

Research Article

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Asymptotic behavior of even-order noncanonical neutral differential equations

<https://doi.org/10.1515/dema-2022-0001>

received November 18, 2021; accepted December 3, 2021

Abstract: In this article, we study the asymptotic behavior of even-order neutral delay differential equation

$$(a \cdot (u + \rho \cdot u \circ \tau)^{(n-1)})'(\ell) + h(\ell)u(g(\ell)) = 0, \quad \ell \geq \ell_0,$$

where $n \geq 4$, and in noncanonical case, that is,

$$\int_a^\infty a^{-1}(s)ds < \infty.$$

To the best of our knowledge, most of the previous studies were concerned only with the study of n -order neutral equations in canonical case. By using comparison principle and Riccati transformation technique, we obtain new criteria which ensure that every solution of the studied equation is either oscillatory or converges to zero. Examples are presented to illustrate our new results.

Keywords: differential equations, even-order, oscillatory behavior, noncanonical case

MSC 2020: 34C10, 34K11

1 Introduction

The neutral DDEs have many interesting applications in various branches of applied science, as these equations appear in the modeling of many technological phenomena, see [1,2]. It is well known that the modeling of natural and technological phenomena produces differential equations, often of higher-order; see, for instance, the papers [3,4]. Oscillation theory is a branch of qualitative theory that investigates the oscillatory and non-oscillatory behavior of solutions to differential equations.

In this work, we consider the even-order NDDE

$$(a \cdot (u + \rho \cdot u \circ \tau)^{(m-1)})'(\ell) + h(\ell)u(g(\ell)) = 0, \quad (1)$$

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where $\ell \geq \ell_0$, $m \geq 4$ is an even integer and a, ρ, τ, h , and g are continuous real-valued functions on $[\ell_0, \infty)$. We also assume that $a \in C^1([\ell_0, \infty), (0, \infty))$, $a'(\ell) > 0$, $\rho(\ell) \in [0, \rho_0]$, $\rho_0 < 1$ is a constant, $h \geq 0$, $h \equiv 0$ on any half-line $[L, \infty)$ for all $L \geq \ell_0$, $\tau(\ell) \leq \ell$, $g(\ell) \leq \ell$, $\lim_{\ell \rightarrow \infty} \tau(\ell) = \lim_{\ell \rightarrow \infty} g(\ell) = \infty$ and $\int_{\ell_0}^{\infty} a^{-1}(s) ds < \infty$.

A solution of (1) is a function $u \in C([\ell_u, \infty), \mathbb{R})$, $\ell_u \geq \ell_0$, which satisfies the properties $u + \rho u(\tau) \in C^{(m-1)}([\ell_u, \infty), \mathbb{R})$, $a(u + \rho u(\tau))^{(m-1)} \in C^1([\ell_u, \infty), \mathbb{R})$ and u satisfies (1) on $[\ell_u, \infty)$. We consider only the proper solutions u of (1), that is, u is not identically zero eventually. A solution u of (1) is called *oscillatory* if it is neither positive nor negative, ultimately; otherwise, it is called *nonoscillatory*.

For several decades, an growing interest in presenting criteria for oscillation of different classes of DDE has been observed. Recently, the works [5–9] have developed many techniques and approaches for studying the oscillations of delay and advanced second-order equations. However, neutral second-order equations have been studied in many techniques through works [10–15]. The development in the study of second-order DDEs was reflected on the study of even-order equations, see for delay [16–21] and for neutral [22–28]. On the other hand, the works ([29,30] for third-order, and [31–33] for odd-order) contributed to the development of the oscillatory theory of odd-order delay differential equations.

Zhang et al. [19] investigated the oscillatory behavior of a higher-order differential equation

$$(a(\ell)(u^{(m-1)}(\ell))^{\alpha})' + h(\ell)u^{\beta}(g(\ell)) = 0, \quad (2)$$

where α, β are ratios of odd natural numbers and

$$\int_{\ell_0}^{\infty} a^{-1/\alpha}(s) ds < \infty. \quad (3)$$

Zhang et al. [19] obtained results which ensure that every solution u of (2) is either oscillatory or satisfies $\lim_{\ell \rightarrow \infty} u(\ell) = 0$. Zhang et al. [20] studied the oscillation of (2) for $\alpha \geq \beta$ and improved the results reported in [19].

