^{2m} \exp\left(\frac{-2\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)\theta_n^2. $$

By using the Cauchy inequality $(a+b)^2 \ge 4ab$ for any $a, b \ge 0$, we find that

$$ (1+k\lambda_n+(1-\beta)\lambda_n^p)^2 \ge 4(1-\beta)(1+k\lambda_n)\lambda_n^p. $$

Hence, we derive that

$$ \frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} \le \frac{\beta}{4(1-\beta)}\lambda_n^{-p} \le \frac{2\beta}{(1-\beta)}\lambda_n^{-p}. \tag{2.12} $$

Hence, by using this inequality, we infer that

$$\|\mathbb{J}_{\beta,p}(t)\theta\|^2_{\mathbb{H}^m(\mathbb{M})} \le \sum_{n=1}^{\infty}\left(\frac{2\beta}{1-\beta}\lambda_n^{-p}\right)^2 \lambda_n^{2m}\exp\left(\frac{-2\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)\theta_n^2$$
$$= \frac{4\beta^2}{(1-\beta)^2}\sum_{n=1}^{\infty}\lambda_n^{2m-2p}\theta_n^2 = \frac{4\beta^2}{(1-\beta)^2}\|\theta\|^2_{\mathbb{H}^{m-p}(\mathbb{M})}.$$

So, the proof of Lemma 2.3 is completed. Let us continue to introduce the following lemma, showing the continuity of the derivative order for mild solutions.

Lemma 2.4. *Let $\gamma > 0$ and $\mu > 0$. Then we obtain the following estimate*

$$\|\mathbb{Q}_{\beta,p}(t)f - \mathbb{Q}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} \le C_6(t^\gamma|\beta-\beta'|^\gamma\|f\|_{\mathbb{H}^{m-p+p\gamma-\gamma}(\mathbb{M})} + |\beta-\beta'|t^{-\mu}\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}). \tag{2.13}$$

Remark 2.1. Since the fact that $m-p+p\gamma-\gamma \le m+\mu-\mu p$, we deduce that the following Sobolev embedding

$$\mathbb{H}^{m+\mu-\mu p}(\mathbb{M}) \hookrightarrow \mathbb{H}^{m-p+p\gamma-\gamma}(\mathbb{M}). \tag{2.14}$$

This follows from Lemma 2.4 that

$$\|\mathbb{Q}_{\beta,p}(t)f - \mathbb{Q}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} \le \overline{C}_6 t^{-\mu}(|\beta-\beta'|^\gamma + |\beta-\beta'|)\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}. \tag{2.15}$$

Remark 2.2. Since the proof of Lemma 2.4, it is easy to obtain the following estimate

$$\|\mathbb{Q}_{\beta,p}(t)f - \mathbb{Q}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} \lesssim |\beta-\beta'|\|f\|_{\mathbb{H}^m(\mathbb{M})} \tag{2.16}$$

for any $f \in \mathbb{H}^m(\mathbb{M})$.

Proof. From (2.7), we derive that

$$\begin{aligned}
&\mathbb{Q}_{\beta,p}(t)f(x) - \mathbb{Q}_{\beta',p}(t)f(x)\\
&= \sum_{n=1}^{\infty}\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}\left[\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)\right]f_n\psi_n(x)\\
&+ \sum_{n=1}^{\infty}\left[\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p} - \frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right]\exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)f_n\psi_n(x)\\
&= \mathbb{A}_1(t,x) + \mathbb{A}_2(t,x).
\end{aligned} \tag{2.17}$$

The first term $\mathbb{A}_1$ is bounded by

$$\begin{aligned}
&\|\mathbb{A}_1(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})}\\
&= \sum_{n=1}^{\infty}\lambda_n^{2m}\left(\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)^2\left[\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)\right]^2 |f_n|^2.
\end{aligned} \tag{2.18}$$

It is obvious to see that

$$\left(\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)^2 \le \frac{\lambda_n^{-2p}}{(1-\beta)^2}. \tag{2.19}$$

By using the inequality $|e^{-a}-e^{-b}| \le C_\gamma|a-b|^\gamma$ for any $\gamma > 0$, we know that

$$\begin{aligned}
&\left|\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)\right|\\
&\le C_\gamma\left|\frac{\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p} - \frac{\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right|^\gamma\\
&= C_\gamma t^\gamma\lambda_n^{p\gamma}|\beta-\beta'|^\gamma\left[\frac{1+k\lambda_n+\lambda_n^p}{(1+k\lambda_n+(1-\beta)\lambda_n^p)(1+k\lambda_n+(1-\beta')\lambda_n^p)}\right]^\gamma.
\end{aligned} \tag{2.20}$$

In view of the two following observations:

$$\left[\frac{1+k\lambda_n+\lambda_n^p}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right]^\gamma \le \frac{1}{(1-\beta)^\gamma}, \quad \left[\frac{1}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right]^\gamma \le k^{-\gamma}\lambda_n^{-\gamma}, \tag{2.21}$$

we find that

$$\left|\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)-\exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)\right| \le \frac{C(k,\gamma)}{(1-\beta)^\gamma}t^\gamma\lambda_n^{p\gamma-\gamma}|\beta-\beta'|^\gamma. \tag{2.22}$$

By combining (2.18), (2.19), and (2.22), we obtain that

$$\|\mathbb{A}_1(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} \le \frac{|C(k,\gamma)|^2}{(1-\beta)^{2\gamma+2}}t^{2\gamma}|\beta-\beta'|^{2\gamma}\sum_{n=1}^\infty \lambda_n^{2m-2p+2p\gamma-2\gamma}f_n^2. \tag{2.23}$$

Therefore, we arrive at the following estimate:

$$\|\mathbb{A}_1(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \le \frac{C(k,\gamma)}{(1-\beta)^{\gamma+1}}t^\gamma|\beta-\beta'|^\gamma\|f\|_{\mathbb{H}^{m-p+p\gamma-\gamma}(\mathbb{M})}. \tag{2.24}$$

The second term $\mathbb{A}_2$ can be bounded as follows. By using Parseval's equality, we derive that

$$\begin{aligned}&\|\mathbb{A}_2(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})}\\ &= \sum_{n=1}^\infty \lambda_n^{2m}\left[\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}-\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right]^2\exp\left(\frac{-2\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)|f_n|^2.\end{aligned} \tag{2.25}$$

It is not difficult to see that

$$\left|\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}-\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right| = \frac{|\beta-\beta'|(1+k\lambda_n)\lambda_n^p}{(1+k\lambda_n+(1-\beta)\lambda_n^p)(1+k\lambda_n+(1-\beta')\lambda_n^p)}. \tag{2.26}$$

By using the Cauchy inequality $a+b\ge 2\sqrt{ab}$ for any $a,b\ge 0$, we obtain that

$$1+k\lambda_n+(1-\beta)\lambda_n^p \ge 2\sqrt{1+k\lambda_n}\sqrt{1-\beta}\lambda_n^{p/2} \tag{2.27}$$

and

$$1+k\lambda_n+(1-\beta')\lambda_n^p \ge 2\sqrt{1+k\lambda_n}\sqrt{1-\beta'}\lambda_n^{p/2}. \tag{2.28}$$

