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Research Article

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Infinitely many solutions for quasilinear Schrödinger equations with sign-changing nonlinearity without the aid of 4-superlinear at infinity

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Abstract: In this article, we will prove the existence of infinitely many solutions for a class of quasilinear Schrödinger equations without assuming the 4-superlinear at infinity on the nonlinearity. We achieve our goal by using the Fountain theorem.

Keywords: quasilinear Schrödinger, Fountain theorem, Cerami condition

MSC 2020: 35J20, 35J60, 35Q55

1 Introduction

In this article, we are concerned with quasilinear Schrödinger equation of the form

$$\begin{cases} -\Delta u + V(x)u - \Delta(u^2)u = \lambda g(x, u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P}$$

where $g(x, u) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is q-superlinear with $2 < q < \min(4, p)$, $2 < p < 22^*2^* = \frac{2N}{N-2}$, $N \geq 3$, λ is positive parameter, and $V(x) \in C(\mathbb{R}^N, \mathbb{R}^+) \cap L^\infty(\mathbb{R}^N)$.

We would like to mention that quasilinear Schrödinger equation (P) arises in various branches of mathematical physics and has been derived as models of several physics phenomenon see [1–3]. The quasilinear problem has been studied extensively in recent years with a huge variety of conditions on the potential V and the nonlinearity g, see, for example, [4–8] and references therein.

Alves et al. [9] assumed nonlinearity to be 4-superlinear and that the potential V is continuous and satisfies:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq V(0) > 0$, for all $x \in \mathbb{R}^N$.

(V_2) $\lim_{|x| \to +\infty} V(x) = V_\infty$ and $V(x) \leq V_\infty$, for all $x \in \mathbb{R}^N$.

Under these two conditions, they combined the variational method with perturbation arguments to obtain a solution. In [10,11], the authors obtained the multiplicity of solutions and made an assumption on V and also the nonlinearity to be 4-superlinear, where they have used variational methods. Most of the works requires the condition on the nonlinearity to be 4-superlinear in that sense $p > 4$. This condition that

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makes the application of the Fountain theorem is fluid. Indeed, if $4 < p < 22^*$, the boundedness of the Cerami sequences of the energy functional associated with the problem (P) follows by compactness embedding of $H^1(\mathbb{R}^N)$ into $L^{\frac{p}{2}}(\mathbb{R}^N)$. So, the energy functional satisfies the Cerami condition for any c. Consequently, we obtain a set of unbounded of critical values. In contrast, if the nonlinear term $g(x,u)$ changes its sign and without the aid of condition $4 < p < 22^*$, we will have difficulties to obtain the delamination of the Cerami sequences, as well as the convergence, to complete the hypotheses of the Fountain theorem [12]. To the best of our knowledge, there are only a few recent papers that deal with the nonlinearity to be not 4-superlinear and the sign-changing potential case for problem (P). Motivated by papers [10,11], in the present article, we shall consider problem (P) with more generally on the nonlinearity of $g(x,u)$. Namely, when powers of the nonlinearity are $2 < q < \min(4,p)$, $2 < p < 22^*$ and can be allowed to sign-changing. By a dual approach and Fountain theorem, we establish the existence of infinitely many solutions. We state below our hypotheses on the potential V and on the nonlinearity $g(x,u)$ in the subcritical case:

(V_1) $V \in C(\mathbb{R}^N,\mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ and $0 < V_0 = \inf_{x\in\mathbb{R}^N} V(x)$.

(V_2) $(V(x))^{-1} \in L^1(\mathbb{R}^N)$.

(G_1) $g \in C(\mathbb{R}^N \times \mathbb{R},\mathbb{R})$ such that $g(x,-u) = -g(x,u)$ for all $x,u \in \mathbb{R}^N \times \mathbb{R}$, there exit positive functions $\xi \in L^{\left(\frac{4}{q}\right)'}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $\left(\frac{4}{q}\right)'$ is the conjugate of $\frac{4}{q}$ and $\tau \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ such that

$$|g(x,t)| \le \xi(x)|t|^{q-1} + \tau(x)|t|^{p-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

(G_2) Let $G(x,t) = \int_0^t g(x,u)du$, $\lim_{t\to+\infty} \frac{|G(x,t)|}{t^q} = +\infty$ uniformly in x, $2 < q < 4$ and there exists $r_0 > 0$ such that $G(x,t) \ge 0 \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, |t| \ge r_0$.

(G_3) There exists $c_0 > 0$ and $\delta_0 > 0$ such that

$$\left(\frac{|G(x,t)|}{t^2}\right)^{\frac{p}{p-2}} \le c_0 \tilde{G}(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \setminus [-\delta_0,\delta_0],$$

where $\tilde{G}(x,t) = \frac{1}{\mu} g(x,t)t - G(x,t)$ is the positive function and $q < \mu < p$.

The main result is the following:

Theorem 1.1. *Let* $2 < p < 22^*$ *and* $2 < q < \min(4,p)$. *Suppose that* (V_1), (V_2), (G_1), (G_2), *and* (G_3) *are satisfied. Then for all* $\lambda > 0$, *the problem* (P) *has infinitely many solutions.*

Remark 1. Example of nonlinearity meets all requirements of Theorem 1.1.