For fourth-order, Zhang et al. [21] studied

$$(a(u''')^{\alpha})'(\ell) + h(\ell)u^{\alpha}(g(\ell)) = 0$$

and presented some oscillation criteria (including Hille-and Nehari-type criteria). Moreover, Moaaz and Muhib [17] presented criteria for fourth-order DDE

$$(a(u''')^{\alpha})'(\ell) + f(\ell, u(g(\ell))) = 0,$$

under the conditions $f(\ell, u) \geq h(\ell)u^{\beta}$ and α, β are the ratios of odd natural numbers.

For neutral delay equations, Zhang et al. [28] studied the even-order nonlinear NDDE

$$(u + \rho \cdot u \circ \tau)^{(m)}(\ell) + h(\ell)f(u(g(\ell))) = 0,$$

under the conditions $uf(u) > 0$ for all $u \neq 0$, and f is nondecreasing. Moaaz et al. [25] investigated the asymptotic behavior of solutions of the higher-order NDDE

$$(a((u + \rho \cdot u \circ \tau)^{(m-1)})^{\alpha})'(t) + f(\ell, u(g(\ell))) = 0,$$

where $|f(\ell, u)| \geq h(\ell)|u|^{\beta}$.

To the best of our knowledge, the previous studies in the literature which considered the asymptotic behavior of solutions of NDDEs of m -order were concerned only with the canonical form $\int_{\ell_0}^{\infty} a^{-1}(s) ds = \infty$. In this paper, we obtain new conditions for testing oscillation of NDDE (1) in noncanonical case, using Riccati substitution along with comparison principles with first-order DDE. Examples are presented to illustrate our new results.

In the following, we present useful lemmas that will be used throughout the results.

Lemma 1.1. [34, Lemma 2.2.3] Assume that $\varpi \in C^m([\ell_0, \infty), \mathbb{R}^+)$, $\varpi^{(m)}$ is not identically zero on a subray of $[\ell_0, \infty)$ and $\varpi^{(m)}$ is of fixed sign. Suppose that $\varpi^{(m-1)}\varpi^{(m)} \leq 0$ for $\ell \in [\ell_1, \infty)$, where $\ell_1 \geq \ell_0$ large enough. If $\lim_{\ell \rightarrow \infty} \varpi(\ell) \neq 0$, then there exists a $\ell_\lambda \in [\ell_1, \infty)$ such that

$$\varpi \geq \frac{\lambda}{(m-1)!} \ell^{m-1} |\varpi^{(m-1)}|,$$

for every $\lambda \in (0, 1)$ and $\ell \in [\ell_\lambda, \infty)$.

Lemma 1.2. [35, Lemma 1] Let $f \in C^m([\ell_0, \infty), \mathbb{R})$, $f^{(r)} > 0$, $r = 0, 1, \dots, m$ and $f^{(m+1)} \leq 0$ eventually. Then, for every $\eta \in (0, 1)$

$$f(\ell) \geq \frac{\eta^\ell}{m} f'(\ell).$$

Lemma 1.3. [36, Lemma 1.2] Assume that $B \geq 0$, $A > 0$, $w \geq 0$ and $\alpha > 0$. Then,

$$Bw - Aw^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}.$$

Lemma 1.4. [20, Lemma 2.1] Let $f \in C^n([t_0, \infty), (0, \infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for all large t , then there exist a t_x such that $t_x \geq t_0$ and an integer l , $0 \leq l \leq n$, with $n+l$ even for $f^{(n)}(t) \geq 0$, or $n+l$ odd for $f^{(n)}(t) \leq 0$ such that

$$l > 0 \quad \text{implies} \quad f^{(k)}(t) > 0 \quad \text{for } t \geq t_x, \quad k = 0, 1, \dots, l-1$$

and

$$l \leq n-1 \quad \text{implies} \quad (-1)^{l+k} f^{(k)}(t) > 0 \quad \text{for } t \geq t_x, \quad k = l, l+1, \dots, n-1.$$

2 Main results

For the convenience, we use notation $v := u + \rho \cdot u \circ \tau$.

