Research Article

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Sharp sufficient condition for the convergence of greedy expansions with errors in coefficient computation

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Abstract: Generalized approximate weak greedy algorithms (gAWGAs) were introduced by Galatenko and Livshits as a generalization of approximate weak greedy algorithms, which, in turn, generalize weak greedy algorithm and thus pure greedy algorithm. We consider a narrower case of gAWGA in which only a sequence of absolute errors $\{\xi_n\}_{n=1}^{\infty}$ is nonzero. In this case sufficient condition for a convergence of a gAWGA expansion to an expanded element obtained by Galatenko and Livshits can be written as $\sum_{n=1}^{\infty}\xi_n^2 < \infty$. In the present article, we relax this condition and show that the convergence is guaranteed for $\xi_n = o\left(\frac{1}{\sqrt{n}}\right)$. This result is sharp because the convergence may fail to hold for $\xi_n \asymp \frac{1}{\sqrt{n}}$.

Keywords: greedy expansion, prescribed coefficients, Hilbert space, greedy approximation, convergence

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1 Introduction

In this article, we consider generalized approximate weak greedy algorithms (gAWGAs), which were introduced by Galatenko and Livshits [1]. Let us recall the definition of gAWGA.

Definition 1.1. Let H be a Hilbert space over $\mathbb{R}$, D be a symmetric unit-normed dictionary in H (i.e., $\overline{\text{span } D} = H$, all elements in D have a unit norm, and if $g \in D$, then $-g \in D$). In addition, let $\{t_n\}_{n=1}^{\infty} \subset (0, 1]$, $\{q_n\}_{n=1}^{\infty} \subset [0, \infty)$ be weakness sequences and $\{(\varepsilon_n, \xi_n)\}_{n=1}^{\infty} \subset \mathbb{R}^2$ be an error sequence. For an expanded element $f \in H$, coefficients $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R}$, remainders $\{r_n\}_{n=0}^{\infty} \subset H$ and expanding elements $\{e_n\}_{n=1}^{\infty} \subset D$ are defined as follows.

Initially, r_0 is set to f. Next, if $r_{n-1} \in H$ ($n \in \mathbb{N}$) has already been defined, then an (arbitrary) element satisfying $(r_{n-1}, e_n) \geqslant t_n \sup_{e\in D}(r_{n-1}, e) - q_n$ is selected as e_n. We set $c_n = (r_{n-1}, e_n)(1 + \varepsilon_n) + \xi_n$ and define $r_n = r_{n-1} - c_n e_n$.

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The process described earlier is called a gAWGA. The series $\sum_{n=1}^{\infty} c_n e_n$ is called a gAWGA expansion of f in the dictionary D with the weakening sequences $\{t_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and the error sequence $\{(\varepsilon_n, \xi_n)\}_{n=1}^{\infty}$.

It immediately follows from the definition of gAWGA that

$$r_N = f - \sum_{n=1}^{N} c_n e_n \quad (N \in \mathbb{N}),$$

and hence, the convergence of the expansion to an expanded element is equivalent to that of the remainders r_n to zero.

As a selection of an expanding element e_n is potentially not unique, there may exist different realizations of gAWGA expansion for a given expanded element f and a given dictionary D. Furthermore, if $t_n = 1$ and $q_n = 0$ for at least one $n \in \mathbb{N}$, gAWGA expansion may turn out to be nonrealizable due to the absence of an element $e \in D$ which provides $\sup_{e \in D}(r_{n-1}, e)$.

If $q_n = \xi_n = 0$ for every $n \in \mathbb{N}$, then gAWGA coincides with the approximate weak greedy algorithm (AWGA) proposed by Gribonval and Nielsen [2]. If $q_n = \xi_n = \varepsilon_n = 0$ for every $n \in \mathbb{N}$, then gAWGA coincides with the weak greedy algorithm (WGA), which was introduced by Temlyakov in [3]. If $q_n = \xi_n = \varepsilon_n = 0$ and $t_n = 1$ for every $n \in \mathbb{N}$, then gAWGA coincides with the pure greedy algorithms [3], also known as “projection pursuit regression” or “matching pursuit” [4,5].

The error sequence $\{(\varepsilon_n, \xi_n)\}_{n=1}^{\infty}$ can be separated into two sequences, i.e., into a relative error sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ and an absolute error sequence $\{\xi_n\}_{n=1}^{\infty}$. As each computational error can be quantified by its absolute value, here we consider the error sequences $\{(0, \xi_n)\}_{n=1}^{\infty}$. Furthermore, we assume that all t_n and q_n have their default values – 1 and 0, respectively.