By summarizing the aforementioned three results, we immediately have

$$\left|\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p}-\frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right| \le \frac{|\beta-\beta'|}{4\sqrt{1-\beta}\sqrt{1-\beta'}}. \tag{2.29}$$

By using the inequality $e^{-z}\le C_\mu z^{-\mu}$ and in view of $(a+b)^\mu \le C_\mu(a^\mu+b^\mu)$ for any $\mu>0$, we find that

$$\begin{aligned}\exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right) &\le C_\mu\left(\frac{\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)^{-\mu}\\ &\le C_{1,\mu}t^{-\mu}(1-\beta'+\lambda_1^{-p}+k\lambda_n^{1-p})^\mu\\ &\le t^{-\mu}(C_2+C_3\lambda_n^{\mu(1-p)}),\end{aligned} \tag{2.30}$$

where C_2 depends on μ,β',λ_1, and p and C_3 depends on μ,k. By combining (2.25), (2.29), and (2.30), we derive that for any $\mu>0$

$$\|\mathbb{A}_2(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} \le \frac{|\beta-\beta'|^2}{8(1-\beta)(1-\beta')}\sum_{n=1}^\infty \lambda_n^{2m}(C_2^2t^{-2\mu}+C_3^2t^{-2\mu}\lambda_n^{2\mu-2\mu p})|f_n|^2. \tag{2.31}$$

This estimate and Parseval's equality implies immediately that

$$\|\mathbb{A}_2(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} \le C_4^2|\beta-\beta'|^2 t^{-2\mu}(\|f\|^2_{\mathbb{H}^m(\mathbb{M})} + \|f\|^2_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}), \tag{2.32}$$

where $C_4 = \max(C_2, C_3)$ and depends on $\mu, \beta', \lambda_1, p$, and k. By Sobolev embedding $\mathbb{H}^{m+\mu-\mu p}(\mathbb{M}) \hookrightarrow \mathbb{H}^m(\mathbb{M})$, it follows from (2.32) that

$$\|\mathbb{A}_2(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \le C_5|\beta-\beta'|t^{-\mu}\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}, \tag{2.33}$$

where C_5 depends on $\mu, \beta', \lambda_1, p, k$, and m. By combining (2.17), (2.24), and (2.33), we derive that

$$\begin{aligned}\|\mathbb{Q}_{\beta,p}(t)f - \mathbb{Q}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} &\le \|\mathbb{A}_1(t,.)\|_{\mathbb{H}^m(\mathbb{M})} + \|\mathbb{A}_2(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \\ &\le C_6(t^\gamma|\beta-\beta'|^\gamma\|f\|_{\mathbb{H}^{m-p+p\gamma-\gamma}(\mathbb{M})} + |\beta-\beta'|t^{-\mu}\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}).\end{aligned} \tag{2.34}$$

□

Lemma 2.5. *Let any $\mu > 0$. Then we obtain the following estimate*:

$$\|\mathbb{J}_{\beta,p}(t)f - \mathbb{J}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} \lesssim t^{-\mu}|\beta-\beta'|\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}, \tag{2.35}$$

where the hidden constant depends on $k, \mu, p, \lambda_1, m, \beta$, and β'.

Remark 2.3. Since the proof of Lemma 2.5, it is easy to obtain that the following estimate:

$$\|\mathbb{J}_{\beta,p}(t)f - \mathbb{J}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} \lesssim |\beta-\beta'|\|f\|_{\mathbb{H}^m(\mathbb{M})}, \tag{2.36}$$

for any $f \in \mathbb{H}^m(\mathbb{M})$.

Proof. In view of the two following equalities $\mathbb{J}_{\beta,p}(t)$ and $\mathbb{J}_{\beta',p}(t)$, we have

$$\mathbb{J}_{\beta,p}(t)f(x) := \sum_{n=1}^{\infty}\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)f_n\psi_n(x) \tag{2.37}$$

and

$$\mathbb{J}_{\beta',p}(t)f(x) := \sum_{n=1}^{\infty}\frac{\beta'(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)f_n\psi_n(x). \tag{2.38}$$

Two aforementioned equalities provide us that

$$\begin{aligned}&\mathbb{J}_{\beta,p}(t)f(x) - \mathbb{J}_{\beta',p}(t)f(x)\\ &= \sum_{n=1}^{\infty}\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\left[\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)\right]f_n\psi_n(x)\\ &\quad+ \sum_{n=1}^{\infty}\left[\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta'(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\right]\exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)f_n\psi_n(x)\\ &= \mathbb{B}_1(t,x) + \mathbb{B}_2(t,x).\end{aligned} \tag{2.39}$$

In view of (2.22), we have the following bound for the term $\mathbb{B}_1$

$$\begin{aligned}\|\mathbb{B}_1(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} &= \sum_{n=1}^{\infty}\lambda_n^{2m}\left(\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\right)^2\\ &\quad\times\left[\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)\right]^2|f_n|^2\\ &\le \left|\frac{C(k,\gamma)}{(1-\beta)^\gamma}\right|^2 t^{2\gamma}|\beta-\beta'|^{2\gamma}\sum_{n=1}^{\infty}\lambda_n^{2m+2p\gamma-2\gamma}\left(\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\right)^2.\end{aligned} \tag{2.40}$$

By using Cauchy inequality, we find that

$$(1+k\lambda_n+(1-\beta)\lambda_n^p)^2 \ge 2(1-\beta)(1+k\lambda_n)\lambda_n^p. \tag{2.41}$$

Thus, it also implies that

$$\left(\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\right)^2 \le \frac{\beta^2}{4(1-\beta)^2}\lambda_n^{-2p}. \tag{2.42}$$

By combining (2.40) and (2.42), we derive that

$$\|\mathbb{B}_1(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} \le C_7^2 t^{2\gamma}|\beta-\beta'|^{2\gamma}\sum_{n=1}^{\infty}\lambda_n^{2m-2p+2p\gamma-2\gamma}|f_n|^2, \tag{2.43}$$

where C_7 depends on k, γ, and β. Hence, we derive that the following estimate:

$$\|\mathbb{B}_1(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \le C_7|\beta-\beta'|^\gamma t^\gamma \|f\|_{\mathbb{H}^{m-p+p\gamma-\gamma}(\mathbb{M})}. \tag{2.44}$$

Let us move to the consideration for the component $\mathbb{B}_2$. By using Parseval's equality, we also find that

$$\|\mathbb{B}_2(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} = \sum_{n=1}^{\infty}\lambda_n^{2m}\left[\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta'(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\right]^2 \exp\left(\frac{-2\beta'\lambda_n^p t}{1+k\lambda_n+(1-\beta')\lambda_n^p}\right)|f_n|^2. \tag{2.45}$$

It is obvious to see that

$$\begin{aligned}&\left|\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta'(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\right|\\ &\le \left|\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\right| + \left|\frac{|\beta-\beta'|(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\right|\\ &= \mathcal{D}_1+\mathcal{D}_2.\end{aligned} \tag{2.46}$$

Let us assume that $\beta \ge \beta'$. The term $\mathcal{D}_1$ is equal to

$$\mathcal{D}_1 = \frac{\beta|\beta-\beta'|\lambda_n^p(1+k\lambda_n)(2+2k\lambda_n+(2-\beta-\beta')\lambda_n^p)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}. \tag{2.47}$$

