

Research Article

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Duality for convolution on subclasses of analytic functions and weighted integral operators

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Abstract: In this article, we investigate a class of analytic functions defined on the unit open disc $\mathcal{U} = \{z : |z| < 1\}$, such that for every $f \in \mathcal{P}_\alpha(\beta, \gamma)$, $\alpha > 0$, $0 \leq \beta \leq 1$, $0 < \gamma \leq 1$, and $|z| < 1$, the inequality

$$\operatorname{Re} \left\{ \frac{f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z) - \beta}{1-\beta} \right\} > 0$$

holds. We find conditions on the numbers α , β , and γ such that $\mathcal{P}_\alpha(\beta, \gamma) \subseteq SP(\lambda)$, for $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $SP(\lambda)$ denotes the set of all λ -spirallike functions. We also make use of Ruscheweyh's duality theory to derive conditions on the numbers α , β , γ and the real-valued function φ so that the integral operator $V_\varphi(f)$ maps the set $\mathcal{P}_\alpha(\beta, \gamma)$ into the set $SP(\lambda)$, provided φ is non-negative normalized function ($\int_0^1 \varphi(t) dt = 1$) and

$$V_\varphi(f)(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t} dt.$$

Keywords: convolution, Hadamard product, duality principle, λ -spirallike functions, dual set, Gaussian hypergeometric function

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1 Introduction

Let $\mathcal{A}$ be the set of analytic functions defined on the open unit disc $\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}$ possessing the property that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and λ be a real number. Then, by $SP(\lambda)$ we denote the subclass of $\mathcal{A}$ of all λ -spirallike functions for which $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (see, e.g., [1–6]). Every analytic function f in $\mathcal{A}$ is a convex λ -spirallike function on $\mathcal{U}$ if and only if $zf'(z)$ is a λ -spirallike function on $\mathcal{U}$. For $\alpha > 0$, $0 \leq \beta \leq 1$, and $0 < \gamma \leq 1$, we denote by $\mathcal{P}_\alpha(\beta, \gamma)$ the set of all functions f in $\mathcal{A}$ provided

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$$\operatorname{Re} \left\{ \frac{f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z) - \beta}{1-\beta} \right\} > 0 \quad \text{for all } z \in \mathcal{U}.$$

For f and g in $\mathcal{A}$ (or $\mathcal{A}_0$, $\mathcal{A}_0 = \{g : g(z) = \frac{f(z)}{z}; f \in \mathcal{A}\}$), $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the convolution product (Hadamard product) of f and g , denoted by $f * g$, is a function in $\mathcal{A}$ (or $\mathcal{A}_0$) defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}.$$

For a set $\mathcal{V} \subseteq \mathcal{A}_0$, the dual set $\mathcal{V}^*$ is defined as $\mathcal{V}^* = \{g \in \mathcal{A}_0 : (f * g)(z) \neq 0, \forall f \in \mathcal{V}, z \in \mathcal{U}\}$. The second dual or the dual hull of $\mathcal{V}$ is defined by $\mathcal{V}^{**} = (\mathcal{V}^*)^*$. Indeed, we have, $\mathcal{V} \subseteq \mathcal{V}^{**}$ (see, e.g., [7] and [8]). Let $(x)_k$ denote the Pochhammer symbol [6]

$$(x)_0 = 1 \quad \text{and} \quad (x)_k = x(x+1)\dots(x+k-1), \quad \text{for } k \in \mathbb{N}.$$

Then, the Gaussian hypergeometric function is defined by ([1–27])

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.$$

The classical theory of integral transforms and their applications have been studied for a long time, and they are applied in many fields of mathematics [28–32]. Several integral transforms are extended to various spaces of distributions [29], tempered distributions [33], distributions of compact support [34], ultradistributions [35], and many others. In a classical sense, Fourier and Ruscheweyh introduced the integral transform $V_\varphi : \mathcal{A} \longrightarrow \mathcal{A}$,

$$V_\varphi(f)(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t} dt,$$

where φ is a real-valued integrable function satisfying the normalizing condition [4]

$$\int_0^1 \varphi(t) dt = 1.$$

In the literature, the integral operator $V_\varphi(f)$ has been discussed by many authors on various choices of φ (see, e.g., [9, 11, 12, 36, 37]). In what follows, we introduce the function $g := g_{\alpha, \gamma}^\lambda$ as a solution to the differential equation