Lemma 2.1. Assume that $u \in C([\ell_0, \infty), (0, \infty))$ is a solution of (1), eventually. Then, $v > 0$, $(av^{(m-1)})' \leq 0$ and v satisfies one of the following:

- (1) v' , $v^{(m-1)}$ and $(-v^{(m)})$ are positive;
- (2) v' , $v^{(m-2)}$ and $(-v^{(m-1)})$ are positive;
- (3) $(-1)^k v^{(k)}$ are positive, for all $k = 1, 2, \dots, m-1$,

for ℓ large enough.

Proof. Assume that u is an eventually positive solution of (1). It follows from (1) that

$$(a(\ell)v^{(m-1)}(\ell))' = -h(\ell)u(g(\ell)) \leq 0.$$

Now, from above inequality and Lemma 2 that there exist three possible cases (1)–(3) for $\ell \geq \ell_1$ large enough. $\square$

Lemma 2.2. Assume that $u \in C([\ell_0, \infty), (0, \infty))$ is a solution of (1), where v satisfies case (3). If

$$\int_{\ell_0}^{\infty} \int_{\ell}^{\infty} (\zeta - \ell)^{m-3} \left(\frac{1}{a(\zeta)} \int_{\ell_1}^{\zeta} h(s) ds \right) d\zeta d\ell = \infty, \quad (4)$$

then, $\lim_{\ell \rightarrow \infty} u(\ell) = 0$.

Proof. Assume that u is an eventually positive solution of (1), where v satisfies case (3). Then, $\lim_{\ell \rightarrow \infty} v(\ell) = D$. We claim that $D = 0$. Suppose that $D > 0$, and so for all $\varepsilon > 0$, there exists $\ell_1 \geq \ell_0$ such that $u(g(\ell)) \geq D$ for $\ell \geq \ell_1$. Integrating (1) from ℓ_1 to ℓ , we get

$$a(\ell)v^{(m-1)}(\ell) = a(\ell_1)v^{(m-1)}(\ell_1) - \int_{\ell_1}^{\ell} h(s)u(g(s))ds \leq -D \int_{\ell_1}^{\ell} h(s)ds,$$

that is,

$$v^{(m-1)}(\ell) < -D \frac{1}{a(\ell)} \int_{\ell_1}^{\ell} h(s)ds. \quad (5)$$

Integrating (5) twice from ℓ to ∞ , we obtain

$$-v^{(m-2)}(\ell) < -D \int_{\ell}^{\infty} \left(\frac{1}{a(\zeta)} \int_{\ell_1}^{\zeta} h(s)ds \right) d\zeta$$

and

$$v^{(m-3)}(\ell) < -D \int_{\ell}^{\infty} \int_s^{\infty} \left(\frac{1}{a(\zeta)} \int_{\ell_1}^{\zeta} h(s)ds \right) d\zeta ds = -D \int_{\ell}^{\infty} (\zeta - \ell) \left(\frac{1}{a(\zeta)} \int_{\ell_1}^{\zeta} h(s)ds \right) d\zeta. \quad (6)$$

Similarly, integrating (6) $m - 4$ times from ℓ to ∞ , we find

$$v'(\ell) < -D \int_{\ell}^{\infty} (\zeta - \ell)^{m-3} \left(\frac{1}{a(\zeta)} \int_{\ell_1}^{\zeta} h(s)ds \right) d\zeta.$$

Integrating this inequality from ℓ_1 to ∞ , we obtain

$$v(\ell_1) > D \int_{\ell_1}^{\infty} \int_{\varrho}^{\infty} (\zeta - \ell)^{m-3} \left(\frac{1}{a(\zeta)} \int_{\ell_1}^{\zeta} h(s)ds \right) d\zeta d\varrho,$$

which is a contradiction with (4). Thus, $D = 0$. This completes the proof. $\square$

Theorem 2.1. Let (4) hold. If there exists a $\lambda_0 \in (0, 1)$ such that the first-order delay differential equation

$$y'(\ell) + h(\ell) \frac{\lambda_0(1 - \rho(g(\ell)))(g(\ell))^{m-1}}{(m-1)!a(g(\ell))} y(g(\ell)) = 0 \quad (7)$$

is oscillatory and

$$\limsup_{\ell \rightarrow \infty} \int_{\ell_0}^{\ell} \left(\frac{\lambda_1 h(s)(1 - \rho(g(s)))g^{m-2}(s)}{(m-2)!} \delta(s) - \frac{1}{4a(s)\delta(s)} \right) ds = \infty \quad (8)$$

holds for some constant $\lambda_1 \in (0, 1)$, then every nonoscillatory solution u of (1) satisfies $\lim_{t \rightarrow \infty} u(\ell) = \infty$.