It is easy to verify that

$$\frac{2+2k\lambda_n+(2-\beta-\beta')\lambda_n^p}{1+k\lambda_n+(1-\beta')\lambda_n^p} \le \frac{2-\beta-\beta'}{1-\beta'}. \tag{2.48}$$

By using Cauchy inequality, we find that

$$(1+k\lambda_n+(1-\beta)\lambda_n^p)^2 \ge 2(1-\beta)(1+k\lambda_n)\lambda_n^p. \tag{2.49}$$

From three aforementioned observations, we obtain

$$\mathcal{D}_1 \le \frac{\beta|\beta-\beta'|(2-\beta-\beta')}{2(1-\beta)(1-\beta')(1+k\lambda_n+(1-\beta')\lambda_n^p)}. \tag{2.50}$$

It is easy to obtain the following bound for $\mathbb{D}_2$

$$\mathcal{D}_2 \le \frac{|\beta-\beta'|(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2} \le \frac{|\beta-\beta'|}{1+k\lambda_n+(1-\beta')\lambda_n^p}. \tag{2.51}$$

By combining (2.46), (2.50), and (2.51), we derive that

$$\begin{aligned}&\left|\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta'(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta')\lambda_n^p)^2}\right|\\ &\le \frac{|\beta-\beta'|}{1+k\lambda_n+(1-\beta')\lambda_n^p}\left(1+\frac{\beta(2-\beta-\beta')}{2(1-\beta)(1-\beta')}\right) \le C_8|\beta-\beta'|.\end{aligned} \tag{2.52}$$

Here, C_8 depends on β and β'. In view of (2.30) and (2.45) and by a similar explanation as in (2.33), we can claim that

$$\|\mathbb{B}_2(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \le C_9|\beta-\beta'|t^{-\mu}\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}, \tag{2.53}$$

where C_9 depends on $\mu, \beta', \lambda_1, p, k$ and m. By combining (2.44) and (2.53), we deduce that

$$\|\mathbb{J}_{\beta,p}(t)f - \mathbb{J}_{\beta',p}(t)f\|_{\mathbb{H}^m(\mathbb{M})} \le \|\mathbb{B}_1(t,.)\|_{\mathbb{H}^m(\mathbb{M})} + \|\mathbb{B}_2(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \le C_7 t^{\gamma}|\beta - \beta'|^{\gamma}\|f\|_{\mathbb{H}^{m-p+p\gamma-\gamma}(\mathbb{M})} + C_9|\beta - \beta'|t^{-\mu}\|f\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}.$$

By a similar claim as shown in Remark 2.1, we complete the proof of Lemma 2.5. □

3 Linear inhomogeneous case

In this section, we will investigate the continuity of solutions of linear problem with respect to β – the fractional order.

Theorem 3.1. *Let the initial datum $z_0 \in \mathbb{H}^m(\mathbb{M})$ and $G \in L^\infty(0, T; \mathbb{H}^{m+\mu-\mu p}(\mathbb{M}))$ for any $0 < \mu < 1$. Then we have*

$$\|Z_\beta(t,.) - Z_{\beta'}(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \lesssim |\beta - \beta'|\|z_0\|_{\mathbb{H}^m(\mathbb{M})} + \frac{T^{1-\mu}}{1-\mu}|\beta - \beta'|\|G\|_{L^\infty(0,T;\,\mathbb{H}^{m+\mu-\mu p}(\mathbb{M}))}. \tag{3.54}$$

Proof. From (2.6), we obtain the formula of two following mild solutions as follows:

$$Z_\beta(t,x) = \mathbf{Q}_{\beta,p}(t)z_0 + \int_0^t \mathbf{J}_{\beta,p}(t-\nu)G(\nu)\mathrm{d}\nu \tag{3.55}$$

and

$$Z_{\beta'}(t,x) = \mathbf{Q}_{\beta',p}(t)z_0 + \int_0^t \mathbf{J}_{\beta',p}(t-\nu)G(\nu)\mathrm{d}\nu. \tag{3.56}$$

By subtracting (3.55) from (3.56) on each side, we obtain

$$Z_\beta(t,x) - Z_{\beta'}(t,x) = (\mathbf{Q}_{\beta,p}(t) - \mathbf{Q}_{\beta',p}(t))z_0 + \int_0^t (\mathbf{J}_{\beta,p}(t-\nu) - \mathbf{J}_{\beta',p}(t-\nu))G(\nu)\mathrm{d}\nu. \tag{3.57}$$

This implies the following bound:

$$\begin{aligned}\|Z_\beta(t,.) - Z_{\beta'}(t,.)\|_{\mathbb{H}^m(\mathbb{M})} &\le \|(\mathbf{Q}_{\beta,p}(t) - \mathbf{Q}_{\beta',p}(t))z_0\|_{\mathbb{H}^m(\mathbb{M})} \\ &\quad + \left\| \int_0^t (\mathbf{J}_{\beta,p}(t-\nu) - \mathbf{J}_{\beta',p}(t-\nu))G(\nu)\mathrm{d}\nu \right\|_{\mathbb{H}^m(\mathbb{M})}.\end{aligned} \tag{3.58}$$

Case 1. $z_0 \in \mathbb{H}^m(\mathbb{M})$. In this case, we use Remark 2.2 to obtain the following equation:

$$\|(\mathbf{Q}_{\beta,p}(t) - \mathbf{Q}_{\beta',p}(t))z_0\|_{\mathbb{H}^m(\mathbb{M})} \lesssim |\beta - \beta'|\|z_0\|_{\mathbb{H}^m(\mathbb{M})}. \tag{3.59}$$

In view of Lemma 2.5, we bound the second term on the right-hand side of (3.58) as follows:

$$\begin{aligned}\left\| \int_0^t (\mathbf{J}_{\beta,p}(t-\nu) - \mathbf{J}_{\beta',p}(t-\nu))G(\nu)\mathrm{d}\nu \right\|_{\mathbb{H}^m(\mathbb{M})} &\lesssim |\beta - \beta'| \int_0^t (t-\nu)^{-\mu}\|G(\nu)\|_{\mathbb{H}^{m+\mu-\mu p}(\mathbb{M})}\mathrm{d}\nu \\ &\lesssim |\beta - \beta'|\|G\|_{L^\infty(0,T;\,\mathbb{H}^{m+\mu-\mu p}(\mathbb{M}))}\left(\int_0^t (t-\nu)^{-\mu}\mathrm{d}\nu\right).\end{aligned} \tag{3.60}$$

Since $\mu < 1$, we know that the proper integral $\int_0^t (t-\nu)^{-\mu} d\nu$ is convergent. Therefore, we obtain

$$\left\| \int_0^t (\mathbf{J}_{\beta,p}(t-\nu) - \mathbf{J}_{\beta',p}(t-\nu)) G(\nu) d\nu \right\|_{\mathbb{H}^m(\mathbb{M})} \le \frac{t^{1-\mu}}{1-\mu} |\beta - \beta'| \| G \|_{L^\infty(0,T;\, \mathbb{H}^{m+\mu-\mu p}(\mathbb{M}))}. \tag{3.61}$$

By combining (3.58), (3.59), and (3.61), we complete the proof of Theorem 3.1. □

4 Globally nonlinearity Lipschitz case

In this section, we established some results about the global existence and continuity of solutions for Problem (1.1). The main results are given in Theorems 4.1 and 4.3.