Let us note that if $c_n > 0$ for all n, then we can interpret this realization of a greedy expansion as a special case of a greedy algorithm with prescribed coefficients (GAPCs), which was initially introduced by Temlyakov [6,7]. Indeed, we can take the sequence of coefficients $\{c_n\}_{n=1}^{\infty}$ from the realization of gAWGA, and consider GAPC for the same expanded element with this sequence of coefficients (treated as “predefined”). GAPC in this case has a realization with the expanding elements coinciding with element selected in gAWGA, and these realizations of gAWGA and GAPC are identical. Thus, if $\{c_n\}_{n=1}^{\infty}$ in gAWGA satisfy conditions sufficient for the convergence of GAPC, it guarantees convergence of gAWGA as well.

Galatenko and Livshits found sufficient conditions on weakening sequences and error sequences for a convergence of gAWGA expansion [1, Theorem 2]. For the considered case, these conditions take the form $\sum_{n=1}^{\infty} \xi_n^2 < \infty$. In the same article, they also showed [1, Theorem 3] that if $\xi_n \asymp \frac{1}{\sqrt{n}}$, then the convergence can be violated. Thus, there remained a gray zone between l_2 and $\frac{1}{\sqrt{n}}$.

Similar results for GAPC were obtained in [8]. More precisely, it was shown that if the sequence of coefficients satisfies conditions $\sum_{n=1}^{\infty} c_n = \infty$ and $\sum_{n=1}^{\infty} c_n^2 < \infty$, then GAPC expansion converges to an expanded element, but for $c_n \asymp \frac{1}{\sqrt{n}}$, the convergence may fail to hold. However, for GAPC, the specified gray zone was eliminated. Specifically, in [9, Theorem 2.2], the authors showed that the convergence is guaranteed for $\{c_n\}_{n=1}^{\infty}$ satisfying conditions $\sum_{n=1}^{\infty} c_n = \infty$ and $c_n = o\left(\frac{1}{\sqrt{n}}\right)$.

2 Main result

In this article, we present a result for gAWGA, similar to the one proved in [9] for GAPC, which removes the gray zone between l_2 and $\frac{1}{\sqrt{n}}$ for gAWGA. This result can be stated as follows.

Theorem 1. *Let H be a Hilbert space, D be a symmetric unit-normed dictionary in H, weakening sequences $\{t_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be identically equal to 1 and 0, respectively, and a relative error sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ be identically equal to 0. Let an absolute error sequence $\{\xi_n\}_{n=1}^{\infty}$ satisfy the condition $\xi_n = o\left(\frac{1}{\sqrt{n}}\right)$. Then, for every* *element $f \in H$, its gAWGA-expansion in dictionary D with the weakening sequences $\{t_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and the error sequence $\{(\varepsilon_n, \xi_n)\}_{n=1}^{\infty}$ converges to f.*

The proof of Theorem 1 uses certain methods and technique that were used in the proof of the similar result for GAPC [9, Theorem 2.2]. However, a straightforward adaptation of the proof of [9, Theorem 2.2] is insufficient for proving Theorem 1: to obtain the result for gAWGA, we introduce new ideas in steps 2 and, especially, 4 and 5, as well as new Lemmas (specifically, Lemmas 3, 5, 6, and 9).

3 Proof of Theorem 1

In the following text, we write "gAWGA expansion" as a short form of "gAWGA expansion in dictionary D with the weakening sequences $\{t_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and the error sequence $\{(\varepsilon_n, \xi_n)\}_{n=1}^{\infty}$."

We note that for proving Theorem 1, it is sufficient to show that

$$\lim_{n\to\infty} \|r_n\| = 0.$$

We split the proof into nine steps. Steps 1–5 are the preparation for the main part of the proof. Step 6 is the proof for one simple case, and steps 7–9 constitute the proof for the more difficult case.

1. We begin the proof by showing that there exists the limit

$$\lim_{n\to\infty} \|r_n\| < \infty.$$

We split the proof of this fact into two lemmas.

Lemma 1. *Let $\alpha_n = \arccos\frac{(r_{n-1}, e_n)}{\|r_{n-1}\|}$ (i.e., $\alpha_n \in [0, \pi]$ is the angle between r_{n-1} and e_n), and let $h_n = \|r_{n-1}\| \sin\alpha_n$. Then $\{h_n\}_{n=1}^{\infty}$ is a nonincreasing sequence.*

Proof. Let $\beta_n = \min\{\widehat{(r_n, e_n)}, \widehat{(r_n, -e_n)}\}$, where $\widehat{(a, b)}$ denotes the angle between vectors a and b. Then $h_n = \|r_n\| \sin\beta_n$. Since the expansion is greedy with $t_n \equiv 1$ and $q_n \equiv 0$, we have $\alpha_{n+1} \leqslant \beta_n$, and so

$$h_{n+1} = \|r_n\| \sin\alpha_{n+1} \leqslant \|r_n\| \sin\beta_n = h_n.$$ □

Lemma 2. *The limit*

$$\lim_{n\to\infty} \|r_n\| < \infty$$

exists.