Proof. Suppose that (1) has a positive solution u which satisfies $\lim_{\ell \rightarrow \infty} u(\ell) \neq 0$. It follows from (1) that

$$(a(\ell)v^{(m-1)}(\ell))' = -h(\ell)u(g(\ell)) \leq 0. \quad (9)$$

From Lemma 2.1, there are three possible cases for the behavior of v and its derivatives.

Let (1) hold. From Lemma 1.1, we have

$$v(\ell) \geq \frac{\lambda \ell^{m-1}}{(m-1)!} v^{(m-1)}(\ell) \quad (10)$$

for every $\lambda \in (0, 1)$. It follows from the definition of $v(\ell)$ that

$$u(\ell) = v(\ell) - \rho(\ell)u(\tau(\ell)) \geq (1 - \rho(\ell))v(\ell). \quad (11)$$

Combining (9) and (11), we get

$$(a(\ell)v^{(m-1)}(\ell))' \leq -h(\ell)(1 - \rho(g(\ell)))v(g(\ell)). \quad (12)$$

From (10), we obtain

$$(a(\ell)v^{(m-1)}(\ell))' + h(\ell) \frac{\lambda(1 - \rho(g(\ell)))(g(\ell))^{m-1}}{(m-1)!} v^{(m-1)}(g(\ell)) \leq 0.$$

Now, we define the function $y(\ell) = a(\ell)v^{(m-1)}(\ell)$. Clearly, y is a positive solution of the first-order delay differential inequality

$$y'(\ell) + h(\ell) \frac{\lambda(1 - \rho(g(\ell)))(g(\ell))^{m-1}}{(m-1)!a(g(\ell))} y(g(\ell)) \leq 0. \quad (13)$$

Thus, using [40, Theorem 1], equation (7) has also a positive solution for all $\lambda_0 \in (0, 1)$, this contradicts the assumption that (7) is oscillatory.

Let (2) hold. We define ω by

$$\omega(\ell) = \frac{a(\ell)v^{(m-1)}(\ell)}{v^{(m-2)}(\ell)}, \quad \ell \geq \ell_1. \quad (14)$$

Then, $\omega(\ell) < 0$ for $\ell \geq \ell_1$. Noting that $(a(\ell)v^{(m-1)}(\ell))' \leq 0$, we find

$$a(s)v^{(m-1)}(s) \leq a(\ell)v^{(m-1)}(\ell), \quad s \geq \ell \geq \ell_1. \quad (15)$$

Dividing (15) by a and integrating it from ℓ to ∞ , we obtain

$$0 \leq v^{(m-2)}(\ell) + a(\ell)v^{(m-1)}(\ell)\delta(\ell),$$

which yields

$$-\frac{a(\ell)v^{(m-1)}(\ell)\delta(\ell)}{v^{(m-2)}(\ell)} \leq 1.$$

Thus, by (14), we get

$$-\omega(\ell)\delta(\ell) \leq 1. \quad (16)$$

Differentiating (14), we arrive at

$$\omega'(\ell) = \frac{(a(\ell)v^{(m-1)}(\ell))'}{v^{(m-2)}(\ell)} - \frac{a(\ell)(v^{(m-1)}(\ell))^2}{(v^{(m-2)}(\ell))^2},$$

which follows from (1) and (14) that

$$\omega'(\ell) = -\frac{h(\ell)u(g(\ell))}{v^{(m-2)}(\ell)} - \frac{\omega^2(\ell)}{a(\ell)}. \quad (17)$$

From the definition of $v(\ell)$ and the fact that $v'(\ell) > 0$, we get that (11) holds. Hence, it follows from (17) that

$$\omega'(\ell) \leq -\frac{h(\ell)(1 - \rho(g(\ell)))v(g(\ell))}{v^{(m-2)}(\ell)} - \frac{\omega^2(\ell)}{a(\ell)}. \quad (18)$$