Let $2 < q = 2.5 < 4$, $4 < p = 4.5 < 22^*$, and $q < \mu = 4 < p$.

$$V(x) = e^{-|x|^2} + 1 \quad \text{and} \quad g(x,u) = \left(\frac{1+\sin^2|x|}{1+|x|^N}\right)|u|^{\frac{5}{2}}u - \left(\frac{1+\cos^2|x|}{1+|x|^N}\right)|u|^{\frac{1}{2}}u.$$

We take

$$\xi(x) = \frac{2}{1+|x|^2} \in L^{\frac{8}{3}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N),$$

$$\tau(x) = \frac{1+\sin^2|x|}{1+|x|^2} \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N).$$

So,

$$|g(x,u)| \le \frac{2}{1+|x|^N}|u|^{\frac{3}{2}} + \frac{1+\sin^2|x|}{1+|x|^N}|u|^{\frac{7}{2}},$$

by the definition given for $G(x,u)$, we have

$$G(x,u) = \frac{2}{9}\left(\frac{1+\sin^2|x|}{1+|x|^N}\right)|u|^{\frac{9}{2}} - \frac{2}{5}\left(\frac{1+\cos^2|x|}{1+|x|^N}\right)|u|^{\frac{5}{2}} \ge \frac{1}{5} \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},\ |u| \ge 1,$$

as well as

$$\tilde{G}(x,u) = \frac{3}{20}\left(\frac{1+\cos^2|x|}{1+|x|^N}\right)|u|^{\frac{5}{2}} + \frac{1}{36}\left(\frac{1+\sin^2|x|}{1+|x|^N}\right)|u|^{\frac{9}{2}} \geq 0,$$

$$\frac{\left(\frac{|G(x,u)|}{|u|^2}\right)^{\frac{9}{5}}}{\tilde{G}(x,u)} \underset{\infty}{\sim} 36\left(\frac{2}{9}\right)^{\frac{9}{5}} > 0.$$

Notation. In this article, we make use of the following notation:

- $C, C_0, C_1, C_2, \ldots$ denote positive (possibly different) constants.
- For $2 \leq s < \infty$, the usual Lebesgue space is endowed with the norm

$$\|u\|_s = \|u\|_{L^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u(x)|^s \mathrm{d}x\right)^{1/s}, \quad u \in L^s(\mathbb{R}^N).$$

- $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ endowed with the norm

$$\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 + u^2 \mathrm{d}x\right)^{1/2}.$$

- S denotes the best constant that verifies

$$S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x\right)^{1/2^*}}. \tag{1}$$

- For any $s \in (2, 2^*)$, S_s denote by

$$S_s := \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{H^1(\mathbb{R}^N)}}{\|u\|_s}. \tag{2}$$

2 Reformulation of the problem

From the embedding results, it is well known that $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $2 \leq p \leq 2^*$ and is compact when $2 \leq p < 2^*$. For more general facts about the embedded, we refer the reader to the previous articles [13,14].

To find solutions of (P), we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the Euler-Lagrange functional related to problem (P) is given by $J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as follows:

$$J_\lambda(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \mathrm{d}x\right) + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla(u^2)|^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x.$$

Since,

$$\frac{1}{4}\int_{\mathbb{R}^N} |\nabla(u^2)|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \,\mathrm{d}x,$$

we obtain

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+2u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx - \lambda\int_{\mathbb{R}^N}G(x,u)dx.$$

Because of the term $\int_{\mathbb{R}^N}|u|^2|\nabla u|^2\,dx$, the functional J_λ is not defined on space $H^1(\mathbb{R}^N)$. For this, we follow the argument developed by Liu et al. [15] (see also [4]), and we change the variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{(1+2f^2(t))^{1/2}} \quad \text{on } [0,+\infty),$$
$$f(t) = -f(-t) \quad \text{on } [0,+\infty).$$

Let us assemble some properties of the change of variables $f : \mathbb{R} \to \mathbb{R}$, which will be used in the continued from the article.

Lemma 2.1. *The function $f(t)$ and its derivative satisfy the following properties*:

(*f*1) *f is uniquely defined, C^∞ and invertible*;
(*f*2) $|f'(t)| \le 1$ *for all $t \in \mathbb{R}$*;
(*f*3) $|f(t)| \le |t|$ *for all $t \in \mathbb{R}$*;
(*f*4) $\frac{f(t)}{t} \to 1$ *as $t \to 0$*;
(*f*5) $\frac{f(t)}{t^{1/2}} \to 2^{1/4}$ *as $t \to +\infty$*;
(*f*6) $f(t)/2 \le tf'(t) \le f(t)$ *for all $t > 0$*;
(*f*7) $f^2(t)/2 \le tf(t)f'(t) \le f^2(t)$ *for all $t \in \mathbb{R}$*;
(*f*8) $|f(t)| \le 2^{1/4}|t|^{1/2}$ *for all $t \in \mathbb{R}$*;
(*f*9) *There exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & \text{if } |t| \le 1,\\ C|t|^{1/2}, & \text{if } |t| \ge 1;\end{cases}$$

(f10) *For each $\alpha > 0$, there exists a positive constant $C(\alpha)$ such that*

$$|f^2(\alpha t)| \le C(\alpha)|f^2(t)|;$$

(f11) $|f(t)f'(t)| \le 1/2$.