Proof. Similar to Lemma 1, we set $h_n = \|r_{n-1}\| \sin\alpha_n$. In view of Lemma 1, the sequence $\{h_n\}_{n=1}^{\infty}$ is nonincreasing. Hence, there exists a limit $\lim_{n\to\infty} h_n = h$.

For gAWGA, we have

$$\|r_n\|^2 = h_n^2 + \xi_n^2. \tag{1}$$

Hence, $\|r_n\| \longrightarrow h$ as $n \longrightarrow \infty$. □

2. We prove Theorem 1 by contradiction. Let us assume that $\lim_{n\to\infty}\|r_n\| > 0$. If $\|r_k\| = 0$ for some $k > 0$, and then it is obvious that the expansion converges to the expanded element. Otherwise, there exists a number $r > 0$, such that for every $k \in \mathbb{N}$

$$\|r_k\| \geqslant r. \tag{2}$$

Let us note that in the considered case of gAWGA expansion, we can always assume that $c_n \geqslant 0$. Indeed, assume that an expanding element $e_{n+1} \in D$ was selected at step $n + 1$. If $c_{n+1} < 0$, then the same element

e_{n+1} will be selected as e_{n+j} until $(r_n, e_{n+1}) \geqslant (r_{n+j-1}, e_{n+1})$ for some $j > 1$. It follows from the fact that if $r_{n+j-1} = r_n + \alpha e_{n+1}$ with $\alpha > 0$ for $j > 1$, then

$$(r_{n+j-1}, e_{n+1}) = (r_n, e_{n+1}) + \alpha > (r_n + \alpha e_{n+1}, e) = (r_{n+j-1}, e)$$

for every $e \in D \backslash \{e_{n+1}\}$.

But if the sequence $\{\xi_n\}_{n=1}^{\infty}$ satisfies the condition $\xi_n = o\left(\frac{1}{\sqrt{n}}\right)$, then it is obvious that every subsequence $\{\xi_{n_k}\}_{k=1}^{\infty}$ satisfies the condition $\xi_{n_k} = o\left(\frac{1}{\sqrt{k}}\right)$. Therefore, we can combine several consecutive steps (namely, consecutive steps with identical selection of an expanding element) into one and consider the sequence of errors $\{\xi_{n_k}\}_{k=1}^{\infty}$ instead of $\{\xi_n\}_{n=1}^{\infty}$. As a result, we ensure that all $c_n \geqslant 0$. If $c_n = 0$ for some n, then at step n, the remainder stays unchanged, and we can simply exclude such steps from consideration. Therefore, for the same reason, we can assume that $c_n > 0$ for every $n \in \mathbb{N}$.

Note that due to Lemma 2, the assertion of the theorem, i.e., the convergence of r_n to zero, follows from the convergence of a subsequence r_{n_k} to zero.

We need the following lemma.

Lemma 3. *If* $\|r_k\| \geqslant r > 0$ *for all* $k > 0$, *then*

$$\sum_{n=1}^{\infty} c_n = \infty.$$

Proof. We prove this lemma by contradiction. Assume on the contrary that

$$\sum_{n=1}^{\infty} c_n < \infty.$$

Then the series $\sum_{n=1}^{\infty} c_n e_n$ converges. Therefore, since

$$r_n = f - \sum_{k=1}^{n} c_k e_k,$$

there exists the limit $r_n \longrightarrow a \neq 0$. As $\overline{\operatorname{span} D} = H$ and since D is symmetric, there exists $d \in D$ such that $(a, d) = b > 0$. Consequently, there exists a number $n \in \mathbb{N}$ such that $\frac{b}{2} < (r_n, d) \leqslant (r_n, e_{n+1})$, $|\xi_{n+1}| < \frac{b}{4}$, and $c_{n+1} < \frac{b}{4}$ simultaneously. Therefore, for this n, we have

$$\frac{b}{4} > c_{n+1} = (r_n, e_{n+1}) + \xi_{n+1} > \frac{b}{2} - \frac{b}{4} = \frac{b}{4}.$$

This contradiction completes the proof of Lemma 3. □

3. For every nonzero element $f \in H$, we set

$$F_f(g) := \frac{(f, g)}{\|f\|}, \quad r_D(f) := \sup_{g \in D} F_f(g) = \frac{\sup_{g \in D}(f, g)}{\|f\|}.$$

Let S_k be the kth partial sum of the sequence $\{c_n\}_{n=1}^{\infty}$, i.e., $S_k = \sum_{j=1}^{k} c_j$.

We need the following lemma.