$$g(t) + \frac{1-\gamma}{\alpha\gamma} t g'(t) = \frac{2 - e^{-i\lambda} - t e^{-i\lambda}}{e^{-i\lambda}(1+t)} - \frac{2(1 - e^{-i\lambda}) \log(1+t)}{e^{-i\lambda} t},$$

which can be expressed in terms of an integral equation as

$$g(t) = \frac{\alpha\gamma}{1-\gamma} \int_0^1 s^{\frac{\alpha\gamma}{1-\gamma}-1} \left(\frac{2 - e^{-i\lambda} - t e^{-i\lambda}}{e^{-i\lambda}(1+t)} - \frac{2(1 - e^{-i\lambda}) \log(1+t)}{e^{-i\lambda} t} \right) ds. \quad (1)$$

However, we find conditions on α, β , and γ so that $\mathcal{P}_\alpha(\beta, \gamma) \subseteq \operatorname{SP}(\lambda)$, for given $|\lambda| < \frac{\pi}{2}$. We refer to the monographs [38] and [10] for more details on a variety of sufficient conditions on the λ -spirallike functions. For detailed analysis of various integral operators we refer readers to [14–19] and references cited therein. In Section 2, we present several lemmas which simplify our results. Section 3 is devoted for our main results and applications. One more result establishes the inclusion $\mathcal{P}_\alpha(\beta, \gamma) \subseteq \operatorname{SP}(\lambda)$ for $\alpha > 0, 0 \leq \beta < 1, 0 < \gamma < 1, \lambda$ being real number but $|\lambda| < \frac{\pi}{2}$. In another conclusion, we impose conditions on the set $\mathcal{P}_\alpha(\beta, \gamma)$ to be univalent. Several remarks, corollaries, and theorems are also derived in some detail.

2 Preliminary lemmas

The following are preliminary lemmas which are very useful in our next analysis.

Lemma 2.1. (Duality principle, see [8]). *Let $\mathcal{V} \subseteq \mathcal{A}_0$ be a compact subset having the property*

$$f \in \mathcal{V} \Rightarrow \forall |x| \leq 1 : f_x \in \mathcal{V}, f_x(z) = f(xz). \quad (2)$$

*Then, for all continuous linear functionals φ on $\mathcal{A}$ we have $\varphi(\mathcal{V}) = \varphi(\mathcal{V}^{**})$ and $\bar{\text{co}}(\mathcal{V}) \subseteq \bar{\text{co}}(\mathcal{V}^{**})$, where $\bar{\text{co}}$ stands for the closed convex hull of the set.*

Lemma 2.2. [7] *Let $f \in \mathcal{A}$. Then, the function f belongs to $\text{SP}(\lambda)$ if and only if*

$$\frac{1}{z}(f(z) * h(z)) \neq 0, \quad z \in \mathcal{U},$$

where

$$h(z) = \frac{z + \frac{x - e^{-2i\lambda}}{1 + e^{-2i\lambda}}}{(1 - z)^2}, \quad |x| = 1.$$

Lemma 2.3. [13] *Let $0 \leq \gamma < 1$ and $\beta \in \mathbb{R}$ such that $\beta \neq 1$. Let*

$$V_{\beta,\gamma} = \left\{ \gamma(1 - \beta) \frac{1 + xz}{1 - xz} + (1 - \gamma)(1 - \beta) \frac{1 + yz}{1 - yz} + \beta, \quad |x| = |y| = 1, \quad z \in \mathcal{U} \right\}. \quad (3)$$

Then, we have

$$V_{\beta,\gamma}^* = \left\{ f \in \mathcal{A}_0 : \exists \alpha \in \mathbb{R}, \operatorname{Re} \left\{ g(z) - \frac{1 - 2\beta}{2(1 - \beta)} \right\} > 0, \quad g(z) = f_x(z), \quad |x| \leq 1 \right\}$$

and

$$V_{\beta,\gamma}^{**} = \left\{ f \in \mathcal{A}_0; \quad \operatorname{Re} \left\{ \frac{g(z) - \beta}{1 - \beta} \right\} > 0, \quad g(z) = f_x(z), \quad |x| \leq 1 \right\}.$$

Lemma 2.4. *Let $0 \leq \gamma < 1$ and $\beta \in \mathbb{R}, \beta \neq 1$, and $V_{\beta,\gamma}$ be given by (3). Then, we have*

$$\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(f)}{\Gamma_2(f)},$$

for every $g \in V_{\beta,\gamma}^*$ and some $f \in V_{\beta,\gamma}$, where Γ_1, Γ_2 are continuous linear functionals on $V_{\beta,\gamma}$ with $\Gamma_2(V_{\beta,\gamma}) \neq 0$.