Using Lemma 1.1, we get

$$v(\ell) \geq \frac{\lambda \ell^{m-2}}{(m-2)!} v^{(m-2)}(\ell)$$

for every $\lambda \in (0, 1)$ and for all sufficiently large ℓ . Then, (18) becomes

$$\omega'(\ell) \leq -\frac{\lambda h(\ell)(1 - \rho(g(\ell)))g^{m-2}(\ell)v^{(m-2)}(g(\ell))}{(m-2)!v^{(m-2)}(\ell)} - \frac{\omega^2(\ell)}{a(\ell)}.$$

Since $\ell \geq g(\ell)$ and $v^{(m-2)}(\ell)$ are decreasing, we have

$$\omega'(\ell) \leq -\frac{\lambda h(\ell)(1 - \rho(g(\ell)))g^{m-2}(\ell)}{(m-2)!} - \frac{\omega^2(\ell)}{a(\ell)}. \quad (19)$$

Multiplying (19) by $\delta(\ell)$ and integrating it from ℓ_1 to ℓ , we have

$$0 \geq \delta(\ell)\omega(\ell) - \delta(\ell_1)\omega(\ell_1) + \int_{\ell_1}^{\ell} \frac{\omega(s)}{a(s)} ds + \int_{\ell_1}^{\ell} \frac{\delta(s)}{a(s)} \omega^2(s) ds + \int_{\ell_1}^{\ell} \frac{\lambda h(s)(1 - \rho(g(s)))g^{m-2}(s)}{(m-2)!} \delta(s) ds.$$

Setting $A = \delta(s)/a(s)$, $B = 1/a(s)$, and $w = -\omega(s)$, and using Lemma 1.3, we have

$$\int_{\ell_1}^{\ell} \left(\frac{\lambda h(s)(1 - \rho(g(s)))g^{m-2}(s)}{(m-2)!} \delta(s) - \frac{1}{4a(s)\delta(s)} \right) ds \leq \delta(\ell_1)\omega(\ell_1) + 1,$$

due to (16), which contradicts (8).

Assume that case (3) holds. From Lemma 2.2 and (4), we see that $\lim_{\ell \rightarrow \infty} u(\ell) = 0$, which is a contradiction.

This completes the proof. $\square$

Corollary 2.1. Assume that (4) and (8) hold. If

$$\liminf_{\ell \rightarrow \infty} \int_{g(\ell)}^{\ell} h(s) \frac{(1 - \rho(g(s)))(g(s))^{m-1}}{(m-1)!a(g(s))} ds > \frac{1}{e}, \quad (20)$$

for some $\lambda_1 \in (0, 1)$, then every nonoscillatory solution u of (1) satisfies $\lim_{t \rightarrow \infty} u(\ell) = \infty$.

Proof. By [38, Theorem 2.1.1], assumption (20) ensures that the differential equation (7) has no positive solutions. Application of Theorem 2.1 yields the result. $\square$

Remark 2.1. Combining Theorem 2.1 and the results reported in [39] for the oscillation of equation (7), one can derive various oscillation criteria for equation (1).

Example 2.1. We consider the NDDE

$$(e^\ell(u(\ell) + \rho_0 u(\theta\ell)))''' + h_0 e^\ell u(\varepsilon\ell) = 0, \quad (21)$$

where $h_0 > 0$ and $\theta, \varepsilon \in (0, 1)$. Note that, $a(\ell) = e^\ell$, $\rho(\ell) = \rho_0$, $\tau(\ell) = \theta\ell$, $h(\ell) = h_0 e^\ell$ and $g(\ell) = \varepsilon\ell$. It is easy to see that $\delta(\ell) = e^{-\ell}$.

Now, from Corollary 2.1, we have

$$\begin{aligned} \int_{\ell_0}^{\infty} \int_{\varrho}^{\infty} (\varsigma - \ell)^{m-3} \left(\frac{1}{a(\varsigma)} \int_{\ell_1}^{\varsigma} h(s) ds \right) d\varsigma d\varrho &= \int_{\ell_0}^{\infty} \int_{\varrho}^{\infty} (\varsigma - \ell) \left(\frac{1}{e^\varsigma} \int_{\ell_1}^{\varsigma} h_0 e^s ds \right) d\varsigma d\varrho = \infty, \\ \liminf_{\ell \rightarrow \infty} \int_{g(\ell)}^{\ell} h(s) \frac{(1 - \rho(g(s)))(g(s))^{m-1}}{(m-1)!a(g(s))} ds &= \liminf_{\ell \rightarrow \infty} \int_{g(\ell)}^{\ell} h_0 e^s \frac{(1 - \rho_0)(\varepsilon s)^3}{3!e^{\varepsilon s}} ds = \infty > \frac{1}{e} \end{aligned}$$