Lemma 4. *If* $\|r_k\| \geqslant r > 0$ *for all* $k > 0$, *then*

$$\liminf_{n \to \infty} S_n r_D(r_{n-1}) > 0.$$

Proof. As mentioned earlier, assume on the contrary that

$$\liminf_{n \to \infty} S_n r_D(r_{n-1}) = 0.$$

The monotonicity of S_n implies that

$$\liminf_{n\to\infty} S_n r_D(r_n) = 0.$$

Therefore, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$, such that

$$\lim_{k\to\infty} S_{n_k} r_D\left(r_{n_k}\right) = 0.$$

Let us consider a sequence of functionals $\{F_{r_{n_k}}\}_{k=1}^{\infty}$ of norm 1. By the Banach-Alaoglu theorem, the unit sphere is weakly* compact. Hence, there exists a weakly*-converging subsequence $\{F_{r_{n_{k_i}}}\}_{i=1}^{\infty}$. For simplicity, we set $F_{r_{n_{k_i}}} = \mathbf{F}_i$. As noted earlier, there exists the weak*-limit

$$F := \lim_{i\to\infty} \mathbf{F}_i.$$

The dictionary D is symmetric, and hence,

$$\mathbf{F}_i(f) = \mathbf{F}_i\left(r_{n_{k_i}} + \sum_{j=1}^{n_{k_i}} c_j e_j\right) = \|r_{n_{k_i}}\| + \sum_{j=1}^{n_{k_i}} c_j \mathbf{F}_i(e_j) \geqslant r - S_{n_{k_i}} r_D\left(r_{n_{k_i}}\right).$$

Passage to the limit, we have $F(f) \geqslant r$, which implies that $F \neq 0$.

Conversely, for every g from the dictionary D, we have

$$F(g) = \lim_{i\to\infty} \mathbf{F}_i(g) \leqslant \lim_{i\to\infty} r_D\left(r_{n_{k_i}}\right) = 0,$$
$$F(-g) = \lim_{i\to\infty} \mathbf{F}_i(-g) \leqslant \lim_{i\to\infty} r_D\left(r_{n_{k_i}}\right) = 0.$$

Hence, $F(g) = 0$ for all $g \in D$ and, since D is complete, we obtain $F = 0$. The contradiction completes the proof of the lemma. □

4. We split the set of indices $k \in \mathbb{N}$ into two parts.

Let $\widetilde{M}$ be the set of indices k such that $|\xi_k| < \frac{(r_{k-1}, e_k)}{10}$, and let $\widetilde{N} = \mathbb{N} \setminus \widetilde{M}$.

If $k \in \widetilde{N}$, then

$$c_k = (r_{k-1}, e_k) + \xi_k \leqslant 11|\xi_k|. \tag{3}$$

If $k \in \widetilde{M}$, we have

$$\begin{aligned} c_k &= (r_{k-1}, e_k) + \xi_k > \frac{9(r_{k-1}, e_k)}{10}, \\ c_k &= (r_{k-1}, e_k) + \xi_k < \frac{11(r_{k-1}, e_k)}{10}. \end{aligned} \tag{4}$$

We further split $\widetilde{M}$ into the disjoint sets $\widetilde{M}_1, \widetilde{M}_2, \dots$, which satisfy the following conditions:

(1) Every set consists of sequential indices.
(2) The union of any two consecutive sets does not consist of sequential indices.

In other words, $\widetilde{M}_j$ are maximum blocks of consequent elements in $\widetilde{M}$.

Let l_j be the first element of $\widetilde{M}_j$. For every l_j, we find the index $p_j \in \widetilde{M}_j$ (if it exists) such that $c_{i+1} \leqslant \frac{c_i}{2}$ for every $i \in \{l_j, l_j+1, \dots, p_j - 1\}$, but not for $i = p_j$. Let $\overline{M_j}$ be defined as $\widetilde{M}_j$ if such an index does not exist, and as $\{l_j, l_j+1, \dots, p_j - 1, p_j\}$ otherwise; $M_j = \widetilde{M}_j \setminus \overline{M_j}$ (some of these M_j may be empty).

Let N be defined as the union of $\widetilde{N}$ and all $\overline{M_j}$, and let $M = \mathbb{N} \setminus N$. We note that $M = \bigcup_j M_j$.

Thus, we have split the set of indices $k \in \mathbb{N}$ into two sets N and M with the aforementioned properties.

5. Now, consider the sequences

$$x_k = \begin{cases} c_k, & \text{if } k \in M, \\ 0, & \text{otherwise;} \end{cases}$$

$$y_k = \begin{cases} c_k, & \text{if } k \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Let also S_n^M and S_n^N be the nth partial sums of the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, respectively.
We note that

$$\begin{aligned} \|r_n\|^2 &= (r_n, r_n) \\ &= (r_{n-1} - c_n e_n, r_{n-1} - c_n e_n) \\ &= \|r_{n-1}\|^2 - 2c_n(r_{n-1}, e_n) + c_n^2 \\ &= \ldots = \|r_0\|^2 - 2\sum_{k=1}^{n} c_k(r_{k-1}, e_k) + \sum_{k=1}^{n} c_k^2 \\ &= \|r_0\|^2 - 2\sum_{\substack{k\leqslant n,\\ k\in M}} c_k(r_{k-1}, e_k) + \sum_{\substack{k\leqslant n,\\ k\in M}} c_k^2 - 2\sum_{\substack{k\leqslant n,\\ k\in N}} c_k(r_{k-1}, e_k) + \sum_{\substack{k\leqslant n,\\ k\in N}} c_k^2. \end{aligned} \tag{5}$$

