

Research Article

Yongge Tian*

Characterizations of the group invertibility of a matrix revisited

<https://doi.org/10.1515/dema-2022-0171>

received April 6, 2022; accepted October 12, 2022

Abstract: A square complex matrix A is said to be group invertible if there exists a matrix X such that $AXA = A$, $XAX = X$, and $AX = XA$ hold, and such a matrix X is called the group inverse of A . The group invertibility of a matrix is one of the fundamental concepts in the theory of generalized inverses, while group inverses of matrices have many essential applications in matrix theory and other disciplines. The purpose of this article is to reconsider the characterization problem of the group invertibility of a matrix, as well as the constructions of various algebraic equalities in relation to group invertible matrices. The coverage includes collecting and establishing a family of existing and new necessary and sufficient conditions for a matrix to be group invertible and giving many algebraic matrix equalities that involve Moore-Penrose inverses and group inverses of matrices through the skillful use of a series of highly selective formulas and facts about ranks, ranges, and generalized inverses of matrices, as well as block matrix operations.

Keywords: block matrix, group inverse, Moore-Penrose inverse, range, rank

MSC 2020: 15A03, 15A09, 15A24

1 Introduction

Throughout this article, let $\mathbb{C}^{m \times n}$ denote the collection of all $m \times n$ matrices over the field of complex numbers; $\bar{A}$, A^T , and A^* denote the conjugate, transpose, and conjugate transpose of $A \in \mathbb{C}^{m \times n}$, respectively; $r(A)$ and $\mathcal{R}(A)$ stand for the rank and the range (column space) of $A \in \mathbb{C}^{m \times n}$, respectively. The Moore-Penrose inverse of $A \in \mathbb{C}^{m \times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{C}^{n \times m}$ that satisfies the following four Penrose equations:

$$(1) \quad AXA = A, \quad (2) \quad XAX = X, \quad (3) \quad (AX)^* = AX, \quad (4) \quad (XA)^* = XA. \quad (1.1)$$

A square matrix $A \in \mathbb{C}^{m \times m}$ is said to be group invertible if and only if there exists an $X \in \mathbb{C}^{m \times m}$ that satisfies the following three matrix equations:

$$(1) \quad AXA = A, \quad (2) \quad XAX = X, \quad (5) \quad AX = XA. \quad (1.2)$$

In such a case, the matrix X , called the group inverse of A , is unique and is denoted by $X = A^\#$.

It has been recognized that Moore-Penrose inverses and group inverses of matrices are two typical kinds of generalized inverses, which were defined and approached in matrix algebra and applications in the 1950s and thereby belong to the core and influential part of the discipline of generalized inverses (cf. [1–3]). Although the Moore-Penrose inverses and group inverses of matrices have been sufficiently approached for several decades, there still exist many fundamental and challenging problems pertaining to their

* **Corresponding author: Yongge Tian**, College of Business and Economics, Shanghai Business School, Shanghai 201400, China, e-mail: yongge.tian@gmail.com

theoretical and computational properties and performances. Here, we mention two fundamental facts that any complex matrix has a unique Moore-Penrose inverse, but a square complex matrix does not necessarily have a group inverse. In the latter situation, a primary step is to determine the conditions under which (1.2) has a common solution. In fact, there were a number of results that appeared in the literature on the descriptions and parallel definitions of the group invertibility of a square complex matrix, as well as elements in other algebraic systems (cf. [1,3–14]).

As we know, matrices can be classified into different classes entitled with the corresponding names according to the constructions of entries in the matrices, while algebraists often like to explore various classified matrices and their performances using some unified algebraic tools and techniques. Apparently, the matrices that satisfy the group invertible conditions in (1.2) are a special class of square matrices with strong backgrounds in the discipline of generalized inverses. It can be seen that a square matrix is not necessarily group invertible, for example, a nilpotent matrix ($A^2 = 0$), and thus, it is a primary work to describe the group invertibility of a matrix via various reasonable algebraic equalities and facts in order to use group inverses of matrices in different situations. It has been recognized that there are many simple or complicated algebraic equalities and facts that can be used to determine whether a matrix is group invertible.

Apart from the above definition, algebraists knew that the group invertibility of a matrix can be described by some other matrix equalities and facts, and as usual, these equalities and facts can symbolically be represented in the following equivalent mathematical statements:

$$f(A, A^*, A^\dagger) = 0 \Leftrightarrow A \text{ is group invertible,} \quad (1.3)$$

namely, group invertible matrices are a unique class of solutions to the matrix equation on the left-hand side of (1.3). Due to the noncommutativity of matrix algebra, it is possible to construct numerous reasonable algebraic matrix equalities via ordinary common operations of matrices, and therefore, how to propose and delineate the equivalent facts in (1.3) can be regarded as a fundamental problem in the subject area of the group invertibility of a matrix. This is without doubt a fruitful research field in matrix analysis; however, it is also challenging and unexpected, as we know that there do not exist general rules and techniques to construct significant and acceptable matrix equalities that are equivalent to a given mathematical fact or assertion (cf. [15,16]).

The purpose of this article is to provide a very comprehensive analysis of the identification problem regarding the group invertibility of a square complex matrix and then to construct and describe a wide range of matrix equalities and facts in relation to the group inverse of a matrix by using some common ordinary matrix analysis tools, including the matrix rank method and the matrix range method. The rest of this article is organized as follows. In Section 2, the author introduces a group of commonly used facts and results concerning ranks, ranges, and generalized inverses of matrices. In Section 3, the author presents a family of existing and new conditions in relation to the group invertibility of a given square matrix using various matrix equalities that are composed of the Moore-Penrose inverses of the given matrix and its algebraic operations. In Section 4, the author derives a series of mixed matrix equalities composed of Moore-Penrose and group inverses of matrices. Section 5 gives some remarks with regard to the links between group invertible matrices and other kinds of special matrices.

2 Some preliminaries

The author begins with presentations and expositions in the matter of commonly used equalities and facts about matrices and their algebraic operations, which can be found in a number of linear algebra and matrix theory reference books (cf. [1–3]) or, easy to prove by the definitions of ranks, ranges, and generalized inverses of matrices.

Lemma 2.1. Let $A \in \mathbb{C}^{m \times m}$. Then, A is group invertible if and only if $r(A^2) = r(A)$. In this case,

$$A^\# = A(A^3)^\dagger A. \quad (2.1)$$

Lemma 2.2. Let $A \in \mathbb{C}^{m \times n}$. Then, the following rank equalities hold:

$$r(A) = r(\bar{A}) = r(A^T) = r(A^*). \quad (2.2)$$

Let $A \in \mathbb{C}^{m \times m}$. Then, the following equalities

$$\bar{A}^k = (\bar{A})^k, \quad (A^k)^T = (A^T)^k, \quad (A^k)^* = (A^*)^k, \quad (2.3)$$

$$r(A^k) = r(\bar{A}^k) = r((\bar{A})^k) = r((A^k)^T) = r((A^T)^k) = r((A^k)^*) = r((A^*)^k), \quad (2.4)$$

$$r((AA^*)^k A) = r(A), \quad r(((AA^*)^k A)^2) = r(A^2) \quad (2.5)$$

hold for any integer $k \geq 2$.

Lemma 2.3. Let $A \in \mathbb{C}^{m \times n}$. Then, the following equalities hold:

$$A^* = A^*AA^\dagger = A^\dagger AA^*, \quad (2.6)$$

$$(A^\dagger)^* = (A^*)^\dagger, \quad (A^\dagger)^\dagger = A, \quad (A^*)^\dagger A^* = AA^\dagger, \quad A^*(A^*)^\dagger = A^\dagger A, \quad (2.7)$$

$$(AA^*)^\dagger = (A^\dagger)^*A^\dagger, \quad (A^*A)^\dagger = A^\dagger(A^\dagger)^*, \quad (AA^*A)^\dagger = A^\dagger(A^\dagger)^*A^\dagger, \quad (2.8)$$

$$A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^* = A^*(A^*AA^*)^\dagger A^*, \quad (2.9)$$

$$\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^*A) = \mathcal{R}(AA^\dagger) = \mathcal{R}((A^\dagger)^*), \quad (2.10)$$

$$\mathcal{R}(A^*) = \mathcal{R}(A^*A) = \mathcal{R}(A^*AA^*) = \mathcal{R}(A^\dagger) = \mathcal{R}(A^\dagger A), \quad (2.11)$$

$$r(A) = r(A^*) = r(A^\dagger) = r((A^\dagger)^*) = r(AA^*) = r(A^*A) = r(AA^*A) = r(AA^\dagger) = r(A^\dagger A). \quad (2.12)$$

Lemma 2.4. Let $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{m \times n}$. Then, the following matrix rank and range inequalities

$$r(A) \geq r(A^2) \geq \dots \geq r(A^k), \quad (2.13)$$

$$\mathcal{R}(A) \supseteq \mathcal{R}(A^2) \supseteq \dots \supseteq \mathcal{R}(A^k) \quad (2.14)$$

hold for any integer $k \geq 2$ and the following matrix rank equalities

$$r(A^2) = r((A^2)^\dagger) = r((A^\dagger)^2) \quad (2.15)$$

hold. In particular, the following facts

$$r(A) = r(A^2) \Leftrightarrow r(A) = r(A^k) \Leftrightarrow r((A^\dagger)^2) = r(A^\dagger) \Leftrightarrow r((A^\dagger)^k) = r(A^\dagger), \quad (2.16)$$

$$r(A) = r(A^2) \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^2) \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^k), \quad (2.17)$$

$$r(A) = r(A^2) \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}((A^2)^*) \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}((A^k)^*) \quad (2.18)$$

hold for any integer $k \geq 2$, and the following fact

$$r(A) = r(A^2) \Rightarrow r(AB) = r(A^k B) \quad (2.19)$$

holds for any integer $k \geq 2$.

Lemma 2.5. Let $A, B \in \mathbb{C}^{m \times m}$. Then, the following fact

$$r(ABA) = r(A^2) = r(A) \Leftrightarrow r((ABA)^k) = r(A^k) = r(A) \quad (2.20)$$

holds for any integer $k \geq 2$.

Lemma 2.6. Let $A \in \mathbb{C}^{m \times n}$, $B, C \in \mathbb{C}^{n \times p}$, and $D \in \mathbb{C}^{m \times q}$. Then, the following results hold.

- (a) $AA^* = 0 \Leftrightarrow A^*A = 0 \Leftrightarrow A = 0$.
- (b) $A^*AB = A^*AC \Leftrightarrow AB = AC$.
- (c) $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$, $\mathcal{R}(A^*) \supseteq \mathcal{R}(C)$, and $AB = AC \Rightarrow B = C$.
- (d) $r(A(A^*A)^k) = r(A)$ holds for all integers $k \geq 1$.
- (e) $A = 0 \Leftrightarrow (AA^*)^k = 0 \Leftrightarrow A(A^*A)^k = 0$ for some/all integers $k \geq 1$.
- (f) $r(AA^*AB) = r(A^*AB) = r(AB)$.
- (g) $r(AA^\dagger D) = r(A^\dagger D) = r(A^*D)$.

Lemma 2.7. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times p}$. Then, the following seven conditions are equivalent:

- $\langle 1 \rangle \mathcal{R}(A) = \mathcal{R}(B)$.
- $\langle 2 \rangle \mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $r(A) = r(B)$.
- $\langle 3 \rangle \mathcal{R}(A) \supseteq \mathcal{R}(B)$ and $r(A) = r(B)$.
- $\langle 4 \rangle AX = B$ and $A = BY$ hold for some X and Y .
- $\langle 5 \rangle AA^\dagger = BB^\dagger$.
- $\langle 6 \rangle AA^\dagger B = B$ and $A = BB^\dagger A$.
- $\langle 7 \rangle r[A, B] = r(A) = r(B)$.

Recall that the rank of a matrix is an initial concept in the ordinary scope of linear algebra and matrix theory, which is not esoteric but likely to be understood or enjoyed by a beginner in mathematics without special knowledge or interest, while there are many fundamental and useful equalities for ranks of matrices and their operations that occur in various literature on linear algebra and matrix theory. By definition, a basic property of the rank of a matrix is that a matrix is null if and only if its rank is zero. As a direct consequence of this elementary but immanent fact about a matrix, it is straightforward to see that two matrices A and B of the same size are equal if and only if $r(A - B) = 0$. In the light of this basic assertion, it is easy to figure out that if certain nontrivial and analytical formulas for calculating the rank of $A - B$ are obtained, they can reasonably be utilized to interpret essential links between the two matrices and to characterize the matrix quality $A = B$ in an explicit and substantial way. According to this simple and ingenious idea, people established a large number of formulas and facts in relation to ranks of matrices in the past several decades. Now, the matrix rank methodology has been depicted as a useful and resultful finite-dimensional analysis tool in the descriptions and characterizations of algebraic matrix expressions and matrix equalities in comparison with other algebraic analysis techniques in matrix mathematics and applications. Below, the author presents some existing analytical formulas for calculating the ranks of matrices, which can be used to deal with various simple and complicated matrix expressions and matrix equalities that involve generalized inverses.

Lemma 2.8. [17] Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$, and $D \in \mathbb{C}^{l \times k}$. Then, the following rank equality

$$r(D - CA^\dagger B) = r \begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A) \quad (2.21)$$

holds. In particular, if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

$$r(D - CA^\dagger B) = r \begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A). \quad (2.22)$$

Lemma 2.9. [18] Let $A, B \in \mathbb{C}^{m \times n}$, and assume that $AXA = A$ and $BXB = B$ hold for an $X \in \mathbb{C}^{n \times m}$. Then, the following rank equality

$$r(A - B) = r \begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B) \quad (2.23)$$

holds. Therefore,

$$A = B \Leftrightarrow r \begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] = r(A) + r(B) \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(B) \quad \text{and} \quad \mathcal{R}(A^*) = \mathcal{R}(B^*). \quad (2.24)$$

Lemma 2.10. [19,20] Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. Then, the following two rank equalities

$$r((AB)^\dagger - B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger) = r((AB)^\dagger - B^*(A^* ABB^*)^\dagger A^*) = r \begin{bmatrix} ABB^*B \\ AB \end{bmatrix} + r[AA^*AB, AB] - 2r(AB) \quad (2.25)$$

hold. Therefore,

$$(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger \Leftrightarrow (AB)^\dagger = B^*(A^* ABB^*)^\dagger A^* \Leftrightarrow r \begin{bmatrix} ABB^*B \\ AB \end{bmatrix} = r[AA^*AB, AB] = r(AB). \quad (2.26)$$

Lemma 2.11. [21,22] Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$, and $C \in \mathbb{C}^{p \times q}$. Then,

$$\begin{aligned} r((ABC)^\dagger - C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger) &= r((ABC)^\dagger - C^*(A^* ABCC^*)^\dagger A^*) \\ &= r \begin{bmatrix} ABCC^* \\ ABC \end{bmatrix} + r[AA^*ABC, ABC] - 2r(ABC). \end{aligned} \quad (2.27)$$

Therefore,

$$(ABC)^\dagger = C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger \Leftrightarrow (ABC)^\dagger = C^*(A^* ABCC^*)^\dagger A^* \Leftrightarrow r \begin{bmatrix} ABCC^*C \\ ABC \end{bmatrix} = r[AA^*ABC, ABC] = r(ABC). \quad (2.28)$$

One remarkable common feature of Lemmas 2.9–2.11 is that the ranks of the differences of certain matrix expressions involving generalized inverses can be calculated by the ranks of the block matrices constructed from the ordinary algebraic operations of the given matrices. The formulas and facts in these lemmas are easily understandable and acceptable within the domain of generalized inverses of matrices, so that they provide an available method to link matrix rank formulas and matrix equalities, while these kind of skillful techniques are usually referred to as the matrix rank method in the constructions and characterizations of matrix equalities that involve generalized inverses.