and

$$\limsup_{\ell \rightarrow \infty} \int_{\ell_0}^{\ell} \left(\frac{\lambda_1 h(s)(1 - \rho(g(s)))g^{m-2}(s)}{(m-2)!} \delta(s) - \frac{1}{4a(s)\delta(s)} \right) ds = \limsup_{\ell \rightarrow \infty} \int_{\ell_0}^{\ell} \left(\frac{\lambda_1 h_0 e^s (1 - \rho_0)(\varepsilon s)^2}{2!} e^{-s} - \frac{1}{4e^s e^{-s}} \right) ds = \infty.$$

Thus, (4), (20), and (8) are satisfied. Therefore, every solution of (21) is oscillatory or tends to zero.

Example 2.2. Consider the equation

$$(\ell^2(u(\ell) + \rho_0 u(\theta\ell))''')' + \frac{h_0}{\ell^2} u(\varepsilon\ell) = 0, \quad (22)$$

here $h_0 > 0$ and $\theta, \varepsilon \in (0, 1)$. We note that $m = 4$, $a(\ell) = \ell^2$, $\rho(\ell) = \rho_0$, $\tau(\ell) = \theta\ell$, $h(\ell) = h_0/\ell^2$, and $g(\ell) = \varepsilon\ell$. It is easy to see that $\delta(\ell) = 1/\ell$ and (4) holds. Next, (20) reduces to

$$h_0(1 - \rho_0) \ln \frac{1}{\varepsilon} > \frac{3!}{\varepsilon\ell}. \quad (23)$$

Moreover, (8) becomes

$$\lim_{\ell \rightarrow \infty} \sup_{\ell_0} \int_{\ell_0}^{\ell} \left(\frac{h_0 \lambda_1 (1 - \rho_0) \varepsilon^2}{2!} - \frac{1}{4} \right) \frac{1}{s} ds = \infty,$$

which is verified if

$$h_0(1 - \rho_0) > \frac{1}{2\varepsilon^2}. \quad (24)$$

Using Corollary 2.1, if

$$h_0 > M := \max \left\{ \frac{3!}{e(1 - \rho_0)\varepsilon \ln \frac{1}{\varepsilon}}, \frac{1}{2(1 - \rho_0)\varepsilon^2} \right\},$$

then every solution of (22) is oscillatory or tends to zero, where

$$M = \frac{1}{2(1 - \rho_0)\varepsilon^2} \quad \text{if } \varepsilon \in (0, 0.28464]$$

and

$$M = \frac{3!}{e(1 - \rho_0)\varepsilon \ln \frac{1}{\varepsilon}} \quad \text{if } \varepsilon \in (0.28464, 1).$$

It is easy to notice that (20) does not apply in the ordinary case ($g(\ell) = \ell$). So, in the following theorem, we set new conditions for testing the oscillation of (1) when $m = 4$, which apply in the ordinary case.

Theorem 2.2. Assume that $m = 4$ and (4) hold. If

$$\limsup_{\ell \rightarrow \infty} \int_{\ell_0}^{\ell} \left(\frac{\lambda_1 h(s)(1 - \rho(g(s)))g^2(s)}{2!} \delta(s) - \frac{1}{4a(s)\delta(s)} \right) ds = \infty, \quad (25)$$

for some constant $\lambda_1 \in (0, 1)$. Assume further that there exist two positive functions $\zeta(\ell), \vartheta(\ell) \in C^1[\ell_0, \infty)$, such that

$$\int_{\ell_0}^{\infty} \left(\zeta(s)h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{3/\eta} - \frac{1}{2} \frac{(\zeta'(s))^2}{\zeta(s)} \frac{a(s)}{\lambda_2 s^2} \right) ds = \infty \quad (26)$$

and

$$\int_{\ell_0}^{\infty} \left(\vartheta(s) \int_s^{\infty} \left(\frac{1}{a(v)} \int_v^{\infty} h(\varsigma) (1 - \rho(g(\varsigma))) \left(\frac{g(\varsigma)}{\varsigma} \right)^{1/\eta} d\varsigma \right) dv - \frac{(\vartheta'(s))^2}{4\vartheta(s)} \right) ds = \infty \quad (27)$$

for some constant $\lambda_2 \in (0, 1)$. Then, every nonoscillatory solution u of (1) satisfies $\lim_{t \rightarrow \infty} u(\ell) = \infty$.