We also note that

$$2\sum_{\substack{k\leqslant n,\\ k\in M}} c_k(r_{k-1}, e_k) - \sum_{\substack{k\leqslant n,\\ k\in M}} c_k^2 = 2\sum_{\substack{k\leqslant n,\\ k\in M}} ((r_{k-1}, e_k) + \xi_k)(r_{k-1}, e_k) - \sum_{\substack{k\leqslant n,\\ k\in M}} ((r_{k-1}, e_k) + \xi_k)^2 = \sum_{\substack{k\leqslant n,\\ k\in M}} (r_{k-1}, e_k)^2 - \sum_{\substack{k\leqslant n,\\ k\in M}} \xi_k^2. \tag{6}$$

If $k \in M$, then by (4) and since $M \subset \widetilde{M}$, we have

$$(r_{k-1}, e_k)^2 - \xi_k^2 > (r_{k-1}, e_k)^2 - \frac{(r_{k-1}, e_k)^2}{100} = \frac{99(r_{k-1}, e_k)^2}{100} > \frac{99}{100} \cdot \left(\frac{10c_k}{11}\right)(r_{k-1}, e_k) > \frac{c_k(r_{k-1}, e_k)}{2}.$$

Combining this estimate with (5) and (6), we find that

$$\begin{aligned} \|r_n\|^2 &= \|r_0\|^2 - \sum_{\substack{k\leqslant n,\\ k\in M}} (r_{k-1}, e_k)^2 + \sum_{\substack{k\leqslant n,\\ k\in M}} \xi_k^2 - 2\sum_{\substack{k\leqslant n,\\ k\in N}} c_k(r_{k-1}, e_k) + \sum_{\substack{k\leqslant n,\\ k\in N}} c_k^2 \\ &\leqslant \|r_0\|^2 - \frac{1}{2}\sum_{\substack{k\leqslant n,\\ k\in M}} c_k(r_{k-1}, e_k) - 2\sum_{\substack{k\leqslant n,\\ k\in N}} c_k(r_{k-1}, e_k) + \sum_{\substack{k\leqslant n,\\ k\in N}} c_k^2. \end{aligned} \tag{7}$$

Rewriting inequality (7) in terms of the sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$, we have

$$\|r_n\|^2 \leqslant \|r_0\|^2 - \frac{1}{2}\sum_{k=1}^{n} x_k(r_{k-1}, e_k) - 2\sum_{k=1}^{n} y_k(r_{k-1}, e_k) + \sum_{k=1}^{n} y_k^2. \tag{8}$$

Now, using the inequality

$$(r_{k-1}, e_k)^2 - \xi_k^2 \leqslant (r_{k-1}, e_k)^2 \leqslant \left(\frac{10}{9}\right)^2 c_k^2 < 2c_k^2,$$

which holds for $k \in M$, we additionally obtain the complementary estimate:

$$\|r_n\|^2 \geqslant \|r_0\|^2 - 2\sum_{k=1}^{n} x_k^2 - 2\sum_{k=1}^{n} y_k(r_{k-1}, e_k) + \sum_{k=1}^{n} y_k^2. \tag{9}$$

6. There are two possible cases for the sequence $\{y_k\}_{k=1}^{\infty}$: the series $\sum_{k=1}^{\infty} y_k^2$ either converges or diverges.
In the first case, using (4) and passing to the limit in (8), we obtain that

$$\sum_{k=1}^{\infty} x_k^2 < \infty.$$

It implies that

$$\sum_{k=1}^{\infty}(x_k + y_k)^2 = \sum_{k=1}^{\infty} c_k^2 < \infty.$$

At the same time, taking into account Lemma 3, we have $\sum_{k=1}^{\infty} c_k = \infty$.

But due to [8, Theorem 2], which gives sufficient condition for the convergence for GAPC expansion, we have $r_n \longrightarrow 0$. This is a contradiction to (2). Therefore, the first case $\sum_{k=1}^{\infty} y_k^2 < \infty$ is not possible.

7. It remains to show that the case $\sum_{k=1}^{\infty} y_k^2 = \infty$ is also impossible.

We first prove two lemmas, which describe the properties of the sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ that follow from the convergence of $\sum_{k=1}^{\infty} y_k^2$.