Proof. Proofs may be found in [4,8]. ☐

Under this change the functional J_λ will be

$$I_\lambda(v) = \frac{1}{2}\left(\int_{\mathbb{R}^N}|\nabla v|^2 + V(x)f^2(v)dx\right) - \lambda\int_{\mathbb{R}^N}G(x,f(v))dx.$$

Obviously, $I_\lambda \in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and

$$\langle I'_\lambda(v), w\rangle = \int_{\mathbb{R}^N}\nabla v\nabla w dx + \int_{\mathbb{R}^N}V(x)f(v)f'(v)w dx - \lambda\int_{\mathbb{R}^N}g(x,f(v))f'(v)w dx,$$

for any $w \in H^1(\mathbb{R}^N)$.

Moreover, the critical points of I_λ are the weak solutions of the following equation:

$$-\Delta v = \frac{1}{\sqrt{1+2|f(v)|^2}}(g(x,f(v)) - V(x)f(v)) \quad \text{in } \mathbb{R}^N. \tag{3}$$

Considering that if v is a critical point of the functional I_λ, so $u = f(v)$ is a critical point of the functional J_λ, that is, $u = f(v)$ is a solution of problem (P).

3 The boundedness of the Cerami sequences

Lemma 3.1. *Let $r_0 > 0$. There exists $k_0 > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)f^2(v)\mathrm{d}x \ge k_0\|v\|^2_{H^1(\mathbb{R}^N)}, \quad \forall v \in H^1(\mathbb{R}^N) : \|v\|_{H^1(\mathbb{R}^N)} \le r_0.$$

Proof. The proof of analogous results can be found in [4,8]. □

Definition 1. Let X be a Banach space with the norm $\|\cdot\|$ and $F : X \to \mathbb{R}$ a function.

A Gâteaux differentiable function F satisfies the Cerami condition locally at c ($(C)_c$-condition for short) if any sequence $\{v_n\} \subset X$ such that $F(v_n) \to c$ and $(1 + \|v_n\|)F'(v_n) \to 0$, as $n \to +\infty$, has a convergent subsequence.

The first step for the $(C)_c$-condition to hold is bounded.

Lemma 3.2. *Assume that V and $G(x, v)$ satisfy (V_1), (V_2), (G_1), (G_2), (G_3), and $\{v_n\}$ is $(C)_c$-condition for I_λ, then $\{v_n\}$ is bounded in X for any $\lambda > 0$.*

Proof. We have

$$I_\lambda(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|_{H^1(\mathbb{R}^N)})I'_\lambda(v_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \tag{4}$$

We claim that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + V(x)f^2(v_n)\mathrm{d}x := \|v_n\|_f^2 \le C. \tag{5}$$

Next we use Lemma 3.1 to deduce that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Assume by contradiction that $\|v_n\|_f \to +\infty$.

We have from (4)

$$c + o_n(1) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n))\mathrm{d}x - \lambda\int_{\mathbb{R}^N} G(x, f(v_n))\mathrm{d}x,$$

so,

$$\int_{\mathbb{R}^N} G(x, f(v_n))\mathrm{d}x = -\frac{c + o_n(1)}{\lambda} + \frac{1}{2\lambda}\|v_n\|_f^2$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} \frac{G(x, f(v_n))}{\|v_n\|_f^2}\mathrm{d}x = \frac{1}{2\lambda} > 0. \tag{6}$$

Then

$$0 < \frac{1}{2\lambda} = \lim_{n\to+\infty}\left|\int_{\mathbb{R}^N} \frac{G(x, f(v_n))}{\|v_n\|_f^2}\mathrm{d}x\right| \le \lim_{n\to+\infty}\int_{\mathbb{R}^N} \frac{|G(x, f(v_n))|}{\|v_n\|_f^2}\mathrm{d}x. \tag{7}$$

Let $\omega_n = \frac{f(v_n)}{\|v_n\|_f}$. Clearly, $\|\omega_n\|_{H^1(\mathbb{R}^N)} \le 1$.

Up to a subsequence, we may assume that

$$\omega_n \rightharpoonup \omega \quad \text{in } H^1(\mathbb{R}^N).$$

Because $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is compact for all $2 \le r < 2^*$, this implies

$$\omega_n \to \omega \quad \text{in } L^r(\mathbb{R}^N),\ 2 \le r < 2^*.$$

$$\omega_n(x) \to \omega(x) \quad \text{a.e on } \mathbb{R}^N.$$

Since $\mu \ge 4$ and (4), this implies that

$$\begin{aligned}
c + 1 \ge I_\lambda(v_n) - \frac{1}{\mu}\left\langle I'_\lambda(v_n), \frac{f(v_n)}{f'(v_n)}\right\rangle \\
= \int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{1}{\mu}\left(\frac{1+4f^2(v_n)}{1+2f^2(v_n)}\right)\right)|\nabla v_n|^2 \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)f^2(v_n)\mathrm{d}x \\
+ \lambda \int_{\mathbb{R}^N} \frac{1}{\mu}g(x, f(v_n))f(v_n) - G(x, f(v_n))\mathrm{d}x \\
\ge \lambda \int_{\mathbb{R}^N} \tilde{G}(x, f(v_n))\mathrm{d}x,
\end{aligned}$$

then

$$\int_{\mathbb{R}^N} \tilde{G}(x, f(v_n))\mathrm{d}x \le \frac{c+1}{\lambda}. \tag{8}$$