Lemma 2.12. [17] Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$, and $C \in \mathbb{C}^{p \times q}$, and assume that $r(ABC) = r(B)$. Then,

$$(ABC)^\dagger = (BC)^\dagger B(AB)^\dagger. \quad (2.29)$$

Let $A = B = C$ in the above lemmas, we obtain the following rank formulas and facts about the operations of a square matrix and its generalized inverses.

Corollary 2.13. Let $A \in \mathbb{C}^{m \times m}$. Then, the following rank equalities

$$r((A^2)^\dagger - A^\dagger(A^\dagger A^2 A^\dagger)^\dagger A^\dagger) = r((A^2)^\dagger - A^*(A^* A^2 A^*)^\dagger A^*) = r \begin{bmatrix} A^2 A^* A \\ A^2 \end{bmatrix} + r[AA^*A^2, A^2] - 2r(A^2), \quad (2.30)$$

$$r((A^3)^\dagger - A^\dagger(A^\dagger A^3 A^\dagger)^\dagger A^\dagger) = r((A^3)^\dagger - A^*(A^* A^3 A^*)^\dagger A^*) = r \begin{bmatrix} A^3 A^* A \\ A^3 \end{bmatrix} + r[AA^*A^3, A^3] - 2r(A^3), \quad (2.31)$$

$$r((A^* A^2 A^*)^\dagger - (AA^*)^\dagger(A^\dagger A^2 A^\dagger)^\dagger (A^* A)^\dagger) = r \begin{bmatrix} A^2 (A^* A)^2 \\ A^2 \end{bmatrix} + r[(AA^*)^2 A^2, A^2] - 2r(A^2) \quad (2.32)$$

hold. Therefore,

$$\begin{aligned} (A^2)^\dagger &= A^\dagger(A^\dagger A^2 A^\dagger)^\dagger A^\dagger \Leftrightarrow (A^2)^\dagger = A^*(A^* A^2 A^*)^\dagger A^* \\ &\Leftrightarrow (A(A^\dagger)^2 A)^\dagger = A^\dagger A^2 A^\dagger \\ &\Leftrightarrow (A^* A^2 A^*)^\dagger = (A^\dagger)^*(A^2)^\dagger (A^\dagger)^* \\ &\Leftrightarrow r \begin{bmatrix} A^2 A^* A \\ A^2 \end{bmatrix} = r[AA^*A^2, A^2] = r(A^2) \\ &\Leftrightarrow \mathcal{R}(AA^*A^2) = \mathcal{R}(A^2) \quad \text{and} \quad \mathcal{R}((A^2 A^* A)^*) = \mathcal{R}((A^2)^*), \end{aligned} \quad (2.33)$$

$$\begin{aligned}
(A^3)^\dagger &= A^\dagger(A^\dagger A^3 A^\dagger)^\dagger A^\dagger \Leftrightarrow (A^3)^\dagger = A^*(A^* A^3 A^*)^\dagger A^* \\
&\Leftrightarrow (A(A^3)^\dagger A)^\dagger = A^\dagger A^3 A^\dagger \\
&\Leftrightarrow (A^* A^3 A^*)^\dagger = (A^\dagger)^*(A^3)^\dagger (A^\dagger)^* \\
&\Leftrightarrow r \begin{bmatrix} A^3 A^* A \\ A^3 \end{bmatrix} = r[AA^* A^3, A^3] = r(A^3) \\
&\Leftrightarrow \mathcal{R}(AA^* A^3) = \mathcal{R}(A^3) \quad \text{and} \quad \mathcal{R}((A^3 A^* A)^*) = \mathcal{R}((A^3)^*),
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
(A^* A^2 A^*)^\dagger &= (AA^*)^\dagger (A^\dagger A^2 A^\dagger)^\dagger (A^* A)^\dagger \Leftrightarrow (A^\dagger A^2 A^\dagger)^\dagger = AA^* (A^* A^2 A^*)^\dagger A^* A \\
&\Leftrightarrow r \begin{bmatrix} A^2 (A^* A)^2 \\ A^2 \end{bmatrix} = r[(AA^*)^2 A^2, A^2] = r(A^2) \\
&\Leftrightarrow \mathcal{R}((AA^*)^2 A^2) = \mathcal{R}(A^2) \quad \text{and} \quad \mathcal{R}((A^2 (A^* A)^2)^*) = \mathcal{R}((A^2)^*).
\end{aligned} \tag{2.35}$$

Corollary 2.14. Let $A \in \mathbb{C}^{m \times m}$, and assume that $r(A^2) = r(A)$. Then,

$$(A^2)^\dagger = A^\dagger(A^\dagger A^2 A^\dagger)^\dagger A^\dagger, \tag{2.36}$$

$$(A^2)^\dagger = A^*(A^* A^2 A^*)^\dagger A^*, \tag{2.37}$$

$$(A(A^2)^\dagger A)^\dagger = A^\dagger A^2 A^\dagger, \tag{2.38}$$

$$(A^* A^2 A^*)^\dagger = (A^\dagger)^*(A^2)^\dagger (A^\dagger)^*, \tag{2.39}$$

$$(A^* A^2 A^*)^\dagger = (AA^*)^\dagger (A^\dagger A^2 A^\dagger)^\dagger (A^* A)^\dagger, \tag{2.40}$$

$$(A^\dagger A^2 A^\dagger)^\dagger = AA^* (A^* A^2 A^*)^\dagger A^* A, \tag{2.41}$$

$$(A^3)^\dagger = A^\dagger(A^\dagger A^3 A^\dagger)^\dagger A^\dagger, \tag{2.42}$$

$$(A^3)^\dagger = A^*(A^* A^3 A^*)^\dagger A^*, \tag{2.43}$$

$$(A(A^3)^\dagger A)^\dagger = A^\dagger A^3 A^\dagger, \tag{2.44}$$

$$(A^* A^3 A^*)^\dagger = (A^\dagger)^*(A^3)^\dagger (A^\dagger)^*, \tag{2.45}$$

$$(A^3)^\dagger = (A^2)^\dagger A (A^2)^\dagger = A^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger, \tag{2.46}$$

$$A(A^3)^\dagger A = A(A^2)^\dagger A (A^2)^\dagger A = (A^\dagger A^2 A^\dagger)^\dagger A^\dagger (A^\dagger A^2 A^\dagger)^\dagger. \tag{2.47}$$

Proof. Under the condition $r(A^2) = r(A)$, it follows that

$$\begin{aligned}
r(A^3 A^* A) &= r(AA^* A^3) = r(A^2 A^* A) = r(AA^* A^2) \\
&= r((AA^*)^2 A^2) = r((A^2 (A^* A)^2)^*) \\
&= r(A^2) = r(A)
\end{aligned}$$

hold. In this situation, the right-hand sides of (2.33)–(2.35) all hold. Therefore, that the matrix equalities in (2.36)–(2.47) hold according to the equivalent facts in (2.33)–(2.35). $\square$

Manifestly, all the preceding formulas and facts belong to mathematical competencies and conceptions in the area of generalized inverses of matrices, and therefore, they can technically be utilized to construct and simplify a wider range of matrix expressions and equalities that involve ordinary operations of matrices and their generalized inverses.

3 Main results

In this section, the author collects together what has been known about the group invertibility of a matrix and constructs a new family of algebraic equalities and facts concerning the group invertibility of a matrix.

Theorem 3.1. Let $A \in \mathbb{C}^{m \times m}$. Then, the following 268 conditions are equivalent:

- $\langle 1 \rangle$ A is group invertible.
- $\langle 2 \rangle$ $\bar{A}$ is group invertible.
- $\langle 3 \rangle$ A^T is group invertible.
- $\langle 4 \rangle$ A^* is group invertible.
- $\langle 5 \rangle$ $A^\dagger$ is group invertible.
- $\langle 6 \rangle$ $(AA^*)^k A$ is group invertible for some/all integers $k \geq 1$.
- $\langle 7 \rangle$ $A^2(A^2)^\dagger A = A$.
- $\langle 8 \rangle$ $A(A^2)^\dagger A^2 = A$.
- $\langle 9 \rangle$ $A^2(A^2)^\dagger = AA^\dagger$.
- $\langle 10 \rangle$ $(A^2)^\dagger A^2 = A^\dagger A$.
- $\langle 11 \rangle$ $A(A^2)^\dagger A^2 A^\dagger = AA^\dagger$.
- $\langle 12 \rangle$ $A(A^2)^\dagger A^2 A^* = AA^*$.
- $\langle 13 \rangle$ $A^\dagger A^2(A^2)^\dagger A = A^\dagger A$.
- $\langle 14 \rangle$ $A^* A^2(A^2)^\dagger A = A^* A$.
- $\langle 15 \rangle$ $A^2(A^2)^\dagger AA^* A^2(A^2)^\dagger = AA^*$.
- $\langle 16 \rangle$ $(A^2)^\dagger A^2 A^* A(A^2)^\dagger A^2 = A^* A$.
- $\langle 17 \rangle$ $A^2(A^2)^\dagger AA^* A(A^2)^\dagger A^2 = AA^* A$.
- $\langle 18 \rangle$ $(A^2 A^\dagger)^\dagger A^2 A^\dagger = AA^\dagger$.
- $\langle 19 \rangle$ $A^\dagger A^2(A^\dagger A^2)^\dagger = A^\dagger A$.
- $\langle 20 \rangle$ $A^2(A^\dagger A^2)^\dagger = A$.
- $\langle 21 \rangle$ $(A^2 A^\dagger)^\dagger A^2 = A$.
- $\langle 22 \rangle$ $A(A^2 A^\dagger)^\dagger A = A$.
- $\langle 23 \rangle$ $A(A^\dagger A^2)^\dagger A = A$.
- $\langle 24 \rangle$ $A(A^2 A^\dagger)^\dagger = AA^\dagger$.
- $\langle 25 \rangle$ $(A^\dagger A^2)^\dagger A = A^\dagger A$.
- $\langle 26 \rangle$ $A^\dagger(A^2 A^\dagger)^\dagger A^2 = A^\dagger A$.
- $\langle 27 \rangle$ $A^2(A^\dagger A^2)^\dagger A^\dagger = AA^\dagger$.
- $\langle 28 \rangle$ $A^2(A^2)^\dagger A(A^2)^\dagger A^2 = A$.
- $\langle 29 \rangle$ $A(A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger A = A$.
- $\langle 30 \rangle$ $A(A^\dagger A^2)^\dagger A(A^2 A^\dagger)^\dagger A = A$.
- $\langle 31 \rangle$ $A^2(A^\dagger A^2)^\dagger A^\dagger(A^2 A^\dagger)^\dagger A^2 = A$.
- $\langle 32 \rangle$ $A(A^\dagger A^2 A^\dagger)^\dagger = A$.
- $\langle 33 \rangle$ $(A^\dagger A^2 A^\dagger)^\dagger A = A$.
- $\langle 34 \rangle$ $(A^\dagger A^2 A^\dagger)^\dagger A^\dagger = A^\dagger$.
- $\langle 35 \rangle$ $A^\dagger(A^\dagger A^2 A^\dagger)^\dagger = A^\dagger$.
- $\langle 36 \rangle$ $A(A^\dagger A^2 A^\dagger)^\dagger A^\dagger(A^\dagger A^2 A^\dagger)^\dagger A = A$.
- $\langle 37 \rangle$ $A^\dagger A(A^\dagger A^2 A^\dagger)^\dagger A^\dagger(A^\dagger A^2 A^\dagger)^\dagger AA^\dagger = A^\dagger$.
- $\langle 38 \rangle$ $A(A^\dagger A^2 A^\dagger)^\dagger A^\dagger = AA^\dagger$.
- $\langle 39 \rangle$ $A^\dagger(A^\dagger A^2 A^\dagger)^\dagger A = A^\dagger A$.
- $\langle 40 \rangle$ $(A^2 A^\dagger)^\dagger A^3(A^\dagger A^2)^\dagger = A$.
- $\langle 41 \rangle$ $A^2(A^3)^\dagger A^2 = A$.
- $\langle 42 \rangle$ $A(A^2)^\dagger A^3(A^2)^\dagger A = A$.
- $\langle 43 \rangle$ $A(A^2(A^3)^\dagger A^2)^\dagger A = A$.
- $\langle 44 \rangle$ $A(A^\dagger A^3 A^\dagger)^\dagger A = A$.
- $\langle 45 \rangle$ $A^2(A^\dagger A^3 A^\dagger)^\dagger = A$.
- $\langle 46 \rangle$ $(A^\dagger A^3 A^\dagger)^\dagger A^2 = A$.
- $\langle 47 \rangle$ $A(A^\dagger A^3 A^\dagger)^\dagger A(A^\dagger A^3 A^\dagger)^\dagger A = A$.