Lemma 5. *Under the aforementioned conditions,*

$$y_n = o\left(\frac{1}{\sqrt{n}}\right). \tag{10}$$

Proof. From the construction of the set N, it follows that

$$N = \bigcup_j \overline{M_j} \cup \widetilde{N}.$$

Let $n \in \overline{M_j}$ for some index j. Similar to the fourth step of the proof, let l_j be the first element of the set $\widetilde{M_j}$. Then $l_j - 1 \in \widetilde{N}$, and so, by (3) and (4), we have:

$$y_{l_j} = c_{l_j} < \frac{11}{10}\left(r_{l_j-1}, e_{l_j}\right) < 2\left(r_{l_j-2} - c_{l_j-1}e_{l_j-1}, e_{l_j}\right) \leqslant 2\left(r_{l_j-2}, e_{l_j-1}\right) + 2c_{l_j-1} \leqslant 20|\xi_{l_j-1}| + 22|\xi_{l_j-1}| = 42|\xi_{l_j-1}|$$

(if $j = 1$ and $l_j = 1$, then we omit this estimate).

It follows from the selection of the index p_j that for $i \in \{l_j + 1, l_j + 2, \dots, p_j\}$ (or for all $i \in \widetilde{M_j}\backslash l_j$ if such an index p_j does not exist for this j), then $y_i \leqslant \frac{y_{i-1}}{2}$ holds. Thus, the sequence $\{y_i\}_{i \in \widetilde{M_j}\backslash M}$ is decreasing at least as fast as the geometric progression with the common ratio $\frac{1}{2}$.

If $n \in \widetilde{N}$, then by (3), we have:

$$y_n = c_n = (r_{n-1}, e_n) + \xi_n \leqslant 11|\xi_n|.$$

Now Lemma 5 follows from the aforementioned estimates and the fact that $\xi_n = o\left(\frac{1}{\sqrt{n}}\right)$. □

Lemma 6. *Under the aforementioned conditions,*

$$\sum_{n=1}^{\infty} x_n^2 < \infty. \tag{11}$$

Proof. Let us consider an arbitrary nonempty set M_j. The set M_j is finite, since otherwise $\sum_{n=1}^{\infty} y_n < \infty$, which contradicts the condition of the considered case. Combining (1) and (4) for $n \in M_j$, we obtain that

$$\begin{aligned} h_n^2 + \xi_n^2 &= \|r_n\|^2 = (r_n, r_n) = (r_{n-1} - c_n e_n, r_{n-1} - c_n e_n) \\ &= \|r_{n-1}\|^2 - 2c_n(r_{n-1}, e_n) + c_n^2 = h_{n-1}^2 + \xi_{n-1}^2 - 2c_n(r_{n-1}, e_n) + c_n^2 \\ &\leqslant h_{n-1}^2 + \xi_{n-1}^2 - \frac{20}{11}c_n^2 + c_n^2 = h_{n-1}^2 + \xi_{n-1}^2 - \frac{9}{11}c_n^2. \end{aligned} \tag{12}$$

Let q_j and $s_j = p_j + 1$ be the last and the first elements of the set M_j, respectively. Summing (12) over all $n \in M_j$, we obtain

$$h_{q_j}^2 + \xi_{q_j}^2 \leqslant h_{s_j-1}^2 + \xi_{s_j-1}^2 - \frac{9}{11}\sum_{i=s_j}^{q_j} c_i^2. \tag{13}$$

Note that $s_j - 1 = p_j \in \widetilde{M_j}$, and so, $c_{s_j-1} < 2c_{s_j}$ due to the selection of the index p_j. Therefore,

$$|\xi_{s_j-1}| \leqslant \frac{(r_{s_j-2}, e_{s_j-1})}{10} < \frac{c_{s_j-1}}{9} < \frac{2c_{s_j}}{9}. \tag{14}$$

From (13) and (14), we have:

$$\frac{9}{11}\sum_{i=s_j}^{q_j} c_i^2 \leqslant h_{s_j-1}^2 + \xi_{s_j-1}^2 - h_{q_j}^2 - \xi_{q_j}^2 < h_{s_j-1}^2 - h_{q_j}^2 + \frac{4c_{s_j}^2}{81}.$$

As a result,

$$\frac{1}{2}\sum_{i=s_j}^{q_j} c_i^2 < h_{q_j}^2 - h_{s_j-1}^2. \tag{15}$$

Summing (15) over all j (with nonempty M_j) and using the fact that $\{h_n\}_{n=1}^{\infty}$ is monotone (Lemma 1), we find that

$$\sum_{j\in M} c_j^2 = \sum_{n=1}^{\infty} x_n^2 < \infty,$$

which completes the proof of Lemma 6. □

8. In this step, we need to prove two more auxiliary lemmas.

Lemma 7. *Under the conditions of this case,*

$$\liminf_{n\to\infty} S_n^N r_D(r_{n-1}) = 0. \tag{16}$$

Proof. By Lemma 5, we have $\sum_{n=1}^{\infty} y_n = \infty$ and $y_n = o\Big(\frac{1}{\sqrt{n}}\Big)$.