Let us consider

$$\Omega_n(a, b) = \{x \in \mathbb{R}^N / 0 < a \le |f(v_n)| < b\}.$$

By (G_2) and (G_3), there exists $\delta_2 > 0$ such that

$$\tilde{G}(x, f(v_n(x))) > \frac{C^{\frac{p}{p-2}}}{c_0} r^{(q-2)\left(\frac{p}{p-2}\right)}, \quad \forall x \in \Omega_n(r, \infty), \quad \forall r > \delta_2.$$

Moreover, by (8), we can write

$$0 \le \frac{C^{\frac{p}{p-2}}}{c_0} r^{(q-2)\left(\frac{p}{p-2}\right)} \operatorname{meas}(\Omega_n(r, \infty)) \le \int_{\Omega_n(r,\infty)} \tilde{G}(x, f(v_n))\mathrm{d}x \le \int_{\mathbb{R}^N} \tilde{G}(x, f(v_n))\mathrm{d}x \le \frac{c+1}{\lambda}.$$

Thus, this implies that

$$0 \le \operatorname{meas}(\Omega_n(r, \infty)) \le \frac{c_0(c+1)}{\lambda C^{\frac{p}{p-2}} r^{(q-2)\left(\frac{p}{p-2}\right)}}.$$

Then

$$\operatorname{meas}(\Omega_n(r, \infty)) \to 0 \text{ as } r \to +\infty. \tag{9}$$

We have $1 < 2 < 2^*$, there exists $0 < \xi < 1$ such that $2 = \xi + 2^*(1 - \xi)$. By (f_4), the Hölder inequality and Sobolev inequality, thus, we can write

$$\begin{aligned}
&\int_{\Omega_n(r,\infty)\cap(v_n \ge 1)} |f^2(v_n)|\mathrm{d}x \\
&\le \left(\int_{\Omega_n(r,\infty)\cap(v_n \ge 1)} |f(v_n)|\mathrm{d}x\right)^{\xi} \left(\int_{\Omega_n(r,\infty)\cap(v_n \ge 1)} |f(v_n)|^{2^*}\mathrm{d}x\right)^{1-\xi} \\
&\le \left(\int_{\Omega_n(r,\infty)\cap(v_n \ge 1)} V^{-1}(x)\mathrm{d}x\right)^{\frac{\xi}{2}} \left(\int_{\Omega_n(r,\infty)\cap(v_n \ge 1)} V(x)f^2(v_n(x))\mathrm{d}x\right)^{\frac{\xi}{2}} \left(\int_{\Omega_n(r,\infty)\cap(v_n \ge 1)} |v_n|^{2^*}\mathrm{d}x\right)^{1-\xi}
\end{aligned}$$

$$\begin{aligned}
&\leq V_0^{-\frac{\xi}{2}}(\text{meas}(\Omega_n(r,\infty)))^{\frac{\xi}{2}}\|v_n\|_f^{\xi} S^{-2^*(1-\xi)}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^{\frac{2^*(1-\xi)}{2}}\\
&\leq V_0^{-\frac{\xi}{2}}(\text{meas}(\Omega_n(r,\infty)))^{\frac{\xi}{2}}\|v_n\|_f^{\xi} S^{-2^*(1-\xi)}\|v_n\|_f^{2^*(1-\xi)}\\
&\leq V_0^{-\frac{\xi}{2}}(\text{meas}(\Omega_n(r,\infty)))^{\frac{\xi}{2}}\|v_n\|_f^{2}.
\end{aligned} \tag{10}$$

By using the fact that $(a+b)^\kappa \leq a^\kappa + b^\kappa$, if $a, b \geq 0$ and $\kappa \in [0,1]$, (f_4) and (10), we obtain

$$\begin{aligned}
\left(\int_{\Omega_n(r,\infty)}|v_n|\mathrm{d}x\right)^{\kappa} &= \left(\int_{\Omega_n(r,\infty)\cap(|v_n|<1)}|v_n|\mathrm{d}x + \int_{\Omega_n(r,\infty)\cap(|v_n|\geq 1)}|v_n|\mathrm{d}x\right)^{\kappa}\\
&\leq (\text{meas}(\Omega_n(r,\infty)))^{\kappa} + \frac{1}{C^\kappa}\left(\int_{\Omega_n(r,\infty)\cap(|v_n|\geq 1)}|f^2(v_n)|\mathrm{d}x\right)^{\kappa}\\
&\leq (\text{meas}(\Omega_n(r,\infty)))^{\kappa} + \frac{V_0^{-\frac{\xi\kappa}{2}}}{C^\kappa}(\text{meas}(\Omega_n(r,\infty)))^{\frac{\xi\kappa}{2}}\|v_n\|_f^{2\kappa}.
\end{aligned} \tag{11}$$

Since $2 < p < 22^*$, there exists $0 < \theta < 1$ such that $\frac{p}{2} = \theta + 2^*(1-\theta)$.