- ⟨48⟩ $(A^\dagger A^3 A^\dagger)^\dagger A^3 (A^\dagger A^3 A^\dagger)^\dagger = A.$
 ⟨49⟩ $A^2 (A^3 A^\dagger)^\dagger A = A.$
 ⟨50⟩ $A (A^3 A^\dagger)^\dagger A^2 = A.$
 ⟨51⟩ $A (A^\dagger A^3)^\dagger A^2 = A.$
 ⟨52⟩ $A^2 (A^\dagger A^3)^\dagger A = A.$
 ⟨53⟩ $(A^3 A^\dagger)^\dagger A^3 = A.$
 ⟨54⟩ $A^3 (A^\dagger A^3)^\dagger = A.$
 ⟨55⟩ $A^2 (A^3 A^\dagger)^\dagger = A A^\dagger.$
 ⟨56⟩ $(A^\dagger A^3)^\dagger A^2 = A^\dagger A.$
 ⟨57⟩ $A^\dagger (A^3 A^\dagger)^\dagger A^3 = A^\dagger A.$
 ⟨58⟩ $A^3 (A^\dagger A^3)^\dagger A^\dagger = A A^\dagger.$
 ⟨59⟩ $(A^\dagger)^2 ((A^\dagger)^3 A)^\dagger = A^\dagger A.$
 ⟨60⟩ $(A (A^\dagger)^3)^\dagger (A^\dagger)^2 = A A^\dagger.$
 ⟨61⟩ $A (A^\dagger)^2 ((A^\dagger)^3 A)^\dagger = A.$
 ⟨62⟩ $(A (A^\dagger)^3)^\dagger (A^\dagger)^2 A = A.$
 ⟨63⟩ $A ((A^\dagger)^3 A)^\dagger (A^\dagger)^3 = A A^\dagger.$
 ⟨64⟩ $(A^\dagger)^3 (A (A^\dagger)^3)^\dagger A = A^\dagger A.$
 ⟨65⟩ $A (A^3 A^\dagger)^\dagger A^3 (A^\dagger A^3)^\dagger A = A.$
 ⟨66⟩ $A (A^\dagger A^3)^\dagger A^3 (A^3 A^\dagger)^\dagger A = A.$
 ⟨67⟩ $A^2 (A^3 A^\dagger)^\dagger A (A^\dagger A^3)^\dagger A^2 = A.$
 ⟨68⟩ $A^2 (A^\dagger A^3)^\dagger A (A^3 A^\dagger)^\dagger A^2 = A.$
 ⟨69⟩ $A^3 (A^5)^\dagger A^3 = A.$
 ⟨70⟩ $A (A^3)^\dagger A^5 (A^3)^\dagger A = A.$
 ⟨71⟩ $A ((A^3 (A^5)^\dagger A^3))^\dagger A = A.$
 ⟨72⟩ $A^3 (A^4)^\dagger A^3 (A^4)^\dagger A^3 = A.$
 ⟨73⟩ $A^2 (A^4)^\dagger A^5 (A^4)^\dagger A^2 = A.$
 ⟨74⟩ $A (A^4)^\dagger A^7 (A^4)^\dagger A = A.$
 ⟨75⟩ $A^4 (A^5)^\dagger A^3 (A^5)^\dagger A^4 = A.$
 ⟨76⟩ $A^2 (A^5)^\dagger A^7 (A^5)^\dagger A^2 = A.$
 ⟨77⟩ $A (A^5)^\dagger A^9 (A^5)^\dagger A = A.$
 ⟨78⟩ $A^2 (A^4 (A^5)^\dagger A^4)^\dagger A^2 = A.$
 ⟨79⟩ $A (A^4 (A^7)^\dagger A^4)^\dagger A = A.$
 ⟨80⟩ $A^2 (A^5 (A^7)^\dagger A^5)^\dagger A^2 = A.$
 ⟨81⟩ $A (A^5 (A^9)^\dagger A^5)^\dagger A = A.$
 ⟨82⟩ $A ((A^2)^\dagger A^5 (A^2)^\dagger)^\dagger A = A.$
 ⟨83⟩ $A (A (A^3)^\dagger A^5 (A^3)^\dagger A)^\dagger A = A.$
 ⟨84⟩ $A (A^2)^\dagger A^4 (A^5)^\dagger A^4 (A^2)^\dagger A = A.$
 ⟨85⟩ $A (A^2)^\dagger ((A^4)^\dagger A^5 (A^4)^\dagger)^\dagger (A^2)^\dagger A = A.$
 ⟨86⟩ $A (A^2 (A^4)^\dagger A^5 (A^4)^\dagger A^2)^\dagger A = A.$
 ⟨87⟩ $A (A^2)^\dagger A^3 (A^4)^\dagger A^5 (A^4)^\dagger A^3 (A^2)^\dagger A = A.$
 ⟨88⟩ $A (A^2 (A^3 (A^4 (A^5)^\dagger A^4)^\dagger A^3)^\dagger A^2)^\dagger A = A.$
 ⟨89⟩ $A (A^2)^\dagger A^5 (A^7)^\dagger A^5 (A^2)^\dagger A = A.$
 ⟨90⟩ $A (A^2)^\dagger ((A^5)^\dagger A^7 (A^5)^\dagger)^\dagger (A^2)^\dagger A = A.$
 ⟨91⟩ $A (A^2 (A^5 (A^7)^\dagger A^5)^\dagger A^2)^\dagger A = A.$
 ⟨92⟩ $A (A^3)^\dagger A^6 (A^7)^\dagger A^6 (A^3)^\dagger A = A.$
 ⟨93⟩ $A (A^3)^\dagger ((A^6)^\dagger A^7 (A^6)^\dagger)^\dagger (A^3)^\dagger A = A.$
 ⟨94⟩ $A (A^3 (A^6)^\dagger A^7 (A^6)^\dagger A^3)^\dagger A = A.$
 ⟨95⟩ $A (A^3 (A^6 (A^7)^\dagger A^6)^\dagger A^3)^\dagger A = A.$

- ⟨96⟩ $A(A^2)^\dagger A^3(A^4)^\dagger A^6(A^7)^\dagger A^6(A^4)^\dagger A^3(A^2)^\dagger A = A.$
- ⟨97⟩ $A(A^2)^\dagger A^4(A^5)^\dagger A^6(A^7)^\dagger A^6(A^5)^\dagger A^4(A^2)^\dagger A = A.$
- ⟨98⟩ $A^2(A^3)^\dagger A^4(A^5)^\dagger A^6(A^7)^\dagger A^6(A^5)^\dagger A^4(A^3)^\dagger A^2 = A.$
- ⟨99⟩ $A(A^2)^\dagger A^3(A^4)^\dagger A^5(A^6)^\dagger A^7(A^6)^\dagger A^5(A^4)^\dagger A^3(A^2)^\dagger A = A.$
- ⟨100⟩ $A(A^2(A^3(A^4(A^5(A^6(A^7)^\dagger A^6)^\dagger A^5)^\dagger A^4)^\dagger A^3)^\dagger A^2)^\dagger A = A.$
- ⟨101⟩ $(A^2)^\dagger A^2 A^\dagger = A^\dagger.$
- ⟨102⟩ $A^\dagger A^2 (A^2)^\dagger = A^\dagger.$
- ⟨103⟩ $A^\dagger A^2 (A^2)^\dagger (A^\dagger)^* = (A^* A)^\dagger.$
- ⟨104⟩ $(A^\dagger)^* (A^2)^\dagger A^2 A^\dagger = (A A^*)^\dagger.$
- ⟨105⟩ $A^2 (A^2)^\dagger (A A^*)^\dagger A^2 (A^2)^\dagger = (A A^*)^\dagger.$
- ⟨106⟩ $(A^2)^\dagger A^2 (A^* A)^\dagger (A^2)^\dagger A^2 = (A^* A)^\dagger.$
- ⟨107⟩ $A^2 (A^2)^\dagger (A^* A A^*)^\dagger (A^2)^\dagger A^2 = (A^* A A^*)^\dagger.$
- ⟨108⟩ $((A^\dagger)^2 A)^\dagger (A^\dagger)^2 = A^\dagger.$
- ⟨109⟩ $A^\dagger ((A^\dagger)^2 A)^\dagger A^\dagger = A^\dagger.$
- ⟨110⟩ $(A^\dagger)^2 (A(A^\dagger)^2)^\dagger = A^\dagger.$
- ⟨111⟩ $A^\dagger (A(A^\dagger)^2)^\dagger A^\dagger = A^\dagger.$
- ⟨112⟩ $A(A^\dagger)^2 (A(A^\dagger)^2)^\dagger = A A^\dagger.$
- ⟨113⟩ $((A^\dagger)^2 A)^\dagger (A^\dagger)^2 A = A^\dagger A.$
- ⟨114⟩ $(A(A^\dagger)^2)^\dagger A^\dagger = A A^\dagger.$
- ⟨115⟩ $A^\dagger ((A^\dagger)^2 A)^\dagger = A^\dagger A.$
- ⟨116⟩ $A((A^\dagger)^2 A)^\dagger (A^\dagger)^2 = A A^\dagger.$
- ⟨117⟩ $(A^\dagger)^2 (A(A^\dagger)^2)^\dagger A = A^\dagger A.$
- ⟨118⟩ $A^\dagger ((A^\dagger)^2 A)^\dagger A^\dagger (A(A^\dagger)^2)^\dagger A^\dagger = A^\dagger.$
- ⟨119⟩ $A^\dagger (A(A^\dagger)^2)^\dagger A^\dagger ((A^\dagger)^2 A)^\dagger A^\dagger = A^\dagger.$
- ⟨120⟩ $(A^\dagger)^2 (A(A^\dagger)^2)^\dagger A((A^\dagger)^2 A)^\dagger (A^\dagger)^2 = A^\dagger.$
- ⟨121⟩ $A^\dagger (A(A^\dagger)^2 A)^\dagger = A^\dagger.$
- ⟨122⟩ $(A(A^\dagger)^2 A)^\dagger A^\dagger = A^\dagger.$
- ⟨123⟩ $A(A(A^\dagger)^2 A)^\dagger A^\dagger A = A.$
- ⟨124⟩ $A A^\dagger (A(A^\dagger)^2 A)^\dagger A = A.$
- ⟨125⟩ $A^\dagger (A(A^\dagger)^2 A)^\dagger A = A^\dagger A.$
- ⟨126⟩ $A(A(A^\dagger)^2 A)^\dagger A^\dagger = A A^\dagger.$
- ⟨127⟩ $A^\dagger (A(A^\dagger)^2 A)^\dagger A(A(A^\dagger)^2 A)^\dagger A^\dagger = A^\dagger.$
- ⟨128⟩ $A A^\dagger (A(A^\dagger)^2 A)^\dagger A(A(A^\dagger)^2 A)^\dagger A^\dagger A = A.$
- ⟨129⟩ $A(A(A^\dagger)^2 A)^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger = A A^\dagger.$
- ⟨130⟩ $A^\dagger (A^\dagger A^2 A^\dagger)^\dagger (A(A^\dagger)^2 A)^\dagger A = A^\dagger A.$
- ⟨131⟩ $A(A(A^\dagger)^2 A)^\dagger (A^\dagger A^2 A^\dagger)^\dagger = A.$
- ⟨132⟩ $(A^\dagger A^2 A^\dagger)^\dagger (A(A^\dagger)^2 A)^\dagger A = A.$
- ⟨133⟩ $(A(A^\dagger)^2 A)^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger = A^\dagger.$
- ⟨134⟩ $A^\dagger (A^\dagger A^2 A^\dagger)^\dagger (A(A^\dagger)^2 A)^\dagger = A^\dagger.$
- ⟨135⟩ $A^\dagger (A(A^\dagger)^3 A)^\dagger A^\dagger = A^\dagger.$
- ⟨136⟩ $(A^\dagger)^2 (A(A^\dagger)^3 A)^\dagger = A^\dagger.$
- ⟨137⟩ $(A(A^\dagger)^3 A)^\dagger A^\dagger (A^\dagger)^2 = A^\dagger.$
- ⟨138⟩ $A^\dagger (A(A^\dagger)^3 A)^\dagger A^\dagger (A(A^\dagger)^3 A)^\dagger A^\dagger = A^\dagger.$
- ⟨139⟩ $(A(A^\dagger)^3 A)^\dagger (A^\dagger)^3 (A(A^\dagger)^3 A)^\dagger = A^\dagger.$
- ⟨140⟩ $(A^2)^\dagger A^3 (A^2)^\dagger = A^\dagger.$
- ⟨141⟩ $A^\dagger ((A^2)^\dagger A^3 (A^2)^\dagger)^\dagger A^\dagger = A^\dagger.$
- ⟨142⟩ $A^\dagger A^2 (A^3)^\dagger A^2 A^\dagger = A^\dagger.$
- ⟨143⟩ $A^\dagger A^2 (A^3 A^\dagger)^\dagger = A^\dagger.$

- (144) $(A^\dagger A^3)^\dagger A^2 A^\dagger = A^\dagger$.
 (145) $(A^\dagger)^2 ((A^\dagger)^3 A)^\dagger A^\dagger = A^\dagger$.
 (146) $A^\dagger (A(A^\dagger)^3)^\dagger (A^\dagger)^2 = A^\dagger$.
 (147) $(A^\dagger)^2 ((A^\dagger)^3 A)^\dagger A^\dagger = A^\dagger$.
 (148) $A^\dagger (A(A^\dagger)^3)^\dagger (A^\dagger)^2 = A^\dagger$.
 (149) $(A^2)^\dagger ((A^3)^\dagger A)^\dagger A^\dagger (A(A^3)^\dagger)^\dagger (A^2)^\dagger = A^\dagger$.
 (150) $(A^\dagger)^2 ((A^\dagger)^3 A)^\dagger A^\dagger (A(A^\dagger)^3)^\dagger (A^\dagger)^2 = A^\dagger$.
 (151) $A^\dagger (A(A^3)^\dagger)^\dagger (A^3)^\dagger ((A^3)^\dagger A)^\dagger A^\dagger = A^\dagger$.
 (152) $A^\dagger (A(A^\dagger)^3)^\dagger (A^\dagger)^3 ((A^\dagger)^3 A)^\dagger A^\dagger = A^\dagger$.
 (153) $A^\dagger (A^2 (A^5)^\dagger A^2)^\dagger A^\dagger = A^\dagger$.
 (154) $A^\dagger A^3 (A^5)^\dagger A^3 A^\dagger = A^\dagger$.
 (155) $(A^2)^\dagger A^4 (A^5)^\dagger A^4 (A^2)^\dagger = A^\dagger$.
 (156) $A^\dagger A^4 (A^7)^\dagger A^4 A^\dagger = A^\dagger$.
 (157) $A^\dagger ((A^3)^\dagger A^5 (A^3)^\dagger)^\dagger A^\dagger = A^\dagger$.
 (158) $A^\dagger A^2 (A^4)^\dagger A^5 (A^4)^\dagger A^2 A^\dagger = A^\dagger$.
 (159) $A^\dagger A^2 (A^4 (A^5)^\dagger A^4)^\dagger A^2 A^\dagger = A^\dagger$.
 (160) $A^\dagger ((A^2)^\dagger A^4 (A^5)^\dagger A^4 (A^2)^\dagger)^\dagger A^\dagger = A^\dagger$.
 (161) $A^\dagger A^2 (A^3)^\dagger A^4 (A^5)^\dagger A^4 (A^3)^\dagger A^2 A^\dagger = A^\dagger$.
 (162) $A^\dagger ((A^2)^\dagger ((A^3)^\dagger ((A^4)^\dagger A^5 (A^4)^\dagger)^\dagger (A^3)^\dagger)^\dagger (A^2)^\dagger)^\dagger A^\dagger = A^\dagger$.
 (163) $A^\dagger ((A^4)^\dagger A^7 (A^4)^\dagger)^\dagger A^\dagger = A^\dagger$.
 (164) $A^\dagger A^2 (A^5)^\dagger A^7 (A^5)^\dagger A^2 A^\dagger = A^\dagger$.
 (165) $A^\dagger A^2 (A^5 (A^7)^\dagger A^5)^\dagger A^2 A^\dagger = A^\dagger$.
 (166) $A^\dagger ((A^2)^\dagger A^5 (A^7)^\dagger A^5 (A^2)^\dagger)^\dagger A^\dagger = A^\dagger$.
 (167) $A^\dagger A^3 (A^6)^\dagger A^7 (A^6)^\dagger A^3 A^\dagger = A^\dagger$.
 (168) $A^\dagger A^3 (A^6 (A^7)^\dagger A^6)^\dagger A^3 A^\dagger = A^\dagger$.
 (169) $A^\dagger ((A^3)^\dagger A^6 (A^7)^\dagger A^6 (A^3)^\dagger)^\dagger A^\dagger = A^\dagger$.
 (170) $A^\dagger A^2 (A^3)^\dagger A^4 (A^6)^\dagger A^7 (A^6)^\dagger A^4 (A^3)^\dagger A^2 A^\dagger = A^\dagger$.
 (171) $A^\dagger A^2 (A^4)^\dagger A^5 (A^6)^\dagger A^7 (A^6)^\dagger A^5 (A^4)^\dagger A^2 A^\dagger = A^\dagger$.
 (172) $(A^2)^\dagger A^3 (A^4)^\dagger A^5 (A^6)^\dagger A^7 (A^6)^\dagger A^5 (A^4)^\dagger A^3 (A^2)^\dagger = A^\dagger$.
 (173) $A^\dagger A^2 (A^3)^\dagger A^4 (A^5)^\dagger A^6 (A^7)^\dagger A^6 (A^5)^\dagger A^4 (A^3)^\dagger A^2 A^\dagger = A^\dagger$.
 (174) $A^k (A^k)^\dagger A = A$ for some/all integers $k \geq 3$.
 (175) $A (A^k)^\dagger A^k = A$ for some/all integers $k \geq 3$.
 (176) $A^k (A^k)^\dagger = AA^\dagger$ for some/all integers $k \geq 3$.
 (177) $(A^k)^\dagger A^k = A^\dagger A$ for some/all integers $k \geq 3$.
 (178) $A^k (A^k)^\dagger A (A^s)^\dagger A^s = A$ for some/all integers $k, s \geq 2$.
 (179) $A^k (A^k)^\dagger AA^* A^s (A^s)^\dagger = AA^*$ for some/all integers $k, s \geq 2$.
 (180) $(A^k)^\dagger A^k A^* A (A^s)^\dagger A^s = A^* A$ for some/all integers $k, s \geq 2$.
 (181) $A^k (A^k)^\dagger AA^* A (A^s)^\dagger A^s = AA^* A$ for some/all integers $k, s \geq 2$.
 (182) $A^\dagger A^k (A^k)^\dagger (A^\dagger)^* = (A^* A)^\dagger$ for some/all integers $k \geq 2$.
 (183) $(A^\dagger)^* (A^k)^\dagger A^k A^\dagger = (AA^*)^\dagger$ for some/all integers $k \geq 2$.
 (184) $A^k (A^k)^\dagger (AA^*)^\dagger A^s (A^s)^\dagger = (AA^*)^\dagger$ for some/all integers $k, s \geq 2$.
 (185) $(A^k)^\dagger A^k (A^* A)^\dagger (A^s)^\dagger A^s = (A^* A)^\dagger$ for some/all integers $k, s \geq 2$.
 (186) $A^k (A^k)^\dagger (A^* AA^*)^\dagger (A^s)^\dagger A^s = (A^* AA^*)^\dagger$ for some/all integers $k, s \geq 2$.
 (187) $(A^k A^\dagger)^\dagger A^k = A$ for some/all integers $k \geq 3$.
 (188) $A^k (A^\dagger A^k)^\dagger = A$ for some/all integers $k \geq 3$.
 (189) $((A^\dagger)^k A)^\dagger (A^\dagger)^k = A^\dagger$ for some/all integers $k \geq 3$.
 (190) $(A^\dagger)^k (A (A^\dagger)^k)^\dagger = A^\dagger$ for some/all integers $k \geq 3$.