It is clear from the context that the set $V_{\beta,\gamma}$ in (3) does not satisfy the property (2), i.e., if $f \in V_{\beta,\gamma}$ then $f(xz) \in V_{\beta,\gamma}$ for all $|x| \leq 1$, which is the requirement of the Duality Principle. Therefore, the Duality Principle can be stated with a slightly weaker condition, but more complicated, when $V_{\beta,\gamma}$ satisfies [7]. In the present article, we apply the duality principle on the set $V_{\beta,\gamma}$ and hence we will not state it in its most general form.

3 Main results

In this section, we discuss various results involving spirallike functions, hypergeometric functions, and certain class of integral transforms. We commence our section by establishing the following theorem.

Theorem 3.1. Let $\alpha > 0$, $0 \leq \beta < 1$, $0 < \gamma < 1$, $|x| = 1$, and λ be a real number such that $|\lambda| < \frac{\pi}{2}$. Then $P_\alpha(\beta, \gamma) \subseteq \text{SP}(\lambda)$ if and only if

$$F(x, z) = \alpha\gamma \sum_{k=1}^{\infty} \frac{k(x+1) + 1 + e^{-2i\lambda}}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k, \quad (4)$$

where

$$\text{Re}\{F(x, z)\} > \frac{1 + \cos 2\lambda}{2(1-\beta)}. \quad (5)$$

Proof. Let f be a function in the class $\mathcal{P}_\alpha(\beta, \gamma)$ and $g_{\alpha, \gamma}(z) = f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z)$. Then, we have $g_{\alpha, \gamma} \in V_{\beta, \gamma}^{**}$. If $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$, then

$$\begin{aligned} g_{\alpha, \gamma}(z) &= 1 + \sum_{k=1}^{\infty} \left((k+1)a_{k+1} + \frac{1-\gamma}{\alpha\gamma} k(k+1)a_{k+1} \right) z^k \\ &= 1 + \sum_{k=1}^{\infty} (k+1)a_{k+1} \left(1 + \frac{1-\gamma}{\alpha\gamma} k \right) z^k \\ &= \sum_{k=0}^{\infty} (k+1)a_{k+1} \left(1 + \frac{1-\gamma}{\alpha\gamma} k \right) z^k \\ &= \sum_{k=1}^{\infty} k a_k \left(1 + \frac{1-\gamma}{\alpha\gamma} (k-1) \right) z^{k-1}. \end{aligned}$$

Hence, we have

$$\frac{f(z)}{z} = \sum_{k=1}^{\infty} a_k z^{k-1} = g_{\alpha, \gamma}(z) * \sum_{k=1}^{\infty} \frac{\alpha\gamma}{k(\alpha\gamma + (1-\gamma)(k-1))} z^{k-1}.$$

Therefore, we obtain a one-to-one correspondence between $P_\alpha(\beta, \gamma)$ and $V_{\beta, \gamma}^{**}$. Thus, by aid of Theorem 2.2, $P_\alpha(\beta, \gamma) \subseteq \text{SP}(\lambda)$ if and only if

$$g_{\alpha, \gamma}(z) * \sum_{k=1}^{\infty} \frac{\alpha\gamma z^{k-1}}{k(\alpha\gamma + (1-\gamma)(k-1))} * \frac{1 + \frac{x - e^{-2i\lambda}}{1 + e^{-2i\lambda}} z}{(1-z)^2} \neq 0, \quad \forall g \in V, \forall |x| = 1, \quad \forall z \in \mathcal{U}. \quad (6)$$

For $z \in \mathcal{U}$, let us consider the continuous linear functional $\lambda_z : \mathcal{A}_0 \rightarrow \mathbb{C}$ such that

$$\lambda_z(h) = h(z) * \sum_{k=1}^{\infty} \frac{\alpha\gamma z^{k-1}}{k(\alpha\gamma + (1-\gamma)(k-1))} * \frac{1 + \frac{x - e^{-2i\lambda}}{1 + e^{-2i\lambda}} z}{(1-z)^2} \neq 0.$$