In view of these properties, the proof of the lemma can be carried out so as in [9, Lemma 1]. For the sake of completeness, we provide all the details below.

Assume on the contrary that there exists a number $c > 0$ and $p \in \mathbb{N}$ such that

$$r_D(r_{k-1})S_k^N \geqslant c \quad (k \geqslant p).$$

Without the loss of generality, let us assume that $p = 1$ (we can achieve that by shifting the sequence of the remainders; if $S_k^N \equiv 0$, then the conclusion of the lemma is obvious).

From (8), we obtain

$$\begin{aligned} \|r_n\|^2 &\leqslant \|r_0\|^2 - \frac{1}{2}\sum_{k=1}^{n} x_k(r_{k-1}, e_k) - 2\sum_{k=1}^{n} y_k(r_{k-1}, e_k) + \sum_{k=1}^{n} y_k^2 \\ &\leqslant \|r_n\|^2 - 2\sum_{k=1}^{n} \frac{y_k}{S_k^N}\|r_{k-1}\| r_D(r_{k-1})S_k^N + \sum_{k=1}^{n} y_k^2 \\ &\leqslant \|r_n\|^2 - 2cr\sum_{k=1}^{n} \frac{y_k}{S_k^N} + \sum_{k=1}^{n} y_k^2. \end{aligned} \tag{17}$$

It is known (see Abel-Dini theorem [10]) that if $\sum_{k=1}^{\infty} y_k = \infty$, then $\sum_{k=1}^{\infty} \frac{y_k}{S_k^N} = \infty$. Also, in our case, we have $\sum_{k=1}^{\infty} y_k^2 = \infty$.

Next, since there exists a number $a > 0$ such that $y_k < \frac{a}{\sqrt{k}}$ for all $k \in \mathbb{N}$, there exists a number $A > 0$ such that $S_k^N < A\sqrt{k}$ for all k. As $y_k = o\Big(\frac{1}{\sqrt{k}}\Big)$, there exists a function $f(k)$ (here, we assume that $f(k)$ might be equal to ∞) such that $y_k = \frac{1}{\sqrt{k}f(k)}$ and $f(k) \to \infty$ as $k \to \infty$. Therefore,

$$\sum_{k=1}^{n} \frac{y_k}{S_k^N} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}f(k)S_k^N} \geqslant \sum_{k=1}^{n} \frac{1}{Akf(k)}. \tag{18}$$

We note that in the considered case

$$\sum_{k=1}^{\infty} y_k^2 = \sum_{k=1}^{\infty} \frac{1}{kf^2(k)} = \infty,$$

and so

$$\sum_{k=1}^{\infty} \frac{1}{kf(k)} = \infty. \tag{19}$$

Combining (17)–(19), we obtain the estimate

$$0 < \|r_n\|^2 \leqslant \|r_0\|^2 - 2cr\sum_{k=1}^{n} \frac{1}{Akf(k)} + \sum_{k=1}^{n} \frac{1}{kf^2(k)} = \|r_0\|^2 - \sum_{k=1}^{n} \frac{1}{kf(k)}\left(\frac{2cr}{A} - \frac{1}{f(k)}\right) \longrightarrow -\infty,$$

as $n \longrightarrow \infty$. This contradiction completes the proof of Lemma 7. □

Lemma 8. *In the considered case,*

$$\liminf_{n\to\infty} S_n^M r_D(r_{n-1}) = 0. \tag{20}$$

Proof. We argue by contradiction. Assume, on the contrary, that there exists a number $\beta > 0$ such that $S_n^M r_D(r_{n-1}) > \beta$ for every $n \in \mathbb{N}$ (similarly to the previous lemma).

Let us note that if $\sum_{k=1}^{\infty} x_k < \infty$, then the inequality $S_n^N > S_n^M$ holds for all sufficiently large indices. Therefore, using Lemma 7, we obtain

$$\liminf_{n\to\infty} S_n r_D(r_{n-1}) = 0,$$

which contradicts the assertion of Lemma 4. Thus,

$$\sum_{k=1}^{\infty} x_k = \infty. \tag{21}$$

Now, we note that from Lemma 2 and inequalities (9) and (11), we have

$$\sum_{k=1}^{\infty} (2y_k(r_{k-1}, e_k) - y_k^2) < \infty. \tag{22}$$

Now by using (2) and applying the Abel-Dini theorem from [10] to the sequence $\{x_k\}_{k=1}^{\infty}$, we find that

$$\sum_{k=1}^{n} x_k(r_{k-1}, e_k) = \sum_{k=1}^{n} \frac{x_k}{S_k^M} S_k^M r_D(r_{k-1})\|r_{k-1}\| > r\beta \sum_{k=1}^{n} \frac{x_k}{S_k^M} \longrightarrow \infty, \quad n \longrightarrow \infty,$$