4.1 Global existence

Let $G : \mathbb{H}^q(\mathbb{M}) \to \mathbb{H}^s(\mathbb{M})$ such that $G(\mathbf{0}) = \mathbf{0}$ and

$$\|G(\theta_1) - G(\theta_2)\|_{\mathbb{H}^s(\mathbb{M})} \le K \|\theta_1 - \theta_2\|_{\mathbb{H}^q(\mathbb{M})}, \tag{4.62}$$

for any $\theta_1, \theta_2 \in \mathbb{H}^q(\mathbb{M})$ and K is a positive constant. Here, $s \le q \le p + s$.

Theorem 4.1. *Let G be defined as mentioned earlier. Let the initial datum* $z_0 \in \mathbb{H}^{q-p}(\mathbb{M})$. *Then problem* (1.1) *has a unique solution* $Z_\beta \in L^r(0, T; \mathbb{H}^q(\mathbb{M}))$, *where* $1 < r < \frac{1}{a}$, $0 < a < 1$. *Moreover, we obtain*

$$\|Z_\beta(t, .)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2T^a e^{d_0 t}}{1-\beta} t^{-a} \|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})} \tag{4.63}$$

for any $d_0 > 0$ *large enough.*

Proof. The key idea of this proof is referenced by us in the article of the second author with his colleagues [40]. To use the Banach fixed-point theorem, we need to define the following operator $\mathbb{P} : \mathbf{Y}_{a,d}((0, T];\mathbb{H}^q(\mathbb{M})) \to \mathbf{Y}_{a,d}((0, T];\mathbb{H}^q(\mathbb{M}))$ with $d > 0$, by

$$\mathbb{P}\theta(t) := \mathbf{Q}_{\beta,p}(t) z_0 + \int_0^t \mathbf{J}_{\beta,p}(t-\nu) G(\theta(\nu)) d\nu. \tag{4.64}$$

Since the property $G(0) = 0$ and combine with Lemma 2.3, we infer that

$$\|\mathbb{P}(\theta(t) = 0)\|_{\mathbb{H}^q(\mathbb{M})} = \|\mathbf{Q}_{\beta,p}(t) z_0\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{1}{1-\beta} \|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}. \tag{4.65}$$

By multiplying both sides by $t^a e^{-\mathrm{d}t}$, we have

$$t^a e^{-\mathrm{d}t} \|\mathbb{P}(\theta(t) = 0)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{t^a e^{-\mathrm{d}t}}{1-\beta} \|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})} \le \frac{T^a}{1-\beta} \|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}. \tag{4.66}$$

The latter estimate provides us that

$$\mathbb{P}(\theta(t) = 0) \in \mathbf{Y}_{a,d}((0, T];\, \mathbb{H}^q(\mathbb{M})),$$

for any $d > 0$. Given any two elements θ_1 and θ_2 in the space $\mathbf{Y}_{a,d}((0, T];\mathbb{H}^q(\mathbb{M}))$. From equation (4.64), we have the following equality:

$$\mathbb{P}\theta_1(t) - \mathbb{P}\theta_2(t) = \int_0^t \mathbf{J}_{\beta,p}(t-\nu)(G(\theta_1(\nu)) - G(\theta_2(\nu)))\mathrm{d}\nu. \tag{4.67}$$

By applying Lemma 2.3, we derive that

$$\|\mathbb{P}\theta_1(t) - \mathbb{P}\theta_2(t)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2\beta}{1-\beta}\int_0^t \|G(\theta_1(\nu)) - G(\theta_2(\nu))\|_{\mathbb{H}^{q-p}(\mathbb{M})}\mathrm{d}\nu. \tag{4.68}$$

It is obvious to see that Sobolev embedding

$$\mathbb{H}^s(\mathbb{M}) \hookrightarrow \mathbb{H}^{q-p}(\mathbb{M}), \tag{4.69}$$

since we note that $q \le p + s$. This fact combined with the globally Lipschitz property of G yields to

$$\|\mathbb{P}\theta_1(t) - \mathbb{P}\theta_2(t)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2\beta}{1-\beta}\int_0^t \|G(\theta_1(\nu)) - G(\theta_2(\nu))\|_{\mathbb{H}^s(\mathbb{M})}\mathrm{d}\nu \le \frac{2\beta}{1-\beta}\int_0^t \|\theta_1(\nu) - \theta_2(\nu)\|_{\mathbb{H}^q(\mathbb{M})}\mathrm{d}\nu. \tag{4.70}$$

Hence, we derive that

$$t^a e^{-\mathrm{d}t}\|\mathbb{P}\theta_1(t) - \mathbb{P}\theta_2(t)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2\beta}{1-\beta}t^a\int_0^t \nu^{-a}e^{-d(t-\nu)}\nu^a e^{-d\nu}\|\theta_1(\nu) - \theta_2(\nu)\|_{\mathbb{H}^q(\mathbb{M})}\mathrm{d}\nu. \tag{4.71}$$

It is obvious to see that

$$\|\theta_1 - \theta_2\|_{\mathbf{Y}_{a,d}((0,T];\, \mathbb{H}^q(\mathbb{M}))} = \sup_{0\le\nu\le T} \nu^a e^{-d\nu}\|\theta_1(\nu) - \theta_2(\nu)\|_{\mathbb{H}^q(\mathbb{M})}.$$

This implies immediately that

$$t^a e^{-\mathrm{d}t}\|\mathbb{P}\theta_1(t) - \mathbb{P}\theta_2(t)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2\beta}{1-\beta}\left(t^a\int_0^t \nu^{-a}e^{-d(t-\nu)}\mathrm{d}\nu\right)\|\theta_1 - \theta_2\|_{\mathbf{Y}_{a,d}((0,T];\, \mathbb{H}^q(\mathbb{M}))}. \tag{4.72}$$

By change variable $\nu = t\xi$, we derive that $\mathrm{d}\nu = t\mathrm{d}\xi$. Then we have

$$t^a\int_0^t \nu^{-a}e^{-d(t-\nu)}\mathrm{d}\nu = t\int_0^1 \xi^{-a}e^{-\mathrm{d}t(1-\xi)}\mathrm{d}\xi. \tag{4.73}$$

Let us provide the following lemma, which is presented in [35], Lemma 8, p. 9.