By using (11) and considering $\kappa = \frac{2}{p}$, we can write

$$\begin{aligned}
&\frac{1}{\|v_n\|_f^2},\left(\int_{\Omega_n(r,\infty)}|v_n|^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{2}{p}}\\
&\leq \frac{1}{\|v_n\|_f^2}\left(\int_{\Omega_n(r,\infty)}|v_n|\mathrm{d}x\right)^{\frac{2\theta}{p}}\left(\int_{\Omega_n(r,\infty)}|v_n|^{2^*}\mathrm{d}x\right)^{\frac{2(1-\theta)}{p}}\\
&\leq \frac{S^{-\frac{22^*(1-\theta)}{p}}}{\|v_n\|_f^2}\left(\int_{\Omega_n(r,\infty)}|v_n|\mathrm{d}x\right)^{\frac{2\theta}{p}}\left(\int_{\Omega_n(r,\infty)}|\nabla v_n|^2\mathrm{d}x\right)^{\frac{2^*(1-\theta)}{p}}\\
&\leq \frac{S^{-\frac{22^*(1-\theta)}{p}}}{\|v_n\|_f^{1+\frac{2\theta}{p}}}(\text{meas}(\Omega_n(r,\infty)))^{\frac{2\theta}{p}} + \frac{S^{-\frac{22^*(1-\theta)}{p}}V_0^{-\frac{\xi\theta}{p}}C^{-\frac{2\theta}{p}}}{\|v_n\|_f}(\text{meas}(\Omega_n(r,\infty)))^{\frac{\xi\theta}{p}}.
\end{aligned} \tag{12}$$

Thus, it follows from $\|v_n\|_f > 1$, (G_3), (8), (9), and (12) that

$$\begin{aligned}
&\int_{\Omega_n(r,\infty)}\frac{|G(x,f(v_n))|}{\|v_n\|_f^2}\mathrm{d}x\\
&= \left(\int_{\mathbb{R}^N}\left(\frac{|G(x,f(v_n))|}{f^2(v_n)}\right)^{\frac{p}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{\Omega_n(r,\infty)}|f(v_n)|^p\mathrm{d}x\right)^{\frac{2}{p}}\frac{1}{\|v_n\|_f^2}\\
&\leq 2^{1/4}c_0^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^N}\tilde{G}(x,f(v_n))\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{\Omega_n(r,\infty)}|v_n|^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{2}{p}}\frac{1}{\|v_n\|_f^2}
\end{aligned} \tag{13}$$

$$
\begin{aligned}
&\leq \frac{2^{1/4} c_0^{\frac{p-2}{p}} S^{-\frac{22^*(1-\theta)}{p}}}{\|v_n\|_f^{1+\frac{2\theta}{p}}} \left(\frac{c+1}{\lambda}\right)^{\frac{p-2}{p}} (\operatorname{meas}(\Omega_n(r,\infty)))^{\frac{2\theta}{p}} \\
&\quad + \frac{2^{1/4} c_0^{\frac{p-2}{p}} V_0^{-\frac{\xi\theta}{p}} C^{-\frac{2\theta}{p}} S^{-\frac{22^*(1-\theta)}{p}}}{\|v_n\|_f} \left(\frac{c+1}{\lambda}\right)^{\frac{p-2}{p}} (\operatorname{meas}(\Omega_n(r,\infty)))^{\frac{\xi\theta}{p}} \\
&\leq 2^{1/4} c_0^{\frac{p-2}{p}} S^{-\frac{22^*(1-\theta)}{p}} \left(\frac{c+1}{\lambda}\right)^{\frac{p-2}{p}} (\operatorname{meas}(\Omega_n(r,\infty)))^{\frac{2\theta}{p}} \\
&\quad + 2^{1/4} c_0^{\frac{p-2}{p}} V_0^{-\frac{\xi\theta}{p}} C^{-\frac{2\theta}{p}} S^{-\frac{22^*(1-\theta)}{p}} \left(\frac{c+1}{\lambda}\right)^{\frac{p-2}{p}} (\operatorname{meas}(\Omega_n(r,\infty)))^{\frac{\xi\theta}{p}} \to 0,
\end{aligned}
\tag{13}
$$

as $r \to \infty$ uniformly in n.

By (G_2) and $\|\omega_n\|_{L^2(\mathbb{R}^N)} \to \|\omega\|_{L^2(\mathbb{R}^N)}$, as $n \to +\infty$, we find that for fix $r > 0$

$$
\begin{aligned}
&\int\limits_{\Omega_n(0,r)} \frac{|G(x, f(v_n))|}{\|v_n\|_f^2} \mathrm{d}x \\
&\leq \int\limits_{\Omega_n(0,r)} \frac{\frac{\xi(x)}{q}|f(v_n)|^q + \frac{\tau(x)}{p}|f(v_n)|^p}{\|v_n\|_f^2} \mathrm{d}x \\
&\leq \frac{r^{q-1}}{q\|v_n\|_f} \|\xi\|_{L^2(\mathbb{R}^N)} \|\omega_n\|_{L^2(\mathbb{R}^N)} + \frac{r^{p-1}}{p\|v_n\|_f} \|\tau\|_{L^2(\mathbb{R}^N)} \|\omega_n\|_{L^2(\mathbb{R}^N)} \to 0, \quad \text{as } n \to \infty.
\end{aligned}
\tag{14}
$$

Combining (7), (13), and (14), we have

$$
0 < \frac{1}{2\lambda} \leq \int\limits_{\mathbb{R}^N} \frac{|G(x, f(v_n))|}{\|v_n\|_f^2} \mathrm{d}x = \int\limits_{\Omega_n(0,r)} \frac{|G(x, f(v_n))|}{\|v_n\|_f^2} \mathrm{d}x + \int\limits_{\Omega_n(r,\infty)} \frac{|G(x, f(v_n))|}{\|v_n\|_f^2} \mathrm{d}x \to 0, \quad \text{as } n, r \to \infty.
$$