By the Duality Principle, we obtain $\lambda_z(V) = \lambda_z(V_{\beta, \gamma}^{**})$. Therefore, (6) holds if and only if

$$\left(1 + 2(1-\beta) \sum_{k=1}^{\infty} z^k \right) * \left(1 + \sum_{k=1}^{\infty} \frac{\alpha\gamma z^k}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right) * \left(1 + \sum_{k=1}^{\infty} \left(k + 1 + \frac{x - e^{-2i\lambda}}{1 + e^{-2i\lambda}} k \right) z^k \right) \neq 0.$$

Hence, we have

$$1 + \frac{2(1-\beta)\alpha\gamma}{1 + e^{-2i\lambda}} \sum_{k=1}^{\infty} \frac{k(x+1) + 1 + e^{-2i\lambda}}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \neq 0. \quad (7)$$

Using the properties of the convolution, we reformulate (7) as

$$\alpha\gamma \sum_{k=1}^{\infty} \frac{k(x+1) + 1 + e^{-2i\lambda}}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \neq -\frac{1 + e^{-2i\lambda}}{2(1-\beta)}. \quad (8)$$

For all $z \in \mathcal{U}$, $|x| = 1$, the equality on the right side of (8) takes its value on the line $\text{Re} w \neq -\frac{1 + \cos 2\lambda}{2(1-\beta)}$. So (8) is equivalent to (4). $\square$

Corollary 3.2. The function $F(x, z)$ can be expressed in terms of Gaussian hypergeometric function as

$$F(x, z) = -(1 + e^{-2i\lambda}) + \frac{\alpha\gamma(e^{-2i\lambda} - x)}{\alpha\gamma + \gamma - 1} \frac{1}{z} \log \frac{1}{1-z} + \frac{-(1-\gamma)(1 + e^{-2i\lambda}) + \alpha\gamma(x+1)}{\alpha\gamma + \gamma - 1} {}_2F_1\left(1, \frac{\alpha\gamma}{1-\gamma}, \frac{\alpha\gamma}{1-\gamma} + 1; z\right).$$

Proof. By taking into account definitions and following simple computations we obtain

$$\begin{aligned} F(x, z) &= \frac{\alpha\gamma}{1-\gamma} \sum_{k=1}^{\infty} \frac{k(x+1) + 1 + e^{-2i\lambda}}{(k+1)\left(\frac{\alpha\gamma}{1-\gamma} + k\right)} z^k \\ &= \frac{\alpha\gamma(e^{-2i\lambda} - x)}{\alpha\gamma + \gamma - 1} \sum_{k=1}^{\infty} \frac{1}{k+1} z^k + \frac{-(1-\gamma)(1 + e^{-2i\lambda}) + \alpha\gamma(x+1)}{\alpha\gamma + \gamma - 1} \sum_{k=1}^{\infty} \frac{\frac{\alpha\gamma}{1-\gamma}}{\frac{\alpha\gamma}{1-\gamma} + k} z^k \\ &= -\left(\frac{\alpha\gamma(e^{-2i\lambda} - x)}{\alpha\gamma + \gamma - 1} + \frac{-(1-\gamma)(1 + e^{-2i\lambda}) + \alpha\gamma(x+1)}{\alpha\gamma + \gamma - 1}\right) + \frac{\alpha\gamma(e^{-2i\lambda} - x)}{\alpha\gamma + \gamma - 1} {}_2F_1(1, 1, 2; z) \\ &\quad + \frac{-(1-\gamma)(1 + e^{-2i\lambda}) + \alpha\gamma(x+1)}{\alpha\gamma + \gamma - 1} {}_2F_1\left(1, \frac{\alpha\gamma}{1-\gamma}, \frac{\alpha\gamma}{1-\gamma} + 1; z\right). \end{aligned}$$

That is,

$$\begin{aligned} F(x, z) &= -\frac{\alpha\gamma(e^{-2i\lambda} + 1) - (1-\gamma)(1 + e^{-2i\lambda})}{\alpha\gamma + \gamma + 1} + \frac{\alpha\gamma(e^{-2i\lambda} - x)}{\alpha\gamma} + \gamma - 1 \\ \text{i.e.} &= -(1 + e^{-2i\lambda}) + \frac{\alpha\gamma(e^{-2i\lambda} - x)}{\alpha\gamma + \gamma - 1} \frac{1}{z} \log \frac{1}{1-z} \\ &\quad + \frac{-(1-\gamma)(1 + e^{-2i\lambda}) + \alpha\gamma(x+1)}{\alpha\gamma + \gamma - 1} {}_2F_1\left(1, \frac{\alpha\gamma}{1-\gamma}, \frac{\alpha\gamma}{1-\gamma} + 1; z\right). \quad \square \end{aligned}$$

The following remark expresses a new form of Inequality (5).