Lemma 4.2. *Let $a_1 > -1$, $a_2 > -1$ such that $a_1 + a_2 \ge -1$, $\rho > 0$ and $t \in [0, T]$. For $h > 0$, the following limit holds*

$$\lim_{\rho\to\infty}\left(\sup_{t\in[0,T]} t^h\int_0^1 \nu^{a_1}(1-\nu)^{a_2}e^{-\rho t(1-\nu)}\mathrm{d}\nu\right) = 0.$$

Let us look into the right-hand side of (4.73). Since $a < 1$, we note that

$$-a > -1.$$

According to Lemma 4.2, we obtain the following statement:

$$\lim_{d\to+\infty}\sup_{0\le t\le T}\left(t\int_0^1 \xi^{-a}e^{-\mathrm{d}t(1-\xi)}\mathrm{d}\xi\right) = 0.$$

Hence, there exists a positive $d_0 > 0$ such that

$$\sup_{0\le t\le T}\left(t^a\int_0^t v^{-a}e^{-d_0(t-v)}\mathrm{d}v\right) \le \frac{1}{2}\cdot \tag{4.74}$$

Combining (4.72) and (4.74), we deduce that

$$t^a e^{-d_0 t}\|\mathbb{P}\theta_1(t) - \mathbb{P}\theta_2(t)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{1}{2}\|\theta_1 - \theta_2\|_{\mathbf{Y}_{a,d_0}((0,T];\ \mathbb{H}^q(\mathbb{M}))}. \tag{4.75}$$

Thus, we provide the following confirmation:

$$\|\mathbb{P}\theta_1 - \mathbb{P}\theta_2\|_{\mathbf{Y}_{a,d_0}((0,T];\ \mathbb{H}^q(\mathbb{M}))} \le \frac{1}{2}\|\theta_1 - \theta_2\|_{\mathbf{Y}_{a,d_0}((0,T];\ \mathbb{H}^q(\mathbb{M}))}. \tag{4.76}$$

From here on, we can conclude that $\mathbb{P}$ is the mapping from $\mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))$ to itself. By applying the Banach fixed-point theorem, we deduce that $\mathbb{P}$ has a fixed point $Z_\beta \in \mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))$. It means that Z_β is the mild solution to the Problem (1.1). Let us claim the regularity of the mild solution Z_β. Indeed, from (4.64) and (4.76), we obtain the following bound:

$$\begin{aligned}\|Z_\beta\|_{\mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))} &= \|\mathbb{P}Z_\beta(t,x)\|_{\mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))}\\ &\le \sup_{0\le t\le T} t^a e^{-d_0 t}\|\mathbb{P}(\theta(t)=0)\|_{\mathbb{H}^q(\mathbb{M})} + \|\mathbb{P}Z_\beta(t,x) - \mathbb{P}(\theta=0)\|_{\mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))}\\ &\le \frac{T^a}{1-\beta}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})} + \frac{1}{2}\|Z_\beta\|_{\mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))}.\end{aligned} \tag{4.77}$$

This aforementioned inequality implies that

$$\|Z_\beta\|_{\mathbf{Y}_{a,d_0}((0,T];\mathbb{H}^q(\mathbb{M}))} \le \frac{2T^a}{1-\beta}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}. \tag{4.78}$$

Thus, we can provide that

$$\|Z_\beta(t,.)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2T^a e^{d_0 t}}{1-\beta}t^{-a}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}, \tag{4.79}$$

which shows that the regularity property

$$Z_\beta \in L^r(0,T;\mathbb{H}^q(\mathbb{M})),$$

where $1 < r < \frac{1}{a}$, $0 < a < 1$. This completes the proof of our theorem. □

4.2 Continuous dependence

Theorem 4.3. *Let $z_0 \in \mathbb{H}^q(\mathbb{M})$ and $G : \mathbb{H}^q(\mathbb{M}) \to \mathbb{H}^q(\mathbb{M})$ such that $G(\mathbf{0}) = \mathbf{0}$ and*

$$\|G(\theta_1) - G(\theta_2)\|_{\mathbb{H}^q(\mathbb{M})} \le K\|\theta_1 - \theta_2\|_{\mathbb{H}^q(\mathbb{M})}, \tag{4.80}$$

for any $\theta_1, \theta_2 \in \mathbb{H}^q(\mathbb{M})$ and K is a postive constant. Then we obtain

$$\|Z_\beta(t,.) - Z_{\beta'}(t,.)\|_{\mathbb{H}^q(\mathbb{M})} \le \exp(Kt)|\beta-\beta'|\left(\|z_0\|_{\mathbb{H}^q(\mathbb{M})} + \frac{2Ke^{d_0 T}}{(1-\beta)(1-a)}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}\right). \tag{4.81}$$

Proof. Since Z_β and $Z_{\beta'}$ are two mild solutions to Problem (1.1) corresponding to β and β'. Since (4.64), we have

$$Z_\beta(t,x) := \mathbf{Q}_{\beta,p}(t)z_0 + \int_0^t \mathbf{J}_{\beta,p}(t-\nu)G(Z_\beta(\nu))\mathrm{d}\nu \tag{4.82}$$

and

$$Z_{\beta'}(t,x) := \mathbf{Q}_{\beta',p}(t)z_0 + \int_0^t \mathbf{J}_{\beta',p}(t-\nu)G(Z_{\beta'}(\nu))\mathrm{d}\nu. \tag{4.83}$$

By combining (4.82) and (4.83), we have that

$$\begin{aligned} Z_\beta(t,x) - Z_{\beta'}(t,x) &= (\mathbf{Q}_{\beta,p}(t) - \mathbf{Q}_{\beta',p}(t))z_0 + \int_0^t (\mathbf{J}_{\beta,p}(t-\nu) - \mathbf{J}_{\beta',p}(t-\nu))G(Z_\beta(\nu))\mathrm{d}\nu \\ &\quad + \int_0^t \mathbf{J}_{\beta',p}(t-\nu)(G(Z_\beta(\nu)) - G(Z_{\beta'}(\nu)))\mathrm{d}\nu \\ &= \mathbb{J}_1(t,x) + \mathbb{J}_2(t,x) + \mathbb{J}_3(t,x). \end{aligned} \tag{4.84}$$

By using Remark 2.2, we have immediately that for any $0 \le t \le T$,

$$\|\mathbb{J}_1(t,.)\|_{\mathbb{H}^q(\mathbb{M})} = \|(\mathbf{Q}_{\beta,p}(t) - \mathbf{Q}_{\beta',p}(t))z_0\|_{\mathbb{H}^q(\mathbb{M})} \lesssim |\beta - \beta'|\|z_0\|_{\mathbb{H}^q(\mathbb{M})}. \tag{4.85}$$

Let us now treat the second term $\mathbb{J}_2$. In view of Remark 2.3, one has

$$\|\mathbb{J}_2(t,.)\|_{\mathbb{H}^q(\mathbb{M})} \lesssim |\beta - \beta'| \int_0^t \|G(Z_\beta(\nu))\|_{\mathbb{H}^q(\mathbb{M})}\mathrm{d}\nu. \tag{4.86}$$

By using the globally Lipschitz property of G as shown in (4.80) and combining with (4.63), we arrive at

$$\|G(Z_\beta(\nu))\|_{\mathbb{H}^q(\mathbb{M})} \le K\|Z_\beta(\nu)\|_{\mathbb{H}^q(\mathbb{M})} \le \frac{2KT^a e^{d_0T}}{1-\beta}\nu^{-a}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})} \tag{4.87}$$

for any $0 \le \nu \le T$. By combining (4.86) and (4.87), we infer that

$$\begin{aligned} \|\mathbb{J}_2(t,.)\|_{\mathbb{H}^q(\mathbb{M})} &\lesssim \frac{2KT^a e^{d_0T}}{1-\beta}|\beta - \beta'|\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}\left(\int_0^t \nu^{-a}\mathrm{d}\nu\right) \\ &= \frac{2KT^a e^{d_0T}}{(1-\beta)(1-a)}|\beta - \beta'|t^{1-a}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})} \\ &\le \frac{2Ke^{d_0T}}{(1-\beta)(1-a)}|\beta - \beta'|\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}. \end{aligned} \tag{4.88}$$