- ⟨191⟩ $A^\dagger(A^k A^\dagger)^\dagger A^k = A^\dagger A$ for some/all integers $k \geq 3$.
- ⟨192⟩ $A^k(A^\dagger A^k)^\dagger A^\dagger = AA^\dagger$ for some/all integers $k \geq 3$.
- ⟨193⟩ $A((A^\dagger)^k A)^\dagger (A^\dagger)^k = AA^\dagger$ for some/all integers $k \geq 3$.
- ⟨194⟩ $(A^\dagger)^k(A(A^\dagger)^k)^\dagger A = A^\dagger A$ for some/all integers $k \geq 3$.
- ⟨195⟩ $A^\dagger(A^k A^\dagger)^\dagger A^k(A^\dagger)^s(A(A^\dagger)^s)^\dagger A = A^\dagger A$ for some/all integers $k, s \geq 2$.
- ⟨196⟩ $A((A^\dagger)^k A)^\dagger (A^\dagger)^k A^s(A^\dagger A^s)^\dagger A^\dagger = AA^\dagger$ for some/all integers $k, s \geq 2$.
- ⟨197⟩ $A^{k+1}(A^{k+s+1})^\dagger A^{s+1} = A$ for some/all integers $k, s \geq 1$.
- ⟨198⟩ $A^\dagger A^{k+1}(A^{k+s+1})^\dagger A^{s+1} A^\dagger = A^\dagger$ for some/all integers $k, s \geq 1$.
- ⟨199⟩ $(A^{k+1})^\dagger A^{k+s+1}(A^{s+1})^\dagger = A^\dagger$ for some/all integers $k, s \geq 1$.
- ⟨200⟩ $A(A^{k+1})^\dagger A^{k+s+1}(A^{s+1})^\dagger A = A$ for some/all integers $k, s \geq 1$.
- ⟨201⟩ $A((A^2 A^\dagger)^\dagger A)^k = A$ for some/all integers $k \geq 2$.
- ⟨202⟩ $A((A^\dagger A^2)^\dagger A)^k = A$ for some/all integers $k \geq 2$.
- ⟨203⟩ $A^\dagger(((A^\dagger)^2 A)^\dagger A^\dagger)^k = A^\dagger$ for some/all integers $k \geq 2$.
- ⟨204⟩ $A^\dagger((A(A^\dagger)^2)^\dagger A^\dagger)^k = A^\dagger$ for some/all integers $k \geq 2$.
- ⟨205⟩ $(A(A^2 A^\dagger)^\dagger)^k = AA^\dagger$ for some/all integers $k \geq 2$.
- ⟨206⟩ $(A^\dagger((A^\dagger)^2 A)^\dagger)^k = A^\dagger A$ for some/all integers $k \geq 2$.
- ⟨207⟩ $((A^\dagger A^2)^\dagger A)^k = A^\dagger A$ for some/all integers $k \geq 2$.
- ⟨208⟩ $((A(A^\dagger)^2)^\dagger A^\dagger)^k = AA^\dagger$ for some/all integers $k \geq 2$.
- ⟨209⟩ $(A(A^2 A^\dagger)^\dagger)^k ((A(A^\dagger)^2)^\dagger A^\dagger)^k = AA^\dagger$ for some/all integers $k \geq 2$.
- ⟨210⟩ $(A^\dagger((A^\dagger)^2 A)^\dagger)^k ((A^\dagger A^2)^\dagger A)^k = A^\dagger A$ for some/all integers $k \geq 2$.
- ⟨211⟩ $A^k(A^k)^\dagger A + A^{k+1}(A^{k+1})^\dagger A = 2A$ for some/all integers $k \geq 2$.
- ⟨212⟩ $A(A^k)^\dagger A^k + A(A^{k+1})^\dagger A^{k+1} = 2A$ for some/all integers $k \geq 2$.
- ⟨213⟩ $A^k(A^{2k-1})^\dagger A^k + A^{k+1}(A^{2k+1})^\dagger A^{k+1} = 2A$ for some/all integers $k \geq 2$.
- ⟨214⟩ $(A^k)^\dagger A^k A^\dagger + (A^{k+1})^\dagger A^{k+1} A^\dagger = 2A^\dagger$ for some/all integers $k \geq 2$.
- ⟨215⟩ $A^\dagger A^k (A^k)^\dagger + A^\dagger A^{k+1} (A^{k+1})^\dagger = 2A^\dagger$ for some/all integers $k \geq 2$.
- ⟨216⟩ $(A^k)^\dagger A^{2k-1} (A^k)^\dagger + (A^{k+1})^\dagger A^{2k+1} (A^{k+1})^\dagger = 2A^\dagger$ for some/all integers $k \geq 2$.
- ⟨217⟩ $A^k(A^k)^\dagger A + A^{k-1}(A^{k-1})^\dagger A + \dots + A^2(A^2)^\dagger A = (k-1)A$ for some/all integers $k \geq 2$.
- ⟨218⟩ $A(A^k)^\dagger A^k + A(A^{k-1})^\dagger A^{k-1} + \dots + A(A^2)^\dagger A^2 = (k-1)A$ for some/all integers $k \geq 2$.
- ⟨219⟩ $A^k(A^k)^\dagger + A^{k-1}(A^{k-1})^\dagger + \dots + A^2(A^2)^\dagger = (k-1)AA^\dagger$ for some/all integers $k \geq 2$.
- ⟨220⟩ $(A^k)^\dagger A^k + (A^{k-1})^\dagger A^{k-1} + \dots + (A^2)^\dagger A^2 = (k-1)A^\dagger A$ for some/all integers $k \geq 2$.
- ⟨221⟩ $r(A^2) = r(A)$.
- ⟨222⟩ $r((AA^*A)^2) = r(A)$.
- ⟨223⟩ $r(A^2 A^* A^2) = r(A)$.
- ⟨224⟩ $r(A^2(A^*)^2 A^2) = r(A)$.
- ⟨225⟩ $r((A^\dagger)^2) = r(A)$.
- ⟨226⟩ $r((A^\dagger)^2(A^\dagger)^*(A^\dagger)^2) = r(A)$.
- ⟨227⟩ $r((A^\dagger)^2((A^\dagger)^*)^2(A^\dagger)^2) = r(A)$.
- ⟨228⟩ $r(A^* A^\dagger A^*) = r(A)$.
- ⟨229⟩ $r(A^\dagger A^* A^\dagger) = r(A)$.
- ⟨230⟩ $r((A^* A^\dagger A^*)^2) = r(A)$.
- ⟨231⟩ $r((A^\dagger A^* A^\dagger)^2) = r(A)$.
- ⟨232⟩ $r(AA^* A^\dagger A^* A) = r(A)$.
- ⟨233⟩ $r(AA^\dagger A^* A^\dagger A) = r(A)$.
- ⟨234⟩ $r(A(A^* A^\dagger A^*)^\dagger A) = r(A)$.
- ⟨235⟩ $r(A(A^\dagger A^* A^\dagger)^\dagger A) = r(A)$.
- ⟨236⟩ $r(A^k) = r(A)$ for some/all integers $k \geq 3$.
- ⟨237⟩ $r(A^k A^* A^k) = r(A)$ for some/all integers $k \geq 3$.

- ⟨238⟩ $r(A^k(A^*)^kA^k) = r(A)$ for some/all integers $k \geq 3$.
 ⟨239⟩ $r((AA^*A)^k) = r(A)$ for some/all integers $k \geq 3$.
 ⟨240⟩ $r((A^\dagger)^k) = r(A)$ for some/all integers $k \geq 3$.
 ⟨241⟩ $r((A^*A^\dagger)^k) = r(A)$ for some/all integers $k \geq 1$.
 ⟨242⟩ $r((A^\dagger A^*)^k) = r(A)$ for some/all integers $k \geq 1$.
 ⟨243⟩ $r((A^*A^\dagger A^*)^k) = r(A)$ for some/all integers $k \geq 2$.
 ⟨244⟩ $r((A^\dagger A^*A^\dagger)^k) = r(A)$ for some/all integers $k \geq 2$.
 ⟨245⟩ $r((AA^*A^\dagger A^*A)^k) = r(A)$ for some/all integers $k \geq 2$.
 ⟨246⟩ $r((AA^\dagger A^*A^\dagger A)^k) = r(A)$ for some/all integers $k \geq 2$.
 ⟨247⟩ $\mathcal{R}(A^2) = \mathcal{R}(A)$.
 ⟨248⟩ $\mathcal{R}(A^2A^*A^2) = \mathcal{R}(A)$.
 ⟨249⟩ $\mathcal{R}(A^2(A^*)^2A^2) = \mathcal{R}(A)$.
 ⟨250⟩ $\mathcal{R}((A^2)^*) = \mathcal{R}(A^*)$.
 ⟨251⟩ $\mathcal{R}((A^2)^*A(A^2)^*) = \mathcal{R}(A^*)$.
 ⟨252⟩ $\mathcal{R}((A^2)^*A^2(A^2)^*) = \mathcal{R}(A^*)$.
 ⟨253⟩ $\mathcal{R}((A^\dagger)^2) = \mathcal{R}(A^*)$.
 ⟨254⟩ $\mathcal{R}(A^k) = \mathcal{R}(A)$ for all integers $k \geq 3$.
 ⟨255⟩ $\mathcal{R}(A^kA^*A^k) = \mathcal{R}(A)$ for some/all integers $k \geq 3$.
 ⟨256⟩ $\mathcal{R}(A^k(A^*)^kA^k) = \mathcal{R}(A)$ for some/all integers $k \geq 3$.
 ⟨257⟩ $\mathcal{R}((A^k)^*) = \mathcal{R}(A^*)$ for some/all integers $k \geq 3$.
 ⟨258⟩ $\mathcal{R}((A^k)^*A(A^k)^*) = \mathcal{R}(A^*)$ for some/all integers $k \geq 3$.
 ⟨259⟩ $\mathcal{R}((A^k)^*A^k(A^k)^*) = \mathcal{R}(A^*)$ for some/all integers $k \geq 3$.
 ⟨260⟩ $\mathcal{R}((AA^*A)^k) = \mathcal{R}(A)$ for some/all integers $k \geq 3$.
 ⟨261⟩ $\mathcal{R}((A^*AA^*)^k) = \mathcal{R}(A^*)$ for some/all integers $k \geq 3$.
 ⟨262⟩ $\mathcal{R}((A^\dagger)^k) = \mathcal{R}(A^*)$ for some/all integers $k \geq 3$.
 ⟨263⟩ $\mathcal{R}((A^*A^\dagger)^k) = \mathcal{R}(A^*)$ for some/all integers $k \geq 1$.
 ⟨264⟩ $\mathcal{R}((A^\dagger A^*)^k) = \mathcal{R}(A^*)$ for some/all integers $k \geq 1$.
 ⟨265⟩ $\mathcal{R}((A^*A^\dagger A^*)^k) = \mathcal{R}(A^*)$ for some/all integers $k \geq 1$.
 ⟨266⟩ $\mathcal{R}((A^\dagger A^*A^\dagger)^k) = \mathcal{R}(A^*)$ for some/all integers $k \geq 1$.
 ⟨267⟩ $\mathcal{R}((AA^*A^\dagger A^*A)^k) = \mathcal{R}(A)$ for some/all integers $k \geq 1$.
 ⟨268⟩ $\mathcal{R}((AA^\dagger A^*A^\dagger A)^k) = \mathcal{R}(A)$ for some/all integers $k \geq 1$.

Proof. The author intends to present in detail the proofs of main equivalent facts through the skillful and convenient use of the matrix rank method and also gives a variety of discussions and explanations about similar or parallel equalities and facts.

The equivalences of Conditions ⟨1⟩–⟨6⟩ and Condition ⟨221⟩ follow from Lemmas 2.1, (2.2)–(2.5), (2.12), and (2.15).