Remark 3.3. The following inequality holds.

$$\frac{1 + \cos 2\lambda}{2\alpha\gamma(1-\beta)} + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \frac{1}{\alpha\gamma + (1-\gamma)k} z^k \right\} \geq \left| \sum_{k=1}^{\infty} \frac{k}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right| + \left| \sum_{k=1}^{\infty} \frac{1}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right|.$$

Proof. Following the previous analysis, we write

$$\begin{aligned} &\frac{1 + \cos 2\lambda}{2\alpha\gamma(1-\beta)} + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \frac{k}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right\} + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \frac{1}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right\} \\ &\geq \operatorname{Re} \left\{ -x \sum_{k=1}^{\infty} \frac{k}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right\} + \operatorname{Re} \left\{ -e^{-2i\lambda} \sum_{k=1}^{\infty} \frac{1}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right\}. \end{aligned}$$

This indeed satisfies the above inequality when

$$\frac{1 + \cos 2\lambda}{2\alpha\gamma(1-\beta)} + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \frac{1}{\alpha\gamma + (1-\gamma)k} z^k \right\} \geq \left| \sum_{k=1}^{\infty} \frac{k}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right| + \left| \sum_{k=1}^{\infty} \frac{1}{(k+1)(\alpha\gamma + (1-\gamma)k)} z^k \right|.$$

Therefore, we obtain

$$\frac{1 + \cos 2\lambda}{2\alpha\gamma(1-\beta)} + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \frac{1}{\alpha\gamma + (1-\gamma)k} z^k \right\} \geq \left| \sum_{k=1}^{\infty} \frac{1}{\alpha\gamma + (1-\gamma)k} z^k \right|. \quad \square$$

Theorem 3.4. Let $f \in P_\alpha(\beta, \gamma)$, $\alpha > 0$, $0 \leq \beta \leq 1$, and $0 < \gamma \leq 1$. Define

$$K_{\alpha, \gamma} = \int_0^1 \frac{dt}{1 - t^{\frac{1-\gamma}{\alpha\gamma}}}. \quad (9)$$

Then, the function f belongs to $P_\alpha(0, 1)$ and, hence, it is univalent for

$$\beta \geq \frac{1 - 2K_{\alpha, \gamma}}{2(1 - K_{\alpha, \gamma})}.$$

Proof. Let $\alpha > 0$ and $\gamma > 0$. Define

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \left(1 + \frac{1-\gamma}{\alpha\gamma} n\right) z^n$$

and

$$\psi(z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha\gamma}{\alpha\gamma + (1-\gamma)n} z^n = 1 + \sum_{n=1}^{\infty} \frac{z^n}{1 + \frac{1-\gamma}{\alpha\gamma} n} = \int_0^1 \frac{1}{1 - t^{\frac{1-\gamma}{\alpha\gamma}} z} dt.$$

By using change of variables, we rewrite $\psi(z)$ in the form

$$\psi(z) = \begin{cases} \frac{1}{1-z}, & \gamma = 1, \\ \frac{\alpha\gamma}{1-\gamma} \int_0^1 \frac{s^{\frac{\alpha\gamma}{1-\gamma}-1} - 1}{1-sz} ds, & 0 < \gamma < 1. \end{cases} \quad (10)$$

In view of these representations, we can write

$$f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z) = f'(z) * \phi(z) \quad \text{and} \quad \left(f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z)\right) * \psi(z) = f'(z). \quad (11)$$

Now, we let $f \in P_\alpha(\beta, \gamma)$. Then, in view of the Duality Principle (see Lemma 2.1), we may restrict our attention to the function $f \in P_\alpha(\beta, \gamma)$ for which

$$f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z) = \gamma(1-\beta) \frac{1+xz}{1-xz} + (1-\beta)(1-\gamma) \frac{1+yz}{1-yz} + \beta.$$