Finally, we continue to consider the third term $\mathbb{J}_3$. By using Remark 2.3 and (4.80), we bound it as follows:

$$\begin{aligned} \|\mathbb{J}_3(t,.)\|_{\mathbb{H}^q(\mathbb{M})} &\le \left\| \int_0^t \mathbf{J}_{\beta',p}(t-\nu)(G(Z_\beta(\nu)) - G(Z_{\beta'}(\nu)))\mathrm{d}\nu \right\|_{\mathbb{H}^q(\mathbb{M})} \\ &\le \int_0^t \|G(Z_\beta(\nu)) - G(Z_{\beta'}(\nu))\|_{\mathbb{H}^q(\mathbb{M})}\mathrm{d}\nu \\ &\le K\int_0^t \|Z_\beta(\nu) - Z_{\beta'}(\nu)\|_{\mathbb{H}^q(\mathbb{M})}\mathrm{d}\nu. \end{aligned} \tag{4.89}$$

By combining (4.84), (4.85), (4.86), and (4.89), we derive that

$$\begin{aligned}\|Z_\beta(t,.) - Z_{\beta'}(t,.)\|_{\mathbb{H}^q(\mathbb{M})} &\le \|\mathbb{J}_1(t,.)\|_{\mathbb{H}^q(\mathbb{M})} + \|\mathbb{J}_2(t,.)\|_{\mathbb{H}^q(\mathbb{M})} + \|\mathbb{J}_3(t,.)\|_{\mathbb{H}^q(\mathbb{M})}\\ &\lesssim |\beta-\beta'|\|z_0\|_{\mathbb{H}^q(\mathbb{M})} + \frac{2Ke^{d_0T}}{(1-\beta)(1-a)}|\beta-\beta'|\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})} + K\int_0^t \|Z_\beta(v)\\ &\quad - Z_{\beta'}(v)\|_{\mathbb{H}^q(\mathbb{M})}\mathrm{d}v.\end{aligned} \tag{4.90}$$

By applying Gronwall's inequality, we obtain that

$$\|Z_\beta(t,.) - Z_{\beta'}(t,.)\|_{\mathbb{H}^q(\mathbb{M})} \lesssim \exp(Kt)|\beta-\beta'|\left(\|z_0\|_{\mathbb{H}^q(\mathbb{M})} + \frac{2Ke^{d_0T}}{(1-\beta)(1-a)}\|z_0\|_{\mathbb{H}^{q-p}(\mathbb{M})}\right). \tag{4.91}$$

□

5 The convergence to parabolic diffusion equation

This section consists of two parts. The first part is to prove that the solution of equation (1.1) converges to the solution of the corresponding parabolic equation. The second part examines the continuity of the solution with respect to the order of derivatives.

Theorem 5.1. *Let $z_0 \in \mathbb{H}^q(\mathbb{M})$ and $G : \mathbb{H}^q(\mathbb{M}) \to \mathbb{H}^q(\mathbb{M})$ such that $G(\mathbf{0}) = \mathbf{0}$ and*

$$\|G(\theta_1) - G(\theta_2)\|_{\mathbb{H}^q(\mathbb{M})} \le K\|\theta_1-\theta_2\|_{\mathbb{H}^q(\mathbb{M})}, \tag{5.92}$$

for any $\theta_1, \theta_2 \in \mathbb{H}^q(\mathbb{M})$, and K is a postive constant. Let us assume that $\frac{1}{2} \le p < 1$. Let $Z_{k,\beta}$ be the mild solution to Problem (1.1) *with $k > 0$. Let Z^* be the mild solution to following parabolic diffusion*:

$$\begin{cases} {}_{\mathrm{CF}}D_t^\beta Z + \mathbb{L}^p Z = G(Z), & \text{in } \mathbb{M}\times(0,T],\\ Z(x,0) = z_0(x), & \text{in } \mathbb{M},\\ Z(x,t) = 0, & \text{in } \partial\mathbb{M}.\end{cases} \tag{5.93}$$

Then we obtain the following estimate:

$$\|Z_{k,\beta} - Z^*_\beta\|_{L^\infty(0,T;\mathbb{H}^m(\mathbb{M}))} \lesssim k^{1-\frac{\theta}{2}}\|z_0\|_{\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})} + \left(k^{1-\frac{\theta}{2}} + k\right)\|Z^*_\beta\|_{L^\infty(0,T;\mathbb{H}^m(\mathbb{M}))}. \tag{5.94}$$

Proof. From (2.6), we can rewrite the formula of two mild solutions $Z_{k,\beta}$ and Z^* as follows:

$$Z_{k,\beta}(t,x) = \mathbb{R}_{k,\beta,p}(t)z_0(x) + \int_0^t \mathbb{F}_{k,\beta,p}(t-v)G(Z_{k,\beta}(v,x))\mathrm{d}v \tag{5.95}$$

and

$$Z^*_\beta(t,x) = \overline{\mathbb{R}}_{k,\beta,p}(t)z_0(x) + \int_0^t \overline{\mathbb{F}}_{\beta,p}(t-v)G(Z^*_\beta(v,x))\mathrm{d}v, \tag{5.96}$$

where the aforementioned operators are defined as follows:

$$\mathbb{F}_{k,\beta,p}(t)f(x) := \sum_{n=1}^\infty \frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x), \tag{5.97}$$

$$\overline{\mathbb{F}}_{\beta,p}(t)f(x) := \sum_{n=1}^\infty \frac{\beta}{(1+(1-\beta)\lambda_n^p)^2}\exp\left(\frac{-\beta\lambda_n^p t}{1+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x), \tag{5.98}$$

$$\mathbb{R}_{k,\beta,p}(t)f(x) := \sum_{n=1}^{\infty} \frac{1+k\lambda_n}{1+k\lambda_n+(1-\beta)\lambda_n^p} \exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x), \tag{5.99}$$

$$\overline{\mathbb{R}}_{k,\beta,p}(t)f(x) := \sum_{n=1}^{\infty} \frac{1}{1+(1-\beta)\lambda_n^p} \exp\left(\frac{-\beta\lambda_n^p t}{1+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x). \tag{5.100}$$

To make it clearer, the proof is divided into three steps as follows.

Step 1. Estimate of $\|\mathbb{F}_{k,\beta,p}(t)f(x) - \overline{\mathbb{F}}_{\beta,p}(t)f(x)\|_{\mathbb{H}^m(\mathbb{M})}$.