The equivalences of Conditions ⟨221⟩, ⟨236⟩, ⟨247⟩, ⟨250⟩, ⟨254⟩, and ⟨257⟩, follow from (2.16)–(2.18).

By Lemma 2.6(f) and Condition ⟨221⟩, the rank equalities

$$r((AA^*A)^2) = r(AA^*A^2A^*A) = r(A^2) = r(A)$$

hold, as required for Condition ⟨222⟩. Conversely, Condition ⟨222⟩ implies

$$r(A) = r((AA^*A)^2) = r(AA^*A^2A^*A) \leq r(A^2).$$

Combining this matrix rank inequality with the well-known matrix rank inequality $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition ⟨221⟩.

By (2.19), Condition ⟨221⟩, and Lemma 2.6(d), we obtain

$$r(A^2A^*A^2) = r(AA^*A) = r(A),$$

as required for Condition (223). Conversely, Condition (223) implies

$$r(A) = r(A^2 A^* A^2) \leq r(A^2).$$

Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

By Condition (221) and Lemma 2.6(d), we obtain

$$r(A^2(A^*)^2 A^2) = r(A^2) = r(A),$$

as required for Condition (224). Conversely, Condition (224) implies

$$r(A) = r(A^2(A^*)^2 A^2) \leq r(A^2).$$

Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

Replacing A with $A^\dagger$ in Conditions (221), (223), and (224) and noting (2.12) lead to the equivalences of Conditions (225), (226), and (227).

By (2.18), (2.21), Condition (221), and elementary block matrix operations, we obtain the following matrix equalities:

$$r(A^* A^\dagger A^*) = r \begin{bmatrix} A^* A A^* & (A^*)^2 \\ (A^*)^2 & 0 \end{bmatrix} - r(A) = r \begin{bmatrix} 0 & (A^*)^2 \\ (A^*)^2 & 0 \end{bmatrix} - r(A) = 2r(A^2) - r(A) = r(A),$$

as required for Condition (228). Conversely, Condition (228) implies

$$r(A) = r(A^* A^\dagger A^*) \leq r(A^\dagger A^*) = r(A^* A^*) = r(A^2)$$

by Lemma 2.6(g). Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

Replacing A with $A^\dagger$ in (228) and noting (2.12) lead to the equivalence of Conditions (228) and (229).

Applying Lemma 2.6(d) to Conditions (228) and (229) leads to the equivalences of Conditions (228)–(231).

By (2.21), Conditions (221), (236), and (254), and elementary block matrix operations, we obtain that the following matrix rank equalities

$$\begin{aligned} r((A^* A^\dagger A^*)^2) &= r(A^* A^\dagger (A^*)^2 A^\dagger A^*) = r \begin{bmatrix} A^* A A^* & (A^*)^2 \\ A^* A^\dagger (A^*)^3 & 0 \end{bmatrix} - r(A) \\ &= r \begin{bmatrix} 0 & (A^*)^2 \\ A^* A^\dagger (A^*)^3 & 0 \end{bmatrix} - r(A) \\ &= r(A^* A^\dagger (A^*)^3) \\ &= r \begin{bmatrix} A^* A A^* & (A^*)^4 \\ (A^*)^2 & 0 \end{bmatrix} - r(A) \\ &= r \begin{bmatrix} 0 & (A^*)^4 \\ (A^*)^2 & 0 \end{bmatrix} - r(A) \\ &= r(A^4) + r(A^2) - r(A) = r(A) \end{aligned}$$

hold, as required for Condition (230). Conversely, Condition (230) implies

$$r(A) = r((A^* A^\dagger A^*)^2) = r(A^* A^\dagger (A^*)^2 A^\dagger A^*) \leq r(A^2).$$

Combining this matrix rank inequality with $r(A) \geq r(A^2)$ implies $r(A) = r(A^2)$, as required for Condition (221).

Replacing A with $A^\dagger$ in (230) and noting (2.12) lead to the equivalence of Conditions (230) and (231).

By (2.18), (2.21), Condition (221), and elementary block matrix operations, we obtain that

$$\begin{aligned}
r(AA^*A^\dagger A^*A) &= r \begin{bmatrix} A^*AA^* & (A^*)^2A \\ A(A^*)^2 & 0 \end{bmatrix} - r(A) \\
&= r \begin{bmatrix} 0 & (A^*)^2A \\ A(A^*)^2 & 0 \end{bmatrix} - r(A) \\
&= r(A(A^*)^2) + r((A^*)^2A) - r(A) = r(A)
\end{aligned}$$

hold, as required for Condition (232). Conversely, Condition (228) implies

$$r(A) = r(A^*A^\dagger A^*) \leq r(A^\dagger A^*) = r(A^*A^*) = r(A^2)$$

by Lemma 2.6(g). Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

By Lemma 2.6(g), and Conditions (221) and (236), we obtain

$$r(AA^\dagger A^*A^\dagger A) = r(A^\dagger A^*A^\dagger) = r((A^*)^3) = r(A),$$

as required for Condition (233). Conversely, Condition (233) implies

$$r(A) = r(AA^\dagger A^*A^\dagger A) \leq r(A^\dagger A^*) = r(A^*A^*) = r(A^2)$$

by Lemma 2.6(g). Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

By (2.18), (2.21), Conditions (221) and (228), and elementary block matrix operations, we obtain

$$\begin{aligned}
r(A(A^*A^\dagger A^*)^\dagger A) &= r \begin{bmatrix} A(A^\dagger)^*AA^*A^\dagger A^*A(A^\dagger)^*A & A(A^\dagger)^*A^2 \\ A^2(A^\dagger)^*A & 0 \end{bmatrix} - r(A^*A^\dagger A^*) \\
&= r \begin{bmatrix} 0 & A(A^\dagger)^*A^2 \\ A^2(A^\dagger)^*A & 0 \end{bmatrix} - r(A) \\
&= r(A(A^\dagger)^*A^2) + r(A^2(A^\dagger)^*A) - r(A) = r(A),
\end{aligned}$$

as required for Condition (234). Conversely, Condition (234) implies

$$r(A) = r(A(A^*A^\dagger A^*)^\dagger A) \leq r((A^*A^\dagger A^*)^\dagger) \leq r(A^\dagger A^*) = r(A^2)$$

by Lemma 2.6(g). Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

By (2.18), (2.21), Conditions (221) and (229), and elementary block matrix operations, we obtain that

$$\begin{aligned}
r(A(A^\dagger A^*A^\dagger)^\dagger A) &= r \begin{bmatrix} (A^\dagger)^*A(A^\dagger)^*A^\dagger A^*A^\dagger (A^\dagger)^*A(A^\dagger)^* & ((A^\dagger)^*A)^2 \\ (A(A^\dagger)^*)^2 & 0 \end{bmatrix} - r(A^\dagger A^*A^\dagger) \\
&= r \begin{bmatrix} 0 & ((A^\dagger)^*A)^2 \\ (A(A^\dagger)^*)^2 & 0 \end{bmatrix} - r(A) \\
&= r(((A^\dagger)^*A)^2) + r((A(A^\dagger)^*)^2) - r(A) = r(A)
\end{aligned}$$

hold, as required for Condition (235). Conversely, Condition (235) implies

$$r(A) = r(A(A^\dagger A^*A^\dagger)^\dagger A) \leq r((A^\dagger A^*A^\dagger)^\dagger) \leq r(A^\dagger A^*) = r(A^2)$$

by Lemma 2.6(g). Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition (221).

The equivalences of Conditions (221)–(244), (236)–(239), (247)–(252), and (254)–(261) follow from (2.16), (2.17), and (2.18).

Replacing A with $A^\dagger$ in (221), (223), and (224) and noting (2.12) lead to the equivalences of Conditions (221) and (225)–(227).

Replacing A with $A^\dagger$ in (236), (223), (224), and (236) and noting (2.12) lead to the equivalences of Conditions (221), (225)–(227), and (240).

The equivalences of Conditions $\langle 230 \rangle$ – $\langle 233 \rangle$ with $\langle 243 \rangle$ – $\langle 246 \rangle$ and $\langle 265 \rangle$ – $\langle 268 \rangle$ follow from (2.16), (2.17), and (2.18).

By (2.19), Conditions $\langle 221 \rangle$ and $\langle 228 \rangle$, and Lemma 2.6(g), we obtain

$$\begin{aligned} r((A^*A^\dagger)^2) &= r(A^*A^\dagger A^*A^\dagger) = r(A^*A^\dagger(A^*)^2) = r(A^*A^\dagger(A^*)) = r(A^*A^\dagger) = r(A), \\ r((A^\dagger A^*)^2) &= r(A^\dagger A^*A^\dagger A^*) = r((A^*)^2 A^\dagger A^*) = r(A^*A^\dagger(A^*)) = r(A^\dagger A^*) = r(A). \end{aligned}$$

Hence,

$$r((A^*A^\dagger)^k) = r(A^*A^\dagger) = r(A) \quad \text{and} \quad r((A^\dagger A^*)^k) = r(A^\dagger A^*) = r(A)$$

hold by (2.16), as required for Conditions $\langle 241 \rangle$ and $\langle 242 \rangle$. Conversely, Conditions $\langle 241 \rangle$ and $\langle 242 \rangle$ imply

$$r(A) = r((A^\dagger A^*)^k) \leq r(A^\dagger A^*) = r(A^2)$$

by Lemma 2.6(g). Combining this matrix rank inequality with $r(A) \geq r(A^2)$ leads to $r(A) = r(A^2)$, as required for Condition $\langle 221 \rangle$.

The equivalences of Conditions $\langle 241 \rangle$ and $\langle 242 \rangle$ with $\langle 263 \rangle$ and $\langle 264 \rangle$ follow from (2.17).

The equivalences of Conditions $\langle 7 \rangle$ and $\langle 8 \rangle$ with Conditions $\langle 247 \rangle$ and $\langle 250 \rangle$ follow from Lemma 2.7(1) and $\langle 6 \rangle$.

The equivalences of Conditions $\langle 9 \rangle$ and $\langle 10 \rangle$ with Conditions $\langle 247 \rangle$ and $\langle 250 \rangle$ follow from Lemma 2.7(1) and $\langle 5 \rangle$.

Observe from Conditions $\langle 9 \rangle$ – $\langle 220 \rangle$ that the products on the left-hand sides involve the terms A^k for $k \geq 3$, while the ranks of the terms on the right-hand sides of these equalities are all equal to $r(A)$. These facts imply that $r(A^k) = r(A)$ for $k \geq 3$. Conditions $\langle 9 \rangle$ – $\langle 220 \rangle$ thereby imply Condition $\langle 221 \rangle$ by (2.16). What we need to do next is to show that Condition $\langle 221 \rangle$ implies each of Conditions $\langle 9 \rangle$ – $\langle 220 \rangle$.

Pre-multiplying the matrix equality in Condition $\langle 7 \rangle$ with $A^\dagger$ yields Condition $\langle 13 \rangle$; and pre-multiplying the matrix equality in Condition $\langle 7 \rangle$ with A^* yields Condition $\langle 14 \rangle$; post-multiplying the matrix equality in Condition $\langle 8 \rangle$ with $A^\dagger$ yields Condition $\langle 11 \rangle$; pre-multiplying the matrix equality in Condition $\langle 7 \rangle$ with A^* yields Condition $\langle 12 \rangle$.

Post-multiplying both sides of the matrix equality in Condition $\langle 7 \rangle$ with the conjugate transpose equality of Condition $\langle 7 \rangle$ yields Condition $\langle 15 \rangle$; pre-multiplying both sides of the matrix equality in Condition $\langle 8 \rangle$ with the conjugate transpose equality of Condition $\langle 8 \rangle$ yields Condition $\langle 16 \rangle$.

Substitution of Conditions $\langle 7 \rangle$ and $\langle 8 \rangle$ into $\langle 17 \rangle$ leads to the equivalences of Conditions $\langle 1 \rangle$ and $\langle 17 \rangle$.

Under Condition $\langle 221 \rangle$, we obtain

$$\mathcal{R}(A^\dagger A^2) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^\dagger) \quad \text{and} \quad \mathcal{R}((A^2 A^\dagger)^*) = \mathcal{R}((A A^\dagger)^*) = \mathcal{R}((A^\dagger)^*)$$

hold by Lemma 2.3. Therefore, we obtain from Lemma 2.7(1) and $\langle 5 \rangle$ that

$$A^\dagger A^2 (A^\dagger A^2)^\dagger = A^\dagger A \quad \text{and} \quad (A^2 A^\dagger)^\dagger A^2 A^2 A^\dagger = A A^\dagger,$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 18 \rangle$, and $\langle 19 \rangle$ through Condition $\langle 221 \rangle$.

Post- and pre-multiplying the matrix equalities in Conditions $\langle 18 \rangle$ and $\langle 19 \rangle$ with A , respectively, yield Conditions $\langle 20 \rangle$ and $\langle 21 \rangle$, respectively.

By Conditions $\langle 7 \rangle$, $\langle 8 \rangle$, $\langle 18 \rangle$, and $\langle 19 \rangle$, we obtain

$$\begin{aligned} A(A^2 A^\dagger)^\dagger A &= A(A^2 A^\dagger)^\dagger A^2 (A^2)^\dagger A = A(A^2 A^\dagger)^\dagger (A^2 A^\dagger) A (A^2)^\dagger A = A^2 (A^2)^\dagger A = A, \\ A(A^\dagger A^2)^\dagger A &= A(A^2)^\dagger A^2 (A^\dagger A^2)^\dagger A = A(A^2)^\dagger A (A^\dagger A^2) (A^\dagger A^2)^\dagger A = A(A^2)^\dagger A^2 = A, \end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 21 \rangle$, and $\langle 22 \rangle$ through Conditions $\langle 7 \rangle$, $\langle 8 \rangle$, $\langle 18 \rangle$, and $\langle 19 \rangle$.

Post- and pre-multiplying the matrix equalities in Conditions $\langle 22 \rangle$ and $\langle 23 \rangle$ with $A^\dagger$, respectively, yield Conditions $\langle 24 \rangle$ and $\langle 25 \rangle$, respectively.

Pre- and post-multiplying the matrix equalities in Conditions $\langle 22 \rangle$ and $\langle 23 \rangle$ with $A^\dagger$, respectively, yield Conditions $\langle 26 \rangle$ and $\langle 27 \rangle$, respectively.

By Conditions $\langle 7 \rangle$ and $\langle 8 \rangle$, we obtain

$$A^2(A^2)^\dagger A(A^2)^\dagger A^2 = A(A^2)^\dagger A^2 = A,$$

thus establishing the equivalence of Conditions (1) and (28) through Conditions (7) and (8).

By Conditions (22) and (23), we obtain

$$\begin{aligned} A(A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger A &= A(A^\dagger A^2)^\dagger A = A, \\ A(A^\dagger A^2)^\dagger A(A^2 A^\dagger)^\dagger A &= A(A^2 A^\dagger)^\dagger A = A, \end{aligned}$$

thus establishing the equivalences of Conditions (1), (29), and (30) through Conditions (22) and (23).