Thus, in view of (11), the preceding observation reveals

$$f'(z) = \left(\gamma(1-\beta) \frac{1+xz}{1-xz} + (1-\beta)(1-\gamma) \frac{1+yz}{1-yz} + \beta\right) * \psi(z). \quad (12)$$

Hence, equation (12) is equivalent to

$$\begin{aligned} f'(z) &= \left(\gamma \frac{1+xz}{1-xz} + (1-\gamma) \frac{1+yz}{1-yz}\right) * ((1-\beta)\psi(z) + \beta), \\ &= \left(\gamma \frac{1+xz}{1-xz} + (1-\gamma) \frac{1+yz}{1-yz}\right) * \left(\int_0^1 \left((1-\beta) \frac{1}{1-t^{\frac{1-\gamma}{\alpha\gamma}} z} + \beta\right) dt\right) \\ &= \left(\gamma \frac{1+xz}{1-xz} + (1-\gamma) \frac{1+yz}{1-yz}\right) * G(z), \end{aligned} \quad (13)$$

where

$$G(z) = \int_0^1 \left((1-\beta) \frac{1}{1-t^{\frac{1-\gamma}{\alpha\gamma}} z} + \beta\right) dt.$$

Therefore, we have

$$\operatorname{Re} G(z) \geq \int_0^1 \left((1-\beta) \frac{1}{1-t^{\frac{1-\gamma}{\alpha\gamma}}} + \beta \right) dt = (1-\beta)K_{\alpha,\gamma} + \beta,$$

where $K_{\alpha,\gamma}$ satisfies (9). Note that if $\beta \geq (1-2K_{\alpha,\gamma})/2(1-K_{\alpha,\gamma})$, then $\operatorname{Re} G(z) \geq 1/2$. Also, it is well-known that functions with real parts greater than $1/2$ preserve the closed convex hull under convolution [10, p.23]. Therefore, from (13) we write

$$\begin{aligned} f'(z) &= \gamma \left(\frac{2}{1-xz} - 1 \right) * G(z) + (1-\gamma) \left(\frac{2}{1-yz} - 1 \right) * G(z) \\ &= 2\gamma G(xz) - \gamma + 2(1-\gamma)G(yz) - (1-\gamma) \\ &= 2\gamma G(xz) + 2(1-\gamma)G(yz) - 1. \end{aligned}$$

But, since $\operatorname{Re} f'(z) > 0$, we have $f \in P_{\alpha}(0, 1)$. □

Theorem 3.5. Let $f \in P_{\alpha}(\beta, \gamma)$, $\alpha > 0$, $0 \leq \beta \leq 1$, $0 < \gamma \leq 1$. Let the following integral equation

$$\frac{\beta}{1-\beta} = - \int_0^1 \varphi(t)g(t)dt \quad (14)$$

hold. Then, for every real λ , $|\lambda| < \frac{\pi}{2}$, we have $V_{\varphi}(f) \in \operatorname{SP}(\lambda)$ if and only if

$$\operatorname{Re} \int_0^1 \Pi_{\gamma}(t) \left(\frac{h(tz)}{tz} - \frac{-t + e^{-i\lambda}(1+t)}{e^{-i\lambda}(1+t)^2} \right) dt \geq 0,$$

where

$$\Pi_{\gamma}(t) = \int_t^1 \Lambda_{\gamma}(s) s^{\frac{\alpha\gamma}{1-\gamma}-2} ds, \quad \Lambda_{\gamma}(t) = \int_t^1 \frac{\varphi(\sigma)}{\frac{\alpha\gamma}{1-\gamma}} d\sigma (\gamma > 0)$$

and

$$h(z) = \frac{z + \frac{x - e^{-2i\lambda}}{1 + e^{-2i\lambda}}}{(1-z)^2}, \quad |x| = 1, \quad z \in \mathcal{U}.$$

Proof. Let $\alpha > 0$, $\gamma > 0$, and $F(z) = V_{\varphi}(f)$. Then $F \in \operatorname{SP}(\lambda)$ if and only if

$$0 \neq \frac{F(z)}{z} * \frac{h(z)}{z} = \frac{1}{z} \int_0^1 \varphi(t) \frac{f(tz)}{t} dt * \frac{h(z)}{z} = \int_0^1 \frac{\varphi(t)dt}{1-tz} * \frac{f(z)}{z} * \frac{h(z)}{z}. \quad (15)$$