From the two aforementioned equalities, we find that

$$\begin{aligned}
&\mathbb{F}_{k,\beta,p}(t)f(x) - \overline{\mathbb{F}}_{\beta,p}(t)f(x)\\
&= \sum_{n=1}^{\infty}\left[\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta}{(1+(1-\beta)\lambda_n^p)^2}\right]\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) f_n\psi_n(x)\\
&\quad + \sum_{n=1}^{\infty}\frac{\beta}{(1+(1-\beta)\lambda_n^p)^2}\left[\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta\lambda_n^p t}{1+(1-\beta)\lambda_n^p}\right)\right] f_n\psi_n(x)\\
&= \mathbb{S}_1(t,x) + \mathbb{S}_2(t,x).
\end{aligned} \tag{5.101}$$

To consider the term $\mathbb{S}_1$, we have

$$\begin{aligned}
&\left|\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta}{(1+(1-\beta)\lambda_n^p)^2}\right|\\
&= \left|\frac{\beta(1+k\lambda_n)(1+(1-\beta)\lambda_n^p)^2 - \beta(1+k\lambda_n+(1-\beta)\lambda_n^p)^2}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2(1+(1-\beta)\lambda_n^p)^2}\right|\\
&\le \frac{k\lambda_n((1-\beta)^2\lambda_n^{2p}+1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2(1+(1-\beta)\lambda_n^p)^2}.
\end{aligned} \tag{5.102}$$

Since $p \ge \frac{1}{2}$, we know that

$$k\lambda_n((1-\beta)^2\lambda_n^{2p}+1+k\lambda_n) = k\lambda_n^{1+2p}((1-\beta)^2+\lambda_n^{-2p}+k\lambda_n^{1-2p}) \le k\lambda_n^{1+2p}((1-\beta)^2+\lambda_1^{-2p}+k\lambda_1^{1-2p}). \tag{5.103}$$

By using the inequality $1+y > y^{\frac{\theta}{2}}$ for any $y>0$ and $1<\theta<2$, we obtain that

$$(1-\beta)^2\lambda_n^{2p}+1+k\lambda_n > 1+k\lambda_n > k^{\frac{\theta}{2}}\lambda_n^{\frac{\theta}{2}}. \tag{5.104}$$

Thus, we derive that

$$\begin{aligned}
(1+k\lambda_n+(1-\beta)\lambda_n^p)^2(1+(1-\beta)\lambda_n^p)^2 &\ge k^{\frac{\theta}{2}}\lambda_n^{\frac{\theta}{2}}(1+k\lambda_n+(1-\beta)\lambda_n^p)(1+(1-\beta)\lambda_n^p)^2\\
&\ge k^{\frac{\theta}{2}}\lambda_n^{\frac{\theta}{2}}(1-\beta)\lambda_n^p(1-\beta)^2\lambda_n^{2p}\\
&= k^{\frac{\theta}{2}}(1-\beta)^3\lambda_n^{3p+\frac{\theta}{2}}.
\end{aligned} \tag{5.105}$$

By combining (5.103) and (5.105), we derive that

$$\begin{aligned}
\text{The right-hand side of (5.102)} &\le k^{1-\frac{\theta}{2}}\frac{(1-\beta)^2+\lambda_1^{-2p}+k\lambda_1^{1-2p}}{(1-\beta)^3}\lambda_n^{1-p-\frac{\theta}{2}}\\
&\le k^{1-\frac{\theta}{2}}\frac{(1-\beta)^2+\lambda_1^{-2p}+k\lambda_1^{1-2p}}{(1-\beta)^3}\lambda_1^{1-p-\frac{\theta}{2}},
\end{aligned} \tag{5.106}$$

where in the last line, we note that $1-p \le \frac{1}{2} < \frac{\theta}{2}$. By using Parseval's equality, we infer that

$$\begin{aligned}
\|\mathbb{S}_1(t,.)\|_{\mathbb{H}^m(\mathbb{M})}^2 &= \sum_{n=1}^{\infty}\lambda_n^{2m}\left[\frac{\beta(1+k\lambda_n)}{(1+k\lambda_n+(1-\beta)\lambda_n^p)^2} - \frac{\beta}{(1+(1-\beta)\lambda_n^p)^2}\right]^2\exp\left(\frac{-2\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)|f_n|^2\\
&\lesssim k^{2-\theta}\sum_{n=1}^{\infty}\lambda_n^{2m}|f_n|^2\\
&= k^{2-\theta}\|f\|_{\mathbb{H}^m(\mathbb{M})}^2.
\end{aligned} \tag{5.107}$$

To treat the second term $\mathbb{S}_2$, we use the inequality $|e^{-a} - e^{-b}| \le |a - b|$ to obtain that

$$\begin{aligned}\left|\exp\left(\frac{-\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta\lambda_n^p t}{1 + (1-\beta)\lambda_n^p}\right)\right| &\le \left|\frac{\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p} - \frac{\beta\lambda_n^p t}{1 + (1-\beta)\lambda_n^p}\right| \\ &\le \frac{\beta k t\lambda_n^{1+p}}{(1 + k\lambda_n + (1-\beta)\lambda_n^p)(1 + (1-\beta)\lambda_n^p)} \\ &\le \beta k t\lambda_n^{1-p}.\end{aligned} \tag{5.108}$$

This estimate together with $p > \frac{1}{2}$ yield that

$$\begin{aligned}\|\mathbb{S}_2(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} &\le \sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n^m}{(1 + (1-\beta)\lambda_n^p)^2}\right)^2\left[\exp\left(\frac{-\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta\lambda_n^p t}{1 + (1-\beta)\lambda_n^p}\right)\right]^2 |f_n|^2 \\ &\le \frac{\beta^4 k^2 T^2}{(1-\beta)^4}\sum_{n=1}^{\infty}\lambda_n^{2m}\lambda_n^{2-6p}|f_n|^2 \\ &\le \frac{\beta^4 k^2 T^2\lambda_1^{2-6p}}{(1-\beta)^4}\|f\|^2_{\mathbb{H}^m(\mathbb{M})}.\end{aligned} \tag{5.109}$$

Combining (5.107) and (5.109), we deduce that

$$\|\mathbb{F}_{k,\beta,p}(t)f(x) - \overline{\mathbb{F}}_{\beta,p}(t)f(x)\|_{\mathbb{H}^m(\mathbb{M})} \le \|\mathbb{S}_1(t,.)\|_{\mathbb{H}^m(\mathbb{M})} + \|\mathbb{S}_2(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \lesssim \left(k^{1-\frac{\theta}{2}} + k\right)\|f\|_{\mathbb{H}^m(\mathbb{M})}. \tag{5.110}$$

Step 2. Estimate of $\|\mathbb{R}_{k,\beta,p}(t)f(x) - \overline{\mathbb{R}}_{\beta,p}(t)f(x)\|_{\mathbb{H}^m(\mathbb{M})}$.

From the aforementioned equalities, we find that

$$\begin{aligned}&\mathbb{R}_{k,\beta,p}(t)f(x) - \overline{\mathbb{R}}_{k,\beta,p}(t)f(x) \\ &= \sum_{n=1}^{\infty}\left[\frac{1 + k\lambda_n}{1 + k\lambda_n + (1-\beta)\lambda_n^p} - \frac{1}{1 + (1-\beta)\lambda_n^p}\right]\exp\left(\frac{-\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p}\right)f_n\psi_n(x) \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{1 + (1-\beta)\lambda_n^p}\left[\exp\left(\frac{-\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p}\right) - \exp\left(\frac{-\beta\lambda_n^p t}{1 + (1-\beta)\lambda_n^p}\right)\right]f_n\psi_n(x) \\ &= \mathbb{S}_3(t,x) + \mathbb{S}_4(t,x).\end{aligned} \tag{5.111}$$

It is obvious to see that

$$\mathbb{S}_3(t,x) = \sum_{n=1}^{\infty}\frac{k(1-\beta)\lambda_n^{1+p}}{(1 + k\lambda_n + (1-\beta)\lambda_n^p)(1 + (1-\beta)\lambda_n^p)}\exp\left(\frac{-\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p}\right)f_n\psi_n(x). \tag{5.112}$$