This is inconsistent with (6). Hence, there exists $M > 0$ such that $\|v_n\|_f \leq M$ for all $n \in \mathbb{N}$. Then by Lemma 3.1, the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Lemma 3.3. *Assume that V and $G(x, v)$ satisfy (V_1), (V_2), (G_1), (G_2), and (G_3). Then, the functional I_λ satisfies the $(C)_c$-condition for all $c > 0$ and for all $\lambda > 0$.*

Proof. Consider a $(C)_c$-sequence $\{v_n\}$ for I_λ, that is,

$$
I_\lambda(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|_{H^1(\mathbb{R}^N)}) I'_\lambda(v_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).
$$

From Lemma 3.2 $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Going if necessary to a subsequence, we can assume that

$$
\begin{cases}
v_n \rightharpoonup v, & \text{in } H^1(\mathbb{R}^N), \\
v_n \to v, & \text{in } L^r(\mathbb{R}^N),\ 2 \leq r < 2^*.
\end{cases}
\tag{15}
$$

From (V_2), we conclude that $\int_{|x|>R} V^{-1}\mathrm{d}x \to 0$ as $R \to +\infty$.

$$
\begin{aligned}
\int\limits_{|x|>R} |v_n - v|\mathrm{d}x &\leq \left(\int\limits_{|x|>R} V^{-1}\mathrm{d}x\right)^{1/2} \left(\int\limits_{|x|>R} V(x)|v_n - v|^2\mathrm{d}x\right)^{1/2} \\
&\leq \left(\int\limits_{|x|>R} V^{-1}\mathrm{d}x\right)^{1/2} \|v_n - v\|_X \\
&\leq M\left(\int\limits_{|x|>R} V^{-1}\mathrm{d}x\right)^{1/2} \to 0, \quad \text{as } R \to +\infty \text{ uniformly in } n.
\end{aligned}
\tag{16}
$$

Moreover, since $v_n \to v$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, $\int_{|x|\le R} |v_n - v|\mathrm{d}x \to 0$, as $n \to +\infty$, which implies that

$$\int_{\mathbb{R}^N} |v_n - v|\mathrm{d}x \to 0, \quad \text{as } n \to +\infty. \tag{17}$$

By using (17), $\frac{p}{2} = \theta + 2^*(1-\theta)$ for some $0 < \theta < 1$ and $\|v_n - v\|^{2^*(1-\theta)}_{H^1(\mathbb{R}^N)} \le M$ for some $M > 0$, we can write

$$\begin{aligned}
\|v_n - v\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |v_n - v|^{\frac{p}{2}}\mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^N} |v_n - v|\mathrm{d}x\right)^{\theta}\left(\int_{\mathbb{R}^N} |v_n - v|^{2^*}\mathrm{d}x\right)^{1-\theta} \\
&\le S^{-2^*(1-\theta)}\left(\int_{\mathbb{R}^N} |v_n - v|\mathrm{d}x\right)^{\theta}\left(\int_{\mathbb{R}^N} |\nabla(v_n - v)|^2\mathrm{d}x\right)^{\frac{2^*(1-\theta)}{2}} \\
&\le S^{-2^*(1-\theta)}\left(\int_{\mathbb{R}^N} |v_n - v|\mathrm{d}x\right)^{\theta}\|v_n - v\|^{2^*(1-\theta)}_{H^1(\mathbb{R}^N)} \\
&\le MS^{-2^*(1-\theta)}\left(\int_{\mathbb{R}^N} |v_n - v|\mathrm{d}x\right)^{\theta} \to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{18}$$

With similar arguments in Lemma 3.1, we have

There exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 + V(x)(f(v_n)f'(v_n) - f(v)f'(v))(v_n - v)\mathrm{d}x \ge C\|v_n - v\|^2_{H^1(\mathbb{R}^N)}. \tag{19}$$

Because $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $I'_\lambda(v_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n \to +\infty$, we have

$$\langle I'_\lambda(v_n) - I'_\lambda(v), v_n - v\rangle \to 0 \text{ as } n \to +\infty.$$

By the Hölder inequality, (15), (18), (19), (f_2), (G_1), $2 < p < 22^*$, and $2 < q < \min(4, p)$, we can write

$$\begin{aligned}
o_n(1) &= \langle I'_\lambda(v_n) - I'_\lambda(v), v_n - v\rangle \\
&= \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 + V(x)(f(v_n)f'(v_n) - f(v)f'(v))(v_n - v)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} (g(x, f(v_n))f'(v_n) - g(x, f(v))f'(v))(v_n \\
&\quad - v)\mathrm{d}x \\
&\ge C\|v_n - v\|^2_{H^1(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} (\xi(x)|f(v_n)|^{q-1} + \tau(x)|f(v_n)|^{p-1} + \xi(x)|f(v)|^{q-1} + \tau(x)|f(v)|^{p-1})|v_n - v|\mathrm{d}x \\
&\ge C\|v_n - v\|^2_{H^1(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} \left(\xi(x)|v_n|^{\frac{q}{2}-1} + \tau(x)|v_n|^{\frac{p}{2}-1} + \xi(x)|v|^{\frac{q}{2}-1} + \tau(x)|v|^{\frac{p}{2}-1}\right)|v_n - v|\mathrm{d}x \\
&\ge C\|v_n - v\|^2_{H^1(\mathbb{R}^N)} - \lambda\|\xi\|_{L^{\left(\frac{4}{q}\right)'}(\mathbb{R}^N)}\left(\|v_n\|^{\frac{q-2}{2}}_{L^2(\mathbb{R}^N)} + \|v\|^{\frac{q-2}{2}}_{L^2(\mathbb{R}^N)}\right)\|v_n - v\|_{L^2(\mathbb{R}^N)} \\
&\quad - \lambda\|\tau\|_\infty\left(\|v_n\|^{\frac{p-2}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^N)} + \|v\|^{\frac{p-2}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^N)}\right)\|v_n - v\|_{L^{\frac{p}{2}}(\mathbb{R}^N)} \\
&\ge k_0\|v_n - v\|^2_{H^1(\mathbb{R}^N)} + o_n(1).
\end{aligned}$$