By Conditions (20) and (21), we obtain

$$A^2(A^\dagger A^2)^\dagger A^\dagger (A^2 A^\dagger)^\dagger A^2 = AA^\dagger A = A,$$

thus establishing the equivalences of Conditions (1) and (31) through Conditions (20) and (21).

The equivalences of Conditions (1) and (32)–(39) follow from (2.36) and (2.39).

By Conditions (20) and (21), we obtain

$$(A^2 A^\dagger)^\dagger A^3 (A^\dagger A^2)^\dagger = A^2 (A^\dagger A^2)^\dagger = A,$$

thus establishing the equivalences of Conditions (1) and (40) through Condition (21).

By (2.44), Conditions (7) and (8), we obtain

$$A^2(A^3)^\dagger A^2 = A^2(A^2)^\dagger A(A^2)^\dagger A^2 = A,$$

thus establishing the equivalences of Conditions (1) and (41) through Conditions (7) and (8).

By Conditions (7) and (8), we obtain

$$A(A^2)^\dagger A^3 (A^2)^\dagger A = A(A^2)^\dagger A^2 A^\dagger A^2 (A^2)^\dagger A = AA^\dagger A = A,$$

thus establishing the equivalences of Conditions (1) and (42) through Conditions (7) and (8).

By Condition (41), we obtain

$$A(A^2(A^3)^\dagger A^2)^\dagger A = AA^\dagger A = A,$$

thus establishing the equivalences of Conditions (1) and (43) through Condition (41).

Under Condition (221), it follows that $r(A^\dagger A^3 A^\dagger) = r(A)$ by (2.16), and thereby we obtain from (2.29) that

$$(A^\dagger A^3 A^\dagger)^\dagger = ((A^\dagger A)A(AA^\dagger))^\dagger = (A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger, \quad (3.1)$$

$$(A^\dagger A^3 A^\dagger)^\dagger = (A^3 A^\dagger)^\dagger A^3 (A^\dagger A^3)^\dagger. \quad (3.2)$$

In these cases, we obtain from (2.44) that

$$A(A^\dagger A^3 A^\dagger)^\dagger A = A(A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger A = A,$$

thus establishing the equivalences of Conditions (1) and (44) through Condition (29).

Pre- and post-multiplying both sides of the matrix equalities in Conditions (22) and (23) with A yields

$$A^2(A^2 A^\dagger)^\dagger A = A^2 \quad \text{and} \quad A(A^\dagger A^2)^\dagger A^2 = A^2.$$

In this situation, we obtain from (3.1) and Conditions (19) and (20) that

$$\begin{aligned} A^2(A^\dagger A^3 A^\dagger)^\dagger &= A^2(A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger = A^2(A^\dagger A^2)^\dagger = A, \\ (A^\dagger A^3 A^\dagger)^\dagger A^2 &= (A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger A^2 = (A^\dagger A^2)^\dagger A^2 = A, \end{aligned}$$

thus establishing the equivalences of Conditions (1), (45), and (46) through Conditions (20)–(23).

By Conditions (44)–(46), we obtain

$$\begin{aligned} A(A^\dagger A^3 A^\dagger)^\dagger A(A^\dagger A^3 A^\dagger)^\dagger A &= A(A^\dagger A^3 A^\dagger)^\dagger A = A, \\ (A^\dagger A^3 A^\dagger)^\dagger A^3 (A^\dagger A^3 A^\dagger)^\dagger &= (A^\dagger A^3 A^\dagger)^\dagger A^2 A^\dagger A^2 (A^\dagger A^3 A^\dagger)^\dagger A = AA^\dagger A = A, \end{aligned}$$

thus establishing the equivalences of Conditions (1), (47), and (48) through Conditions (44)–(46).

Under Condition (221), it follows that $r(A^3 A^\dagger) = r(A^\dagger A^3) = r(A)$, and thereby we obtain from (2.29) that

$$\begin{aligned}(A^3A^\dagger)^\dagger &= (AA(AA^\dagger))^\dagger = (A^2A^\dagger)^\dagger A(A^2)^\dagger, \\ (A^\dagger A^3)^\dagger &= ((A^\dagger A)AA)^\dagger = (A^2)^\dagger A(A^\dagger A^2)^\dagger.\end{aligned}$$

In these cases, we obtain from Conditions $\langle 7 \rangle$, $\langle 8 \rangle$, $\langle 22 \rangle$, and $\langle 23 \rangle$ that

$$\begin{aligned}A(A^3A^\dagger)^\dagger A^2 &= A(A^2A^\dagger)^\dagger A(A^2)^\dagger A^2 = A(A^2A^\dagger)^\dagger A = A, \\ A^2(A^3A^\dagger)^\dagger A &= A^2(A^2A^\dagger)^\dagger A(A^2)^\dagger A = A^2(A^2)^\dagger A = A, \\ A(A^\dagger A^3)^\dagger A^2 &= A(A^2)^\dagger A(A^\dagger A^2)^\dagger A^2 = A(A^2)^\dagger A^2 = A, \\ A^2(A^\dagger A^3)^\dagger A &= A^2(A^2)^\dagger A(A^\dagger A^2)^\dagger A = A(A^\dagger A^2)^\dagger A = A,\end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$, and $\langle 49 \rangle$ – $\langle 52 \rangle$ through Conditions $\langle 8 \rangle$, $\langle 22 \rangle$, and $\langle 23 \rangle$.

Under Conditions $\langle 221 \rangle$ and $\langle 229 \rangle$, it follows that $\mathcal{R}(A^3A^\dagger) = \mathcal{R}(A)$ and $\mathcal{R}((A^\dagger A^3)^*) = \mathcal{R}(A^*)$ by Conditions $\langle 241 \rangle$ and $\langle 242 \rangle$. Therefore, we obtain from Lemma 2.7(1) and $\langle 5 \rangle$ that

$$(A^3A^\dagger)^\dagger A^3A^\dagger = AA^\dagger \quad \text{and} \quad A^\dagger A^3(A^\dagger A^3)^\dagger = A^\dagger A.$$

In these cases,

$$\begin{aligned}(A^3A^\dagger)^\dagger A^3 &= (A^3A^\dagger)^\dagger A^3A^\dagger A = AA^\dagger A = A, \\ A^3(A^\dagger A^3)^\dagger &= AA^\dagger A^3(A^\dagger A^3)^\dagger = AA^\dagger A = A,\end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 53 \rangle$, and $\langle 54 \rangle$ through Conditions $\langle 221 \rangle$, $\langle 241 \rangle$, and $\langle 242 \rangle$.

Post- and pre-multiplying Conditions $\langle 49 \rangle$ and $\langle 51 \rangle$ with $A^\dagger$, respectively, and simplifying yield Conditions $\langle 55 \rangle$ and $\langle 56 \rangle$, respectively, thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 55 \rangle$, and $\langle 56 \rangle$ through Conditions $\langle 48 \rangle$ and $\langle 51 \rangle$.

Pre- and post-multiplying Conditions $\langle 50 \rangle$ and $\langle 52 \rangle$ with $A^\dagger$, respectively, yield Conditions $\langle 57 \rangle$ and $\langle 58 \rangle$, respectively, thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 57 \rangle$, and $\langle 58 \rangle$ through Conditions $\langle 50 \rangle$ and $\langle 52 \rangle$.

Replacing A with $A^\dagger$ in Conditions $\langle 55 \rangle$ and $\langle 56 \rangle$ leads to Conditions $\langle 59 \rangle$ and $\langle 60 \rangle$, thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 59 \rangle$, and $\langle 60 \rangle$ through Conditions $\langle 5 \rangle$, $\langle 55 \rangle$, and $\langle 56 \rangle$.

Pre- and post-multiplying Conditions $\langle 59 \rangle$ and $\langle 60 \rangle$ with A , respectively, yield Conditions $\langle 61 \rangle$ and $\langle 62 \rangle$, respectively, thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 61 \rangle$, and $\langle 62 \rangle$ through Conditions $\langle 59 \rangle$ and $\langle 60 \rangle$.

Replacing A with $A^\dagger$ in Conditions $\langle 55 \rangle$ and $\langle 58 \rangle$ leads to Conditions $\langle 63 \rangle$ and $\langle 64 \rangle$, thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 63 \rangle$, and $\langle 64 \rangle$ through Conditions $\langle 5 \rangle$, $\langle 55 \rangle$, and $\langle 58 \rangle$.

By Conditions $\langle 50 \rangle$ – $\langle 52 \rangle$, we obtain

$$\begin{aligned}A(A^3A^\dagger)^\dagger A^3(A^\dagger A^3)^\dagger A &= A^2(A^\dagger A^3)^\dagger A = A, \\ A(A^\dagger A^3)^\dagger A^3(A^3A^\dagger)^\dagger A &= A^2(A^3A^\dagger)^\dagger A = A, \\ A^2(A^3A^\dagger)^\dagger A(A^\dagger A^3)^\dagger A^2 &= A(A^\dagger A^3)^\dagger A^2 = A, \\ A^2(A^\dagger A^3)^\dagger A(A^3A^\dagger)^\dagger A^2 &= A(A^3A^\dagger)^\dagger A^2 = A,\end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$ and $\langle 65 \rangle$ – $\langle 68 \rangle$ through Conditions $\langle 50 \rangle$ – $\langle 52 \rangle$.

Under Condition $\langle 221 \rangle$, it follows that $r(A^k) = r(A)$ by Condition $\langle 236 \rangle$, and thereby we obtain from (2.29) that

$$(A^4)^\dagger = (AA^2A)^\dagger = (A^3)^\dagger A^2(A^3)^\dagger, \quad (3.3)$$

$$(A^5)^\dagger = (A^2AA^2)^\dagger = (A^3)^\dagger A(A^3)^\dagger, \quad (3.4)$$

$$(A^6)^\dagger = (A^2A^2A^2)^\dagger = (A^4)^\dagger A^2(A^4)^\dagger, \quad (3.5)$$

$$(A^7)^\dagger = (A^3AA^3)^\dagger = (A^4)^\dagger A(A^4)^\dagger, \quad (3.6)$$

$$(A^8)^\dagger = (A^3A^2A^3)^\dagger = (A^5)^\dagger A^2(A^5)^\dagger, \quad (3.7)$$

$$(A^9)^\dagger = (A^4AA^4)^\dagger = (A^5)^\dagger A(A^5)^\dagger. \quad (3.8)$$

In this situation, we further obtain from Conditions $\langle 176 \rangle$, $\langle 177 \rangle$, and (3.4) that

$$\begin{aligned} A^3(A^5)^\dagger A^3 &= A^3(A^3)^\dagger A(A^3)^\dagger A^3 = AA^\dagger AA^\dagger A = A, \\ A(A^3(A^5)^\dagger A^3)^\dagger A &= AA^\dagger A = A, \end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 69 \rangle$, and $\langle 71 \rangle$ through Conditions $\langle 176 \rangle$ and $\langle 177 \rangle$.

By Conditions $\langle 176 \rangle$ and $\langle 177 \rangle$, we obtain

$$A(A^3)^\dagger A^5(A^3)^\dagger A = A(A^3)^\dagger A^3 A^\dagger A^3(A^3)^\dagger A = AA^\dagger AA^\dagger A^\dagger A = A,$$

thus establishing the equivalence of Conditions $\langle 1 \rangle$ and $\langle 70 \rangle$ through Conditions $\langle 176 \rangle$ and $\langle 177 \rangle$.

By Conditions $\langle 33 \rangle$, $\langle 70 \rangle$, $\langle 176 \rangle$, $\langle 177 \rangle$, and (3.3), we obtain

$$\begin{aligned} A^3(A^4)^\dagger A^3(A^4)^\dagger A^3 &= A^3(A^3)^\dagger A^2(A^3)^\dagger A^3(A^3)^\dagger A^2(A^3)^\dagger A^3 = A^2(A^3)^\dagger A^2 = A, \\ A^2(A^4)^\dagger A^5(A^4)^\dagger A^2 &= A^2(A^3)^\dagger A^2(A^3)^\dagger A^3 A^\dagger A^3(A^3)^\dagger A^2(A^3)^\dagger A^2 = A^2(A^3)^\dagger A^2 = A, \\ A(A^4)^\dagger A^7(A^4)^\dagger A &= A(A^3)^\dagger A^2(A^3)^\dagger A^3 AA^\dagger A^3(A^3)^\dagger A^2(A^3)^\dagger A = A(A^3)^\dagger A^5(A^3)^\dagger A = A, \end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$ and $\langle 72 \rangle$ – $\langle 74 \rangle$ through Conditions $\langle 33 \rangle$, $\langle 63 \rangle$, $\langle 176 \rangle$, and $\langle 177 \rangle$.

Under Condition $\langle 221 \rangle$, substituting (3.4) into Conditions $\langle 75 \rangle$ – $\langle 78 \rangle$ and then simplifying by Conditions $\langle 174 \rangle$ – $\langle 177 \rangle$ lead to the equivalences of Conditions $\langle 1 \rangle$ and $\langle 75 \rangle$ – $\langle 78 \rangle$.

Under Condition $\langle 221 \rangle$, substituting (3.6) into Conditions $\langle 79 \rangle$ and $\langle 80 \rangle$ and then simplifying by Conditions $\langle 174 \rangle$ – $\langle 177 \rangle$ lead to the equivalences of Conditions $\langle 1 \rangle$, $\langle 79 \rangle$, and $\langle 80 \rangle$.

Under Condition $\langle 221 \rangle$, substituting (3.8) into Condition $\langle 81 \rangle$ and then simplifying by Conditions $\langle 174 \rangle$ – $\langle 177 \rangle$ lead to the equivalence of Conditions $\langle 1 \rangle$ and $\langle 81 \rangle$.

The equivalences of Conditions $\langle 1 \rangle$ and $\langle 81 \rangle$ – $\langle 100 \rangle$ can be derived from (2.44), (3.3)–(3.8), $\langle 174 \rangle$, and $\langle 175 \rangle$.

By Conditions $\langle 7 \rangle$ – $\langle 10 \rangle$, we obtain

$$\begin{aligned} (A^2)^\dagger A^2 A^\dagger &= A^\dagger AA^\dagger = A^\dagger, \\ A^\dagger A^2 (A^2)^\dagger &= A^\dagger AA^\dagger = A^\dagger, \\ A^\dagger A^2 (A^2)^\dagger (A^\dagger)^* &= A^\dagger (A^\dagger)^* = (A^* A)^\dagger, \\ (A^\dagger)^* (A^2)^\dagger A^2 A^\dagger &= (A^\dagger)^* A^\dagger = (AA^*)^\dagger, \\ A^2 (A^2)^\dagger (AA^*)^\dagger A^2 (A^2)^\dagger &= AA^\dagger (AA^*)^\dagger AA^\dagger = (AA^*)^\dagger, \\ (A^2)^\dagger A^2 (A^* A)^\dagger (A^2)^\dagger A^2 &= A^\dagger A (A^* A)^\dagger A^\dagger A = (A^* A)^\dagger, \\ A^2 (A^2)^\dagger (A^* AA^*)^\dagger (A^2)^\dagger A^2 &= AA^\dagger (A^* AA^*)^\dagger A^\dagger A = (A^* AA^*)^\dagger, \end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$ and $\langle 101 \rangle$ – $\langle 107 \rangle$ through Conditions $\langle 7 \rangle$ – $\langle 10 \rangle$.