By a similar approach as in Step 1, we can obtain that

$$\frac{k(1-\beta)\lambda_n^{1+p}}{(1 + k\lambda_n + (1-\beta)\lambda_n^p)(1 + (1-\beta)\lambda_n^p)} \lesssim k^{1-\frac{\theta}{2}}\lambda_n^{1-\frac{\theta}{2}}. \tag{5.113}$$

Thus, we derive the following estimate:

$$\begin{aligned}\|\mathbb{S}_3(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} &= \sum_{n=1}^{\infty}\lambda_n^{2m}\left[\frac{k(1-\beta)\lambda_n^{1+p}}{(1 + k\lambda_n + (1-\beta)\lambda_n^p)(1 + (1-\beta)\lambda_n^p)}\right]^2\exp\left(\frac{-2\beta\lambda_n^p t}{1 + k\lambda_n + (1-\beta)\lambda_n^p}\right)|f_n|^2 \\ &\lesssim k^{2-\theta}\sum_{n=1}^{\infty}\lambda_n^{2m+2-\theta}|f_n|^2 \\ &= k^{2-\theta}\|f\|^2_{\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})}.\end{aligned} \tag{5.114}$$

By using (5.108), we obtain

$$\begin{aligned}\|\mathbb{S}_4(t,.)\|^2_{\mathbb{H}^m(\mathbb{M})} &= \sum_{n=1}^{\infty}\lambda_n^{2m}\left(\frac{1}{1+(1-\beta)\lambda_n^p}\right)^2\left[\exp\left(\frac{-\beta\lambda_n^p t}{1+k\lambda_n+(1-\beta)\lambda_n^p}\right)-\exp\left(\frac{-\beta\lambda_n^p t}{1+(1-\beta)\lambda_n^p}\right)\right]^2|f_n|^2\\ &\lesssim k^2\sum_{n=1}^{\infty}\lambda_n^{2m+2-4p}|f_n|^2\\ &= k^2\|f\|^2_{\mathbb{H}^{m+1-2p}(\mathbb{M})}.\end{aligned} \tag{5.115}$$

It is easy to observe that $m+1-2p < m+1-\frac{\theta}{2}$. Thus, from (5.111), (5.114), and (5.115), we derive that

$$\|\mathbb{R}_{k,\beta,p}(t)f(x)-\overline{\mathbb{R}}_{\beta,p}(t)f(x)\|_{\mathbb{H}^m(\mathbb{M})} \le \|\mathbb{S}_3(t,.)\|_{\mathbb{H}^m(\mathbb{M})}+\|\mathbb{S}_4(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \lesssim k^{1-\frac{\theta}{2}}\|f\|_{\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})}. \tag{5.116}$$

Step 3. Estimate of $\|Z_{k,\beta}(t,.)-Z^*_\beta(t,.)\|_{\mathbb{H}^m(\mathbb{M})}$.

From the two formulas (5.95) and (5.96), we can easily obtain the following calculation:

$$\begin{aligned}Z_{k,\beta}(t,x)-Z^*_\beta(t,x) &= (\mathbb{R}_{k,\beta,p}(t)-\overline{\mathbb{R}}_{\beta,p}(t))z_0+\int_0^t\mathbb{F}_{k,\beta,p}(t-\nu)(G(Z_{k,\beta}(\nu,x))-G(Z^*_\beta(\nu,x)))d\nu\\ &\quad+\int_0^t(\mathbb{F}_{k,\beta,p}(t-\nu)-\overline{\mathbb{F}}_{\beta,p}(t-\nu))G(Z^*_\beta(\nu,x))d\nu\\ &= J_{11}(t,x)+J_{12}(t,x)+J_{13}(t,x).\end{aligned} \tag{5.117}$$

In view of (5.116) and the assumption that $z_0\in\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})$, we have the following bound for J_{11}

$$\|J_{11}(t,.)\|_{\mathbb{H}^m(\mathbb{M})} = \|(\mathbb{R}_{k,\beta,p}(t)-\overline{\mathbb{R}}_{\beta,p}(t))z_0\|_{\mathbb{H}^m(\mathbb{M})} \lesssim k^{1-\frac{\theta}{2}}\|z_0\|_{\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})}. \tag{5.118}$$

For the term J_{12}, by using Lemma 2.3 and the globally Lipschitz property of G, we find that

$$\begin{aligned}\|J_{12}(t,.)\|_{\mathbb{H}^m(\mathbb{M})} &= \left\|\int_0^t\mathbb{F}_{k,\beta,p}(t-\nu)(G(Z_{k,\beta}(\nu,x))-G(Z^*_\beta(\nu,x)))d\nu\right\|_{\mathbb{H}^m(\mathbb{M})}\\ &\le \frac{2\beta}{1-\beta}\int_0^t\|G(Z_{k,\beta}(\nu,x))-G(Z^*_\beta(\nu,x))\|_{\mathbb{H}^m(\mathbb{M})}d\nu\\ &\le \frac{2K\beta}{1-\beta}\int_0^t\|Z_{k,\beta}(\nu,x)-Z^*_\beta(\nu,x)\|_{\mathbb{H}^m(\mathbb{M})}d\nu.\end{aligned} \tag{5.119}$$

For the third term J_{13}, by using 5.110, we obtain the following estimate:

$$\begin{aligned}\|J_{13}(t,.)\|_{\mathbb{H}^m(\mathbb{M})} &\lesssim \left(k^{1-\frac{\theta}{2}}+k\right)\int_0^t\|G(Z^*_\beta(\nu,x))\|_{\mathbb{H}^m(\mathbb{M})}d\nu\\ &\lesssim K\left(k^{1-\frac{\theta}{2}}+k\right)\int_0^t\|Z^*_\beta(\nu,x)\|_{\mathbb{H}^m(\mathbb{M})}d\nu\\ &\lesssim KT\left(k^{1-\frac{\theta}{2}}+k\right)\|Z^*_\beta\|_{L^\infty(0,T;\,\mathbb{H}^m(\mathbb{M}))}.\end{aligned} \tag{5.120}$$

By combining the aforementioned four observations, we deduce that

$$\begin{aligned}\|Z_{k,\beta}(t,.)-Z^*_\beta(t,.)\|_{\mathbb{H}^m(\mathbb{M})} &\lesssim k^{1-\frac{\theta}{2}}\|z_0\|_{\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})}+KT\left(k^{1-\frac{\theta}{2}}+k\right)\|Z^*_\beta\|_{L^\infty(0,T;\mathbb{H}^m(\mathbb{M}))}\\ &\quad+\frac{2K\beta}{1-\beta}\int_0^t\|Z_{k,\beta}(\nu,x)-Z^*_\beta(\nu,x)\|_{\mathbb{H}^m(\mathbb{M})}d\nu.\end{aligned} \tag{5.121}$$

By applying Gronwall's inequality, we can conclude that

$$\|Z_{k,\beta}(t,.) - Z^*_\beta(t,.)\|_{\mathbb{H}^m(\mathbb{M})} \lesssim k^{1-\frac{\theta}{2}}\|z_0\|_{\mathbb{H}^{m+1-\frac{\theta}{2}}(\mathbb{M})} + \left(k^{1-\frac{\theta}{2}} + k\right)\|Z^*_\beta\|_{L^\infty(0,T;\,\mathbb{H}^m(\mathbb{M}))}, \tag{5.122}$$

which allows us to complete (5.94). ☐

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