Therefore, $\|v_n - v\|_{H^1(\mathbb{R}^N)} \to 0$ as $n \to +\infty$. Hence, I_λ satisfies the $(C)_c$-condition for any $c > 0$. This completes the proof. □

4 Proof of the main result

In this section, we show the existence of infinitely many solutions via the Fountain theorem [12].

Let X be a Banach and separable. Since X is separable (see [16]), there exist $\{e_n\}_{n\in\mathbb{N}} \subset X$ and $\{f_n^*\}_{n\in\mathbb{N}} \subset X^*$ with

$$X = \overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}}, \quad X^* = \overline{\operatorname{span}\{f_n^*\}_{n=1}^{\infty}}$$

$$\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\langle , \rangle$ is the duality pairing between X and X^*.

$$\text{Let } X_j = \operatorname{span}\{e_j\}, \quad Y_n = \bigoplus_{j=0}^{n} X_j, \quad Z_n = \overline{\bigoplus_{j=n}^{\infty} X_j}.$$

Lemma 4.1. ([12] Fountain theorem). *Consider an even functional $I_\lambda \in C(X, \mathbb{R})$. If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(1) $a_k = \max\{I_\lambda(v) : v \in Y_k, \|v\|_X = \rho_k\} \leq 0$,

(2) $b_k = \inf\{I_\lambda(v) : v \in Z_k, \|v\|_X = \rho_k\} \to \infty$ as $k \to +\infty$,

(3) *I_λ satisfies the $(C)_c$-condition for every $c > 0$.*

Then I_λ has an unbounded sequence of critical values.

Proof of Theorem 1.1. The functional I_λ is even, $I_\lambda \in C(H^1(\mathbb{R}^N), \mathbb{R})$. By Lemma 3.3 I_λ satisfies the $(C)_c$-condition for every $c > 0$. We only need to verify I_λ satisfying (1) and (2) of Lemma 4.1.

For $k \geq 1$. Denote

$$\alpha_k = \sup_{v\in Z_k, \|v\|_{H^1(\mathbb{R}^N)}=1} \|v\|_{L^{\frac{p}{2}}(\mathbb{R}^N)}.$$

Then we have $\alpha_k \to 0$ as $k \to +\infty$. In fact, suppose to the contrary that there exist $\varepsilon_0 > 0, k_0 \in \mathbb{N}$ and the sequence $\{v_k\}$ in Z_k such that $\|v_k\|_{H^1(\mathbb{R}^N)} = 1$ and $\|v_k\|_{L^{\frac{p}{2}}(\mathbb{R}^N)} \geq \varepsilon_0$ for all $k \geq k_0$. Since the sequence $\{v_k\}$ is bounded in $H^1(\mathbb{R}^N)$, there exist $v \in H^1(\mathbb{R}^N)$ such that $v_k \rightharpoonup v$ in X as $n \to +\infty$ and

$$\langle f_j, v\rangle = \lim_{k\to+\infty} \langle f_j, v_k\rangle = 0 \quad \text{for } j = 1, 2, \ldots.$$

Hence, we obtain $v = 0$. However, we have

$$\varepsilon_0 \leq \lim_{k\to+\infty} \|v_k\|_{L^{\frac{p}{2}}(\mathbb{R}^N)} = \|v\|_{L^{\frac{p}{2}}(\mathbb{R}^N)} = 0,$$

which provides a contradiction.

Fix $r > 0$.

Since $\alpha_k \to 0$ as $k \to +\infty$,

$$\exists\, n_0 > 0 : \forall k \geq n_0 0 < \alpha_k < \left(\frac{k_0 S_{\frac{p}{2}}^{\frac{p}{2}}}{2^{1/4}\lambda\|\tau\|_\infty}\right)^{\frac{2}{p}} r^{\frac{4-p}{p}}. \tag{20}$$