Replacing A with $A^\dagger$ in Conditions $\langle 18 \rangle$ – $\langle 31 \rangle$ leads to Conditions $\langle 108 \rangle$ – $\langle 120 \rangle$, thus establishing the equivalences of Conditions $\langle 1 \rangle$ and $\langle 108 \rangle$ – $\langle 120 \rangle$ through Conditions $\langle 5 \rangle$ and $\langle 18 \rangle$ – $\langle 31 \rangle$.

Replacing A with $A^\dagger$ in Conditions $\langle 32 \rangle$ – $\langle 39 \rangle$ leads to Conditions $\langle 121 \rangle$ – $\langle 134 \rangle$, thus establishing the equivalences of Conditions $\langle 1 \rangle$ and $\langle 121 \rangle$ – $\langle 134 \rangle$ through Conditions $\langle 5 \rangle$, and $\langle 32 \rangle$ – $\langle 39 \rangle$.

Replacing A with $A^\dagger$ in Conditions $\langle 44 \rangle$ – $\langle 48 \rangle$ leads to Conditions $\langle 135 \rangle$ – $\langle 139 \rangle$, thus establishing the equivalences of Conditions $\langle 1 \rangle$ and $\langle 135 \rangle$ – $\langle 139 \rangle$ through Conditions $\langle 5 \rangle$ and $\langle 44 \rangle$ – $\langle 48 \rangle$.

By Conditions $\langle 9 \rangle$ and $\langle 10 \rangle$, we obtain

$$\begin{aligned} (A^2)^\dagger A^3 (A^2)^\dagger &= (A^2)^\dagger A^2 A^\dagger A^2 (A^2)^\dagger = A^\dagger AA^\dagger AA^\dagger = A^\dagger, \\ A^\dagger ((A^2)^\dagger A^3 (A^2)^\dagger)^\dagger A^\dagger &= A^\dagger AA^\dagger = A^\dagger, \end{aligned}$$

thus establishing the equivalences of Conditions $\langle 1 \rangle$, $\langle 140 \rangle$, and $\langle 141 \rangle$ through Conditions $\langle 9 \rangle$ and $\langle 10 \rangle$.

The equivalence of Condition $\langle 1 \rangle$ and each of $\langle 142 \rangle$ – $\langle 173 \rangle$ and $\langle 178 \rangle$ – $\langle 196 \rangle$ can be established analogously.

Under Condition $\langle 221 \rangle$, it follows that $r(A^{2k+1}) = r(A)$ by Condition $\langle 236 \rangle$, and thereby we obtain from (2.29) that

$$(A^{k+s+1})^\dagger = (A^k AA^s)^\dagger = (A^{k+1})^\dagger A(A^{s+1})^\dagger.$$

In this case, we obtain from Conditions $\langle 176 \rangle$ and $\langle 177 \rangle$ that

$$A^{k+1}(A^{k+s+1})^\dagger A^{s+1} = A^{k+1}(A^{k+1})^\dagger A(A^{s+1})^\dagger A^{s+1} = AA^\dagger AA^\dagger A = A,$$

thus establishing the equivalence of Conditions $\langle 1 \rangle$ and $\langle 197 \rangle$ through Conditions $\langle 176 \rangle$ and $\langle 177 \rangle$.

The equivalences of $\langle 1 \rangle$ and the remaining Conditions in $\langle 198 \rangle$ – $\langle 220 \rangle$ can be established analogously. $\square$

Theorem 3.1 and its proof give a detailed collection and derivation of many existing and novel necessary and sufficient conditions for a matrix to be group invertible, and thereby they can be used as useful supplementary issues for people to deal with group invertible matrices under various assumptions.

4 Miscellaneous matrix equalities in relation to a group invertible matrix

The collection of matrix equalities and facts in the two preceding sections embody many dominant features of group invertibility of a matrix, and therefore, they give us much noticeable knowledge and greater understanding about group inverses of matrices. Moreover, the work of this kind also provides certain theoretical orientations about establishing more general matrix equalities for generalized inverses of matrices; in other words, there will be ample opportunity to derive many novel assertions regarding the operations of matrices and their group inverses.

The author first gives some existing knowledge regarding operations of the group inverse of a given matrix (cf. [1,3]).

Lemma 4.1. Assume that $A \in \mathbb{C}^{m \times m}$ is group invertible. Then, the following matrix equalities

$$(A^\#)^\# = A, \quad (A^\#)^* = (A^*)^\#, \quad (A^k)^\# = (A^\#)^k \quad (4.1)$$

hold for any integer $k \geq 2$, and the following rank and range equalities

$$r((A^\dagger)^\#) = r((A^\#)^\dagger) = r(AA^\#) = r(A^\#) = r(A), \quad (4.2)$$

$$\mathcal{R}(((A^\dagger)^\#)^\dagger) = \mathcal{R}(AA^\#) = \mathcal{R}(A^\#) = \mathcal{R}(A), \quad (4.3)$$

$$\mathcal{R}(((A^\#)^\dagger)^\#) = \mathcal{R}((A^\dagger)^\#) = \mathcal{R}((A^\#)^\dagger) = \mathcal{R}((A^\#)^*) = \mathcal{R}(AA^\#)^* = \mathcal{R}(A^*) \quad (4.4)$$

hold.

Based on the basic facts in Lemma 4.1 and the findings in Section 3, we obtain the following results.

Theorem 4.2. Assume that $A \in \mathbb{C}^{m \times m}$ is group invertible. Then,

$$A = A^\#((A^\#)^3)^\dagger A^\#, \quad (4.5)$$

$$A^\# = (A^\dagger A^3 A^\dagger)^\dagger, \quad (4.6)$$

$$A^\# = (A^2 A^\dagger)^\dagger A(A^\dagger A^2)^\dagger, \quad (4.7)$$

$$A^\# = (A^\dagger A^2 A^\dagger)^\dagger A^\dagger (A^\dagger A^2 A^\dagger)^\dagger, \quad (4.8)$$

$$A^\# = (A^3 A^\dagger)^\dagger A^3 (A^\dagger A^3)^\dagger, \quad (4.9)$$

$$A^\# = A(A^\dagger A^3)^\dagger A(A^3 A^\dagger)^\dagger A, \quad (4.10)$$

$$(A^\dagger)^\# = (A(A^\dagger)^3 A)^\dagger, \quad (4.11)$$

$$(A^\#)^\dagger = A^\dagger A^3 A^\dagger, \quad (4.12)$$

$$AA^\# = A(A^2)^\dagger A, \quad (4.13)$$

$$AA^\# = A^\#((A^2)^\dagger)^\dagger A^\#, \quad (4.14)$$

$$A^2 = A(AA^\#)^\dagger A, \quad (4.15)$$

$$A^3 = A(A^\#)^\dagger A, \quad (4.16)$$

$$(A^2)^\dagger = A^\dagger(AA^\#)A^\dagger, \quad (4.17)$$

$$(A^3)^\dagger = A^\dagger A^\# A^\dagger, \quad (4.18)$$

$$AA^\# = AA^*A(AA^*A)^\#, \quad (4.19)$$

$$AA^\dagger = A^\#(A^\#)^\dagger = ((A^\dagger)^\#)^\dagger(A^\dagger)^\#, \quad (4.20)$$

$$A^\dagger A = (A^\#)^\dagger A^\# = (A^\dagger)^\#((A^\dagger)^\#)^\dagger, \quad (4.21)$$

$$AA^\# = A(AA^\#)^\dagger A^\# = A^\#(AA^\#)^\dagger A, \quad (4.22)$$

$$(A^\#)^2 = A^\#(AA^\#)^\dagger A^\#, \quad (4.23)$$

$$(A^\#)^2 = A(A^2)^\dagger A^\# = A^\#(A^2)^\dagger A, \quad (4.24)$$

$$(A^\#)^3 = A(A^2)^\dagger A^\#(A^2)^\dagger A, \quad (4.25)$$

$$(A^\#)^3 = A^\# A^\dagger A^\#, \quad (4.26)$$

$$A = (A^k)^\#((A^{2k+1})^\#)^\dagger(A^k)^\# \text{ for any integer } k \geq 2, \quad (4.27)$$

$$A^\# = (A^{k+1})^\#((A^{2k+1})^\#)^\dagger(A^{k+1})^\# \text{ for any integer } k \geq 2, \quad (4.28)$$

$$A^\# = A^k(A^{2k+1})^\dagger A^k \text{ for any integer } k \geq 2, \quad (4.29)$$

$$AA^\# = A^k(A^k)^\# \text{ for any integer } k \geq 2, \quad (4.30)$$

$$AA^\# = A^k(A^{2k})^\dagger A^k \text{ for any integer } k \geq 2, \quad (4.31)$$

$$A^{2k+1} = (A(A^\#)^\dagger)^k A \text{ for any integer } k \geq 2, \quad (4.32)$$

$$A^{2k+1} = A((A^{2k-1})^\#)^\dagger A \text{ for any integer } k \geq 2, \quad (4.33)$$

$$(A^{2k+1})^\dagger = A^\dagger(A^{2k-1})^\# A^\dagger \text{ for any integer } k \geq 2, \quad (4.34)$$

$$(A^{2k+1})^\dagger = (A^k)^\dagger A^\#(A^k)^\dagger \text{ for any integer } k \geq 2, \quad (4.35)$$

$$(A^{2k+1})^\dagger = (A^\dagger A^\#)^k A^\dagger \text{ for any integer } k \geq 2, \quad (4.36)$$

$$(A^{2k-1})^\# = A(A^{2k+1})^\dagger A \text{ for any integer } k \geq 2, \quad (4.37)$$

$$(A^{2k-1})^\# = (A^\dagger A^{2k+1} A^\dagger)^\dagger \text{ for any integer } k \geq 2, \quad (4.38)$$

$$(A^\#)^{2k+1} = A^\#(A^{2k-1})^\dagger A^\# \text{ for any integer } k \geq 2, \quad (4.39)$$

$$(A^\#)^{2k+1} = (A^\# A^\dagger)^k A^\# \text{ for any integer } k \geq 2, \quad (4.40)$$

$$(A^{3k})^\dagger = (A^k)^\dagger(A^k)^\#(A^k)^\dagger \text{ for any integer } k \geq 2, \quad (4.41)$$

$$(A^\#)^{3k} = (A^\#)^k(A^k)^\dagger(A^\#)^k \text{ for any integer } k \geq 2. \quad (4.42)$$

Proof. Replacing A in (2.1) with $A^\#$ and applying the first and third equalities in (4.1), we obtain (4.5).

It is easy to verify that

$$(A^\dagger A^3 A^\dagger)^\dagger A(A^\dagger A^3 A^\dagger)^\dagger = (A^\dagger A^3 A^\dagger)^\dagger A^\dagger A^3 A^\dagger (A^\dagger A^3 A^\dagger)^\dagger = (A^\dagger A^3 A^\dagger)^\dagger$$

hold. In this case, applying (2.23) to $A^\# - (A^\dagger A^3 A^\dagger)^\dagger$ and then simplifying by (2.10)–(2.12) and (4.2)–(4.4), we obtain

$$\begin{aligned} r(A^\# - (A^\dagger A^3 A^\dagger)^\dagger) &= r \begin{bmatrix} A^\# \\ (A^\dagger A^3 A^\dagger)^\dagger \end{bmatrix} + r[A^\#, (A^\dagger A^3 A^\dagger)^\dagger] - r(AA^\#) - r((A^\dagger A^3 A^\dagger)^\dagger) \\ &= r \begin{bmatrix} A \\ A \end{bmatrix} + r[A, A] - 2r(A) \\ &= 0. \end{aligned}$$

This fact implies $A^\# - (A^\dagger A^3 A^\dagger)^\dagger = 0$, thus establishing (4.6).

Substitution of (3.1) and (3.2) into (4.6) yields (4.7) and (4.9).

Combining (2.1) and (2.47) yields (4.8).

By (2.29), we obtain

$$(A^3)^\dagger = (A(A^\dagger A^3 A^\dagger)A)^\dagger = (A^\dagger A^3)^\dagger A^\dagger A^3 A^\dagger (A^3 A^\dagger)^\dagger = (A^\dagger A^3)^\dagger A (A^3 A^\dagger)^\dagger.$$

Substitution of it into (2.1) yields (4.10).

Replacing A in (4.6) with $A^\dagger$ and applying the second equality in (2.7), we obtain (4.11).

Taking the Moore-Penrose inverse of (4.6) and applying (2.7) yield (4.12).

By (2.1) and Lemma 3.1(7), we obtain

$$AA^\# = A^2(A^3)^\dagger A = A^2(A^2)^\dagger A(A^2)^\dagger A = A(A^2)^\dagger A,$$

thus establishing (4.13).

Replacing A in (4.13) with $A^\#$ and applying the first and third equalities in (4.1) lead to (4.14).

Under (4.3) and (4.4), applying (2.22) to $A^2 - A(AA^\#)^\dagger A$ and then simplifying, we obtain

$$\begin{aligned} r(A^2 - A(AA^\#)^\dagger A) &= r \begin{bmatrix} AA^\# & A \\ A & A^2 \end{bmatrix} - r(AA^\#) \\ &= r \begin{bmatrix} 0 & A \\ A - A^2 A^\# & 0 \end{bmatrix} - r(A) \\ &= r \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} - r(A) = 0. \end{aligned}$$

This fact implies $A^2 - A(AA^\#)^\dagger A = 0$, thus establishing (4.15).

Pre- and post-multiplying both sides of (4.12) with A and then simplifying yield (4.16).

Pre- and post-multiplying both sides of (4.13) with $A^\dagger$ and then simplifying yield (4.17).

Pre- and post-multiplying both sides of (2.1) with $A^\dagger$ and then simplifying yield (4.18).

Obviously, $AA^*A(AA^*A)^\#$ and $AA^\#$ are idempotent matrices by the definition of group inverse. Therefore, by (2.23), we obtain

$$\begin{aligned} r(AA^*A(AA^*A)^\# - AA^\#) &= r \begin{bmatrix} AA^*A(AA^*A)^\# \\ AA^\# \end{bmatrix} + r[AA^*A(AA^*A)^\#, AA^\#] - r(AA^*A(AA^*A)^\#) - r(AA^\#) \\ &= r \begin{bmatrix} AA^*A \\ A \end{bmatrix} + r[AA^*A, A] - 2r(A) \\ &= 0. \end{aligned}$$

This fact implies (4.19).

Applying Lemma 2.7(1) and (5) to (4.3) and (4.4) leads to (4.20) and (4.21).

Pre- and post-multiplying both sides of (4.15) with $(A^\#)^2$, respectively, and then simplifying yield (4.22).

Pre- and post-multiplying both sides of (4.15) with $(A^\#)^2$, simultaneously, and then simplifying yield (4.23).