which together with (8) contradicts (22) . This contradiction proves Lemma 8. □

9. In this step, we finalize the proof of Theorem 1 by proving the following lemma.

Lemma 9. *In the considered case,*

$$\liminf_{n\to\infty} S_n r_D(r_{n-1}) = 0. \tag{23}$$

Proof. Assume on the contrary, that there exists a number $\alpha > 0$ such that for all $n \in \mathbb{N}$, the inequality

$$S_n r_D(r_{n-1}) \geqslant \alpha. \tag{24}$$

In view of (2) and (20), there exists a number l_1 such that

$$S_{l_1}^M\left(r_{l_1-1}, e_{l_1}\right) < \frac{\alpha}{4}.$$

By (2) and (16), there exists a number $k_2 > l_1$ such that

$$S_{k_2}^N\left(r_{k_2-1}, e_{k_2}\right) < \frac{\alpha}{4}.$$

Let k_1 be the largest number such that $k_1 < k_2$ and

$$S_{k_1}^M\left(r_{k_1-1}, e_{k_1}\right) < \frac{\alpha}{4}.$$

Combining these estimates with (24), we obtain

$$S_{k_1}^M\left(r_{k_1-1}, e_{k_1}\right) < \frac{\alpha}{4},$$
$$S_{k_2}^N\left(r_{k_2-1}, e_{k_2}\right) < \frac{\alpha}{4},$$
$$S_{k_2}^M\left(r_{k_2-1}, e_{k_2}\right) > \frac{3\alpha}{4},$$
$$S_{k_1}^N\left(r_{k_1-1}, e_{k_1}\right) > \frac{3\alpha}{4}.$$

As a result, we have

$$\frac{3\alpha}{4} < S_{k_2}^M\left(r_{k_2-1}, e_{k_2}\right) < \frac{\alpha}{4}\frac{S_{k_2}^M}{S_{k_2}^N},$$
$$\frac{3\alpha}{4} < S_{k_1}^N\left(r_{k_1-1}, e_{k_1}\right) < \frac{\alpha}{4}\frac{S_{k_1}^N}{S_{k_1}^M},$$

and hence,

$$S_{k_1}^M < \frac{1}{3}S_{k_1}^N \leqslant \frac{1}{3}S_{k_2}^N < \frac{1}{9}S_{k_2}^M. \tag{25}$$

Lemma 6 implies that in the considered case we have $x_k \to 0,\ (k \to \infty)$, and so by selecting a sufficiently large l_1, we can guarantee that $2x_k < S_k^M$ for every $k \in M$ exceeding k_1.

It is easy to see that, for $x \in \left[0, \frac{1}{2}\right)$, the following inequality

$$4x \geqslant -\ln(1-x)$$

holds.

From (25), we have

$$\sum_{k=k_1+1}^{k_2} \frac{x_k}{S_k^M} \geqslant -\frac{1}{4}\sum_{k=k_1+1}^{k_2} \ln\left(1 - \frac{x_k}{S_k^M}\right) = -\frac{1}{4}\sum_{k=k_1+1}^{k_2} \ln\left(\frac{S_{k-1}^M}{S_k^M}\right) = -\frac{1}{4}\ln\left(\frac{S_{k_1}^M}{S_{k_2}^M}\right) > \frac{\ln 3}{2}. \tag{26}$$

Using (26) and the fact that for $k_1 < k \leqslant k_2$, we have

$$S_k^M(r_{k-1}, e_k) \geqslant \frac{\alpha}{4},$$

and then we obtain that

$$\sum_{k=k_1+1}^{k_2} x_k(r_{k-1}, e_k) = \sum_{k=k_1+1}^{k_2} \frac{x_k}{S_k^M} S_k^M r_D(r_{k-1})\|r_{k-1}\| \geqslant \frac{r\alpha}{4}\sum_{k=k_1+1}^{k_2} \frac{x_k}{S_k^M} > \frac{r\alpha \ln 3}{8}.$$

Now, we find a number $l_2 > k_2$ such that

$$S_{l_1}^M\left(r_{l_1-1}, e_{l_1}\right) < \frac{\alpha}{4}.$$

Similar to the selection of k_1 and k_2, we select k_3 and k_4 ($k_2 < k_3 < k_4$) to satisfy

$$\sum_{k=k_3+1}^{k_4} x_k(r_{k-1}, e_k) > \frac{r\alpha \ln 3}{8}.$$

Continuing this procedure, we obtain that $\sum_{k=1}^{\infty} x_k(r_{k-1}, e_k) = \infty$, which together with (22) contradicts (8). This competes the proof of Lemma 9. □

The assertions of Lemmas 4 and 9 contradict each other, and this contradiction comes from the assumption that $r_n \not\to 0$, $(n \to \infty)$. Thus, $r_n \to 0$ and the proof of Theorem 1 is complete.

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