For all $v \in Z_k$ such that $\|v\|_{H^1(\mathbb{R}^N)} = r$. It follows from the assumptions (G_1), (f_8), $2 < p < 22^*$, $2 < q < \min(4, p)$, and Lemma 3.1, we have

$$
\begin{aligned}
I_\lambda(v) &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)f^2(v)\mathrm{d}x - \frac{2^{1/4}\lambda}{q}\int_{\mathbb{R}^N} \xi(x)|v|^{\frac{q}{2}}\mathrm{d}x - \frac{2^{1/4}\lambda}{p}\int_{\mathbb{R}^N} \tau(x)|v|^{\frac{p}{2}}\mathrm{d}x \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)f^2(v)\mathrm{d}x - \frac{2^{1/4}\lambda}{q}\|\xi\|_{L^{(\frac{4}{q})'}(\mathbb{R}^N)}\|v\|^{\frac{q}{2}}_{L^2(\mathbb{R}^N)} - \frac{2^{1/4}\lambda}{p}\|\tau\|_\infty\|v\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^N)} \\
&\geq \frac{1}{2}k_0\|v\|^2_{H^1(\mathbb{R}^N)} - \frac{2^{1/4}\lambda}{q}S_2^{-\frac{q}{2}}\|\xi\|_{L^{(\frac{4}{q})'}(\mathbb{R}^N)}\|v\|^{\frac{q}{2}}_{H^1(\mathbb{R}^N)} - \frac{2^{1/4}\lambda S_{\frac{p}{2}}^{-\frac{p}{2}}}{p}\alpha_k^{\frac{p}{2}}\|\tau\|_\infty\|v\|^{\frac{p}{2}}_{H^1(\mathbb{R}^N)} \\
&\geq \frac{1}{2}k_0 r^2 - \frac{2^{1/4}\lambda}{q}S_2^{-\frac{q}{2}}\|\xi\|_{L^{(\frac{4}{q})'}(\mathbb{R}^N)} r^{\frac{q}{2}} - \frac{2^{1/4}\lambda S_{\frac{p}{2}}^{-\frac{p}{2}}}{p}\alpha_k^{\frac{p}{2}}\|\tau\|_\infty r^{\frac{p}{2}} \\
&\geq k_0\left(\frac{1}{2} - \frac{1}{p}\right)r^2 - \frac{2^{1/4}\lambda}{q}S_2^{-\frac{q}{2}}\|\xi\|_{L^{(\frac{4}{q})'}(\mathbb{R}^N)} r^{\frac{q}{2}} \to +\infty, \quad \text{as } r \to +\infty,
\end{aligned}
$$

which implies (2).

For the first item (1), let us suppose that this condition is not meant for a certain k. So, there exists a sequence $\{v_n\}$ in Y_k such that

$$
\|v_n\|_{H^1(\mathbb{R}^N)} \to +\infty, \quad \text{and} \quad I_\lambda(v_n) \geq 0. \tag{21}
$$

We consider $\omega_n = \frac{f(v_n)}{\|v_n\|_{H^1(\mathbb{R}^N)}}$. Then it is obvious that $\|\omega_n\|_{H^1(\mathbb{R}^N)} \leq 1$. As Y_k is of finite dimension, there exists $\omega \in Y_k$ with $\omega \neq 0$ such that

$$
\omega_n \to \omega \quad \text{strongly in} \quad Y_k \quad \text{and} \quad w_n(x) \to w(x) \quad \text{a.e on} \quad \mathbb{R}^N \quad \text{as} \quad n \to +\infty.
$$

Let

$$
\Omega_2 = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}.
$$

Then $\operatorname{meas}(\Omega_2) > 0$.

For all $x \in \Omega_2$, we have

$$
|f(v_n(x))| = |\omega_n(x)|\|v_n\|_{H^1(\mathbb{R}^N)} \to +\infty, \quad \text{as} \quad n \to +\infty.
$$

For $r_0 > 0$, we have $\Omega_2 \subset \Omega_n(r_0, \infty)$ for large n.

Hence, for all $x \in \Omega_2$ and by (G_2), we have

$$
\lim_{n\to+\infty} \frac{G(x, f(v_n))}{\|v_n\|^2_{H^1(\mathbb{R}^N)}} = \lim_{n\to+\infty} \frac{G(x, f(v_n))}{f^2(v_n(x))}|w_n(x)|^2\mathrm{d}x = \lim_{n\to+\infty} \frac{G(x, f(v_n))}{|f|(v_n(x))|^q}|w_n(x)|^{q-2}\mathrm{d}x = +\infty.
$$

Thus, it follows from (14), (21), and Fatou's lemma that

$$
\begin{aligned}
0 &\leq \liminf_{n\to+\infty} \frac{I_\lambda(v_n)}{\|v_n\|^2_{H^1(\mathbb{R}^N)}} \\
&\leq \frac{1}{2} - \liminf_{n\to+\infty} \lambda\left(\frac{1}{\|v_n\|^2_X}\int_{\Omega_n(0,r_0)} G(x, f(v_n))\mathrm{d}x - \int_{\Omega_n(r_0,\infty)} \frac{G(x, f(v_n))}{f^2(v_n)}|\omega_n|^2\mathrm{d}x\right) \\
&\leq \frac{1}{2} + \frac{r_0^{q-1}}{q}\|\xi\|_{L^2(\mathbb{R}^N)}\|\omega\|_{L^2(\mathbb{R}^N)} + \frac{r_0^{p-1}}{p}\|\tau\|_{L^2(\mathbb{R}^N)}\|\omega\|_{L^2(\mathbb{R}^N)} - \lambda\int_{\mathbb{R}^N} \liminf_{n\to+\infty} \frac{G(x, f(v_n))}{f^2(v_n)}\chi_{\Omega_n(r_0,\infty)}|\omega_n|^2\mathrm{d}x = -\infty,
\end{aligned}
$$

which provides a contradiction. So, I_λ satisfies (2). All the assumptions of Lemma 4.1 are satisfied. Therefore, this concludes the proof of Theorem 1.1. □

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