Hence, by applying equation (12), equation (15) is equivalent to

$$(1-\beta) \int_0^1 \varphi(t) \frac{1}{z} \int_0^z \left(\frac{h(tw)}{tw} - g(t) \right) dw dt * \psi(z) * \left(\gamma \frac{1+xz}{1-xz} + (1-\gamma) \frac{1+yz}{1-yz} \right) \neq 0, \quad (16)$$

where $g(t)$ is defined by (1). Equation (16) indeed holds if and only if

$$\operatorname{Re} \left\{ (1-\beta) \int_0^1 \varphi(t) \frac{1}{z} \int_0^z \left(\frac{h(tw)}{tw} - g(t) \right) dw dt * \psi(z) \right\} > \frac{1}{2}.$$

Now, by using the fact $(1-\beta) \int_0^1 \varphi(t)(1-g(t))dt = 1$ we derive

$$\operatorname{Re} \left\{ \int_0^1 \varphi(t) \frac{1}{z} \int_0^z \left(\frac{h(tw)}{tw} - \frac{1+g(t)}{2} \right) dw dt * \psi(z) \right\} \geq 0.$$

Using the definition of ψ given by (10), the above expression becomes

$$\operatorname{Re} \left\{ \int_0^1 \varphi(t) \frac{1}{z} \int_0^z \int_0^1 s^{\frac{\alpha\gamma}{1-\gamma}-1} \frac{\alpha\gamma}{1-\gamma} \left(\frac{h(tw)}{tw} - \frac{1+g(t)}{2} \right) ds dw dt \right\} \geq 0.$$

By the definition of g presented in (1), it is easy to see that

$$\frac{1+g(t)}{2} = \frac{\alpha\gamma}{1-\gamma} \int_0^1 s^{\frac{\alpha\gamma}{1-\gamma}-1} \frac{1}{2} \left(1 + \frac{2 - e^{-i\lambda} - e^{-i\lambda}st}{e^{-i\lambda}(1+st)} - \frac{2(1 - e^{-i\lambda}) \log 1 + st}{e^{-i\lambda} st} \right) ds.$$

By substituting in the left-hand side of the previous inequality and using the change of variables $v = st$ in the resulting equation, it follows that

$$\operatorname{Re} \left\{ \int_0^1 \frac{\varphi(t)}{t^{\frac{\alpha\gamma}{1-\gamma}}} \left(\int_0^t v^{\frac{\alpha\gamma}{1-\gamma}-1} \left\{ \frac{1}{z} \int_0^z \frac{h(wv)}{wv} - \frac{t}{2} \left(1 + \frac{2 - e^{-i\lambda} - e^{-i\lambda}v}{e^{-i\lambda}(1+v)} - \frac{2(1 - e^{-i\lambda}) \log 1 + v}{e^{-i\lambda} v} \right) dw \right\} dv \right) dt \right\} \geq 0. \quad (17)$$

By integrating by parts, Inequality (17) yields

$$\operatorname{Re} \left\{ \int_0^1 \Lambda_\gamma(t) t^{\frac{\alpha\gamma}{1-\gamma}-2} \left\{ \frac{1}{z} \int_0^z \frac{h(tw)}{w} - \frac{t}{2} \left(1 + \frac{2 - e^{-i\lambda} - e^{-i\lambda}t}{e^{-i\lambda}(1+t)} - \frac{2(1 - e^{-i\lambda}) \log 1 + t}{e^{-i\lambda} t} \right) dw \right\} dt \right\} \geq 0,$$

where $\Lambda_\gamma(t) = \int_t^1 \frac{\varphi(\sigma)}{\sigma^{\frac{\alpha\gamma}{1-\gamma}}} d\sigma$. Once again, integrating by part suggests the following compact form:

$$\operatorname{Re} \left\{ \int_0^1 \Pi_\gamma(t) \left(\frac{h(tz)}{tz} - \frac{-t + e^{-i\lambda}(1+t)}{e^{-i\lambda}(1+t)^2} \right) dt \right\} \geq 0,$$

where $\Pi_\gamma(t) = \int_t^1 \Lambda_\gamma(s) s^{\frac{\alpha\gamma}{1-\gamma}-2} ds$. Thus, $F \in \operatorname{SP}(\lambda)$. To extend our result from the particular choice of f given in (12) to all of $\mathcal{P}_\alpha(\beta, \gamma)$ we refer to Lemma 2.4. Note that V_φ defines a linear functional and the condition on the λ -spirallikeness $\operatorname{Re} \left(e^{i\lambda} \frac{zF'(z)}{F(z)} > 0 \right) > 0$ is expressed as a quotient of two linear functionals. Therefore, Lemma 2.4 can be applied. Finally, to prove the sharpness, let $f \in \mathcal{P}_\alpha(\beta, \gamma)$ be in the form

$$f'(z) + \frac{1-\gamma}{\alpha\gamma} z f''(z) = \gamma(1-\beta) \frac{1+xz}{1-xz} + (1-\gamma)(1-\beta) \frac{1+yz}{1-yz} + \beta.$$