By (2.1) and (2.44), we obtain

$$\begin{aligned}
(A^\#)^2 &= A(A^3)^\dagger A^2(A^3)^\dagger A \\
&= A(A^2)^\dagger A(A^2)^\dagger A^2(A^2)^\dagger A(A^2)^\dagger A \\
&= A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A, \\
A(A^2)^\dagger A^\# &= A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A, \\
A^\#(A^2)^\dagger A &= A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A,
\end{aligned}$$

thus establishing (4.24).

By (2.1) and (2.44), we obtain

$$\begin{aligned}
(A^\#)^3 &= A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A \\
&= A(A^2)^\dagger A^\#(A^2)^\dagger A \\
&= A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A, \\
A^\# A^\dagger A^\# &= A(A^2)^\dagger A(A^2)^\dagger A A^\dagger A(A^2)^\dagger A(A^2)^\dagger A \\
&= A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A(A^2)^\dagger A,
\end{aligned}$$

thus establishing (4.25) and (4.26).

Pre- and post-multiplying both sides of the equality in Theorem 3.1(197) with $A^\#$ and then simplifying yield (4.29).

Replacing A in (4.29) with $A^\#$ and applying by the first and third equalities in (4.1) lead to (4.27).

Pre- and post-multiplying both sides of the equality in (4.27) with $A^\#$ and then simplifying yield (4.28).

Equation (4.30) follows the third equality in (4.1) and the definition of group inverse.

Replacing A in (4.13) with A^k and applying (4.30) lead to (4.31).

Equation (4.32) follows from (4.16).

By the definitions of the Moore-Penrose inverse, the group generalized inverse, and the third equality in (4.1), we obtain

$$A((A^{2k-1})^\#)^\dagger A(A^{2k+1})^\# A((A^{2k-1})^\#)^\dagger A = A((A^{2k-1})^\#)^\dagger (A^{2k-1})^\# ((A^{2k-1})^\#)^\dagger A = A((A^{2k-1})^\#)^\dagger A.$$

In this case, applying (2.14) to $A^{2k+1} - A((A^{2k-1})^\#)^\dagger A$ and then simplifying by Lemma 4.1 yield

$$\begin{aligned}
r(A^{2k+1} - A((A^{2k-1})^\#)^\dagger A) &= r \left[\begin{array}{c} A^{2k+1} \\ A((A^{2k-1})^\#)^\dagger A \end{array} \right] + r[A^{2k+1}, A((A^{2k-1})^\#)^\dagger A] - r(A^{2k+1}) - r(A((A^{2k-1})^\#)^\dagger A) \\
&= r \left[\begin{array}{c} A \\ A \end{array} \right] + r[A, A] - 2r(A) = 0,
\end{aligned}$$

which implies that (4.33) holds.

By the definitions of the Moore-Penrose inverse and the group inverse and the third equality in (4.1), we obtain that

$$\begin{aligned}
A^\dagger (A^{2k-1})^\# A^\dagger A^{2k+1} A^\dagger (A^{2k-1})^\# A^\dagger &= A^\dagger (A^{2k-1})^\# (A^\dagger A) A^{2k-1} (A A^\dagger) (A^{2k-1})^\# A^\dagger \\
&= A^\dagger (A^{2k-1})^\# A^{2k-1} (A^{2k-1})^\# A^\dagger = A^\dagger (A^{2k-1})^\# A^\dagger
\end{aligned}$$

hold. In this case, applying (2.14) to $(A^{2k+1})^\dagger - A^\dagger (A^{2k-1})^\# A^\dagger$ and then simplifying by Lemma 4.1 lead to

$$\begin{aligned}
r((A^{2k+1})^\dagger - A^\dagger (A^{2k-1})^\# A^\dagger) &= r \left[\begin{array}{c} (A^{2k+1})^\dagger \\ A^\dagger (A^{2k-1})^\# A^\dagger \end{array} \right] + r[(A^{2k+1})^\dagger, A^\dagger (A^{2k-1})^\# A^\dagger] - r((A^{2k+1})^\dagger) - r(A^\dagger (A^{2k-1})^\# A^\dagger) \\
&= r \left[\begin{array}{c} A^* \\ A^* \end{array} \right] + r[A^*, A^*] - 2r(A) = 0,
\end{aligned}$$

which implies that (4.34) holds.

Pre- and post-multiplying both sides of the equality in (4.30) with $(A^k)^\dagger$ and then simplifying yield (4.35).

By (2.29) and (4.18), we obtain

$$(A^{2k+1})^\dagger = (A^{2k-2} A A^2)^\dagger = (A^3)^\dagger A (A^{2k-1})^\dagger = A^\dagger A^\# A^\dagger A (A^{2k-1})^\dagger = A^\dagger A^\# (A^{2k-1})^\dagger,$$

thus establishing (4.36) by induction.

Pre- and post-multiplying both sides of the equality in (4.34) with A and then simplifying yield (4.37).

Replacing A in (4.13) with A^{2k-1} and applying Theorem 3.1(9) and (10) lead to (4.38).

Replacing A in (4.34) with $A^\#$ and applying the third equality in (4.1) lead to (4.39).

Substitution of (4.36) for $2k - 1$ into (4.39) leads to (4.40).

Replacing A in (4.6) with A^k in (4.18) and (4.26) leads to (4.41) and (4.42). $\square$

Finally, the author presents a multiple-dagger equality for the Moore-Penrose inverse of A^k as a generalization of (2.36) and (2.46).

Theorem 4.3. Assume that $A \in \mathbb{C}^{m \times m}$ is group invertible. Then,

$$(A^k)^\dagger = A^\dagger (X^\dagger A^\dagger)^{k-1} \quad (4.43)$$

holds for any integer $k \geq 2$, where $X = A^\dagger A^2 A^\dagger$.

Proof. Rewriting A^k as $A^k = A^{k-2}AA$, and applying (2.29) to the product lead to $(A^k)^\dagger = (A^{k-2}AA)^\dagger = (A^2)^\dagger A(A^{k-1})^\dagger$. Substitution of (2.36) into the right-hand side of the equalities and then simplifying yield $(A^k)^\dagger = A^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger A(A^{k-1})^\dagger = A^\dagger X^\dagger (A^{k-1})^\dagger$. Continuing the process by induction leads to (4.43). $\square$

5 Concluding remarks

The author elaborated a full-range of overview and analysis of the group invertibility of a matrix, and obtained several complex families of matrix equalities associated with group invertible matrices with relative ease using a blend of skillful operations and treatments of ranks, ranges, conjugate transposes, and Moore-Penrose inverses of multiple products of given matrices. The meaning of this detailed study is giving a sufficient exposition and knowledge reservation regarding group invertible matrices from view point of matrix equalities. Unquestionably, given the equivalences of different matrix equalities, we can make use of them as classification tools in the investigations of matrix expressions and matrix equalities that are composed of matrices and their generalized inverses, and therefore, we can take them as a reference and a source of inspiration for deep understanding and exploration of numerous properties and performances of group inverses of matrices and their operations.

It is expected that more specific matrix equalities (matrix equations) for a given square matrix and its algebraic operations can be constructed and then can accordingly be added in the condition lists of Theorems 3.1 and 4.2. Also, motivated by the findings in the preceding sections, it would be of interest to reconsider in depth some other fundamental problems on group inverses of matrices through merging various existing results and using some tricky matrix analysis tools. Here, the author would like to propose some proper problems that are worthy of further, in-depth study in relation to the existence of group inverses of matrices.

- (I) Taking the matrix A in the preceding sections as certain specified matrices, such as block matrices, bi-diagonal matrices, and triangular matrices (cf. [4, 6, 23–25, 13]), we are able to obtain various detailed equalities and facts in relation to group inverses of matrices. In particular, let $B \in \mathbb{C}^{m \times n}$, $C \in \mathbb{C}^{n \times m}$, and let $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{C}^{(m+n) \times (m+n)}$. In this situation, applying the results in the preceding sections to this A will lead to a variety of explicit consequences on the group inverse of the block matrix, which in turn will reveal many novel and unexpected relationships between the two matrices B and C from the viewpoint of group inverses of matrices.
- (II) Let $A \in \mathbb{C}^{m \times m}$, $P \in \mathbb{C}^{n \times m}$, and $Q \in \mathbb{C}^{m \times n}$. Then, the triple matrix product $PAQ \in \mathbb{C}^{n \times n}$ is defined. In this situation, it would be interesting to consider the relationships between the group inverses of A and

PAQ under some further assumptions, such as $r(PAQ) = r(A)$, P and Q are two invertible matrices, or $P = Q^{-1}$, etc.

- (III) The following three groups of established equivalent statements illustrate certain intrinsic connections between matrix equalities that involve the group invertibility condition of a matrix:

$$\begin{aligned} AA^\dagger &= A^\dagger A \text{ (range-Hermitian matrix)} \Leftrightarrow (A^2)^\dagger = (A^\dagger)^2 \text{ and } r(A^2) = r(A), \\ AA^* &= A^* A \text{ (normal matrix)} \Leftrightarrow (A^2)^\dagger = (A^\dagger)^2, \quad A^* A^\dagger = A^\dagger A^*, \text{ and } r(A^2) = r(A), \\ A &= A^* \text{ (Hermitian matrix)} \Leftrightarrow (A^2)^\dagger = (A^\dagger)^2, \quad A^2 A^* = A(A^*)^2, \text{ and } r(A^2) = r(A), \end{aligned}$$

where the rank equality $r(A^2) = r(A)$ is embodied on the right-hand sides of the equivalent facts (cf. [26]). Based on this fact, we can use the results in the preceding sections to derive a great abundance of necessary and sufficient conditions for a matrix to be EP, normal, and Hermitian, respectively.

- (IV) In addition to the group inverse of a square matrix defined by the unique common solution of the three matrix equations in (1.2), algebraists also defined certain weighted group inverses of a rectangular matrix under some general assumptions, and correspondingly, they considered the problems on the existence of the weighted group inverses of a matrix (cf. [27–30]). As an on-going research subject in this respect, it would be of great interest to extend the formulas, results, and facts in the preceding sections to these well-defined weighted group inverses of matrices.

Finally, the author remarks that the whole work in this article involves a wide range of derivations and simplifications of many specified algebraic equalities of matrices, and thus it is really a complex, prolonged, and exhausting task in matrix algebra to give exact descriptions and detailed classifications of various matrix equalities and their equivalent facts. The author also hopes that this study can bring positive influence to the constructions and characterizations of various complicated and tangible algebraic equalities, and that this study can make certain essential advances in analytical methodology in matrix theory and applications.

Acknowledgments: The author wishes to thank an anonymous referee for his/her helpful comments and suggestions on an earlier version of this article.

Conflict of interest: The author states that there is no conflict of interest.

References

- [1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.
- [2] D. S. Bernstein, *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas Revised and Expanded Edition*, Princeton University Press, Princeton, NJ, USA, 2018.
- [3] S. L. Campbell and C. D. Meyer Jr., *Generalized Inverses of Linear Transformations*, SIAM, Philadelphia, 2009.
- [4] C. Bu, J. Zhao, and J. Zheng, *Group inverse for a class 2×2 block matrices over skew fields*, Appl. Math. Comput. **204** (2008), 45–49.
- [5] C. Cao, Y. Wang, and Y. Sheng, *Group inverses for some 2×2 block matrices over rings*, Front. Math. China **11** (2016), 521–538.
- [6] C. Cao, H. Zhang, and Y. Ge, *Further results on the group inverse of some anti-triangular block matrices*, J. Appl. Math. Comput. **46** (2014), 169–179.
- [7] C. Cao, X. Zhang, and X. Tang, *Reverse order law of group inverses of products of two matrices*, Appl. Math. Comput. **158** (2004), 489–495.
- [8] C.-Y. Deng, *Reverse order law for the group inverses*, J. Anal. Math. Appl. **382** (2011), 663–671.
- [9] C.-Y. Deng, *On the group invertibility of operators*, Electron. J. Linear Algebra **31** (2016), 492–510.
- [10] S. K. Mitra, *On group inverses and the sharp order*, Linear Algebra Appl. **92** (1987), 17–37.
- [11] D. Mosić and D. S. Djordjević, *The reverse order law $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution*, RACSAM **109** (2015), 257–265.

- [12] P. Robert, *On the group inverse of a linear transformation*, J. Math. Anal. Appl. **22** (1968), 658–669.
- [13] Y. Sheng, Y. Ge, H. Zhang, and C. Cao, *Group inverses for a class of 2×2 block matrices over rings*, Appl. Math. Comput. **219** (2013), 9340–9346.
- [14] D. Zhang, D. Masic, and T.-Y. Tam, *On the existence of group inverses of Peirce corner matrices*, Linear Algebra Appl. **582** (2019), 482–498.
- [15] P. Basavappa, *On the solutions of the matrix equation $f(X, X^*) = g(X, X^*)$* , Canad. Math. Bull. **15** (1972), 45–49.
- [16] S. A. McCullough and L. Rodman, *Hereditary classes of operators and matrices*, Amer. Math. Monthly **104** (1997), 415–430.
- [17] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products*, Linear Algebra Appl. **211** (1994), 85–100.
- [18] Y. Tian, *Rank equalities related to outer inverses of matrices and applications*, Linear Multilinear Algebra **49** (2002), 269–288.
- [19] Y. Tian, *The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalent equalities*, J. Math. Kyoto Univ. **45** (2005), 841–850.
- [20] Y. Tian, *A family of 512 reverse order laws for generalized inverses of a matrix product: a review*, Heliyon **6** (2020), e04924.
- [21] Y. Tian, *Some mixed-type reverse-order laws for the Moore-Penrose inverse of a triple matrix product*, Rocky Mt. J. Math. **37** (2007), 1327–1347.
- [22] Y. Tian, *Miscellaneous reverse order laws and their equivalent facts for generalized inverses of a triple matrix product*, AIMS Math. **6** (2021), 13845–13886.
- [23] R. E. Hartwig, *Block generalized inverses*, Arch. Rat. Mech. Anal. **61** (1976), 197–251.
- [24] R. E. Hartwig, *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, J. Austral. Math. Soc. **24** (1977), 10–34.
- [25] P. Patrício and R. E. Hartwig, *The $(2,2,0)$ group inverse problem*, Appl. Math. Comput. **217** (2010), 516–520.
- [26] R. E. Hartwig and K. Spindelböck, *Matrices for which A^* and $A^\dagger$ can commute*, Linear Multilinear Algebra **14** (1983), 241–256.
- [27] J. Cen, *On existence of weighted group inverse of rectangular matrix (in Chinese)*, Math. Numer. Sin. **29** (2007), 39–48.
- [28] Y. Chen, *Existence conditions and expressions for weighted group inverses of rectangular matrices*, J. Nanjing Norm. Univ. Nat. Sci. Edn. **31** (2008), no. 3, 1–5.
- [29] R. E. Cline and T. N. E. Greville, *A Drazin inverse for rectangular matrices*, Linear Algebra Appl. **29** (1980), 53–62.
- [30] X. Sheng and G. Chen, *The computation and perturbation analysis for weighted group inverse of rectangular matrices*, J. Appl. Math. Comput. **31** (2008), 33–43.