Let $x = y = 1$. Using a series expansion, we obtain

$$f(z) = z + 2(1-\beta) \sum_{n=2}^{\infty} \frac{\alpha\gamma}{n(\alpha\gamma + (n-1)(1-\gamma))} z^n.$$

Therefore, we can write

$$F(z) = V_\varphi(f)(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t} dt = z + 2(1-\beta) \sum_{n=2}^{\infty} \frac{\alpha\gamma \mu_n}{n(\alpha\gamma + (n-1)(1-\gamma))} z^n,$$

where $\mu_n = \int_0^1 \lambda(t) t^{n-1} dt$. Furthermore, it is easy to write $g(t)$ in (1) in a series expansion as

$$g(t) = 1 + \frac{-2}{e^{-i\lambda}} \sum_{n=1}^{\infty} (-1)^n \frac{n+2 - e^{-i\lambda}}{n+1} \frac{\alpha\gamma}{\alpha\gamma + n(1-\gamma)} t^n. \quad (18)$$

Now by equations (14) and (18), we have

$$\begin{aligned}\frac{\beta}{1-\beta} &= - \int_0^1 \varphi(t)g(t)dt \\ &= - \int_0^1 \varphi(t) \left(1 + \frac{-2}{e^{-i\lambda}} \sum_{n=1}^{\infty} (-1)^n \frac{n+2-e^{-i\lambda}}{n+1} \frac{\alpha\gamma}{\alpha\gamma+n(1-\gamma)} t^n \right) dt \\ &= -1 - \frac{-2}{e^{-i\lambda}} \sum_{n=1}^{\infty} (-1)^n \frac{n+2-e^{-i\lambda}}{n+1} \frac{\alpha\gamma\mu_{n+1}}{\alpha\gamma+n(1-\gamma)}.\end{aligned}$$

Hence, we obtain

$$\frac{1}{1-\beta} = \frac{-2}{e^{-i\lambda}} \sum_{n=2}^{\infty} (-1)^{n-1} \frac{n+1-e^{-i\lambda}\alpha\gamma\mu_n}{n(\alpha\gamma+n(1-\gamma))}.$$

Computing $F'(-1)$, indeed gives

$$\begin{aligned}F'(-1) &= 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\alpha\gamma\mu_n}{(n-1)(1-\gamma) + \alpha\gamma} \\ &= (1-e^{-i\lambda}) + 2(1-\beta)(1-e^{-i\lambda}) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\alpha\gamma\mu_n}{n((n-1)(1-\gamma) + \alpha\gamma)} \\ &= (1-e^{-i\lambda}) \left(1 + 2(1-\beta) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\alpha\gamma\mu_n}{n((n-1)(1-\gamma) + \alpha\gamma)} \right) \\ &= -(1-e^{-i\lambda})F(-1).\end{aligned}$$

Hence, we obtain

$$\operatorname{Re} \left\{ e^{i\lambda} \frac{-F'(-1)}{F(-1)} \right\} < 0.$$

This implies that our result is sharp for the λ -spirallike function. $\square$

4 Concluding remarks

In this article, a class of analytic functions was discussed on a unit open disc $\mathcal{U} = \{z : |z| < 1\}$. Certain conditions on the numbers α, β , and γ were imposed so that $\mathcal{P}_\alpha(\beta, \gamma)$ defines a subset of the set $\operatorname{SP}(\lambda)$ of λ -spirallike functions for all $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Ruscheweyh's duality theory was employed in predicting conditions on the numbers α, β, γ and the real-valued functions φ so that the integral transform $V_\varphi(f)$ maps $\mathcal{P}_\alpha(\beta, \gamma)$ into $\operatorname{SP}(\lambda)$ for nonnegative and normalized functions φ .

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