Proof. Assume that (1) has a nonoscillatory solution u which is eventually positive and $\lim_{\ell \rightarrow \infty} u(\ell) \neq 0$. It follows from (1) and Lemma 2.1 that there exist four possible cases for the behavior of v and its derivatives:

- (i) $v'(\ell) > 0$, $v''(\ell) > 0$, $v'''(\ell) > 0$ and $v^{(4)}(\ell) \leq 0$;
- (ii) $v'(\ell) > 0$, $v''(\ell) < 0$, $v'''(\ell) > 0$ and $v^{(4)}(\ell) \leq 0$;
- (iii) $v'(\ell) < 0$, $v''(\ell) > 0$ and $v'''(\ell) < 0$;
- (iv) $v'(\ell) > 0$, $v''(\ell) > 0$ and $v'''(\ell) < 0$.

Let (i) hold. Define the function $\phi(\ell)$ by

$$\phi(\ell) = \zeta(\ell) \frac{a(\ell)v'''(\ell)}{v(\ell)}.$$

Then, clearly $\phi(\ell)$ is positive for $\ell \geq \ell_1$ and satisfies

$$\phi'(\ell) = \frac{\zeta'(\ell)}{\zeta(\ell)} \phi(\ell) + \zeta(\ell) \left(\frac{(a(\ell)v'''(\ell))'}{v(\ell)} - \frac{a(\ell)v'''(\ell)v'(\ell)}{v^2(\ell)} \right). \quad (28)$$

From (1) and (28), we have

$$\phi'(\ell) = \frac{\zeta'(\ell)}{\zeta(\ell)} \phi(\ell) - \zeta(\ell) \frac{h(\ell)u(g(\ell))}{v(\ell)} - \zeta(\ell) \frac{a(\ell)v'''(\ell)v'(\ell)}{v^2(\ell)}. \quad (29)$$

Using (11) and (29), we get

$$\phi'(\ell) \leq \frac{\zeta'(\ell)}{\zeta(\ell)} \phi(\ell) - \zeta(\ell) \frac{h(\ell)(1 - \rho(g(\ell)))v(g(\ell))}{v(\ell)} - \zeta(\ell) \frac{a(\ell)v'''(\ell)v'(\ell)}{v^2(\ell)}. \quad (30)$$

Now, it follows from Lemmas 1.1 and 1.2 that

$$v'(\ell) \geq \frac{\lambda_2 \ell^2}{2} v'''(\ell) \quad (31)$$

and so

$$\frac{v(g(\ell))}{v(\ell)} \geq \left(\frac{g(\ell)}{\ell} \right)^{3/\eta}, \quad (32)$$

respectively. Substituting (31) and (32) into (30), we get

$$\phi'(\ell) \leq \frac{\zeta'(\ell)}{\zeta(\ell)} \phi(\ell) - \zeta(\ell) h(\ell) (1 - \rho(g(\ell))) \left(\frac{g(\ell)}{\ell} \right)^{3/\eta} - \frac{\lambda_2 \ell^2}{2} \frac{\zeta(\ell) a(\ell) (v'''(\ell))^2}{v^2(\ell)}.$$

From the definition of $\phi(\ell)$, we obtain

$$\phi'(\ell) \leq \frac{\zeta'(\ell)}{\zeta(\ell)} \phi(\ell) - \zeta(\ell) h(\ell) (1 - \rho(g(\ell))) \left(\frac{g(\ell)}{\ell} \right)^{3/\eta} - \frac{\lambda_2 \ell^2}{2 \zeta(\ell) a(\ell)} \phi^2(\ell).$$

Setting $A = \lambda_2 \ell^2 / 2 \zeta(\ell) a(\ell)$, $B = \zeta'(\ell) / \zeta(\ell)$, and $\varsigma = \phi(s)$ and using Lemma 1.3, we have

$$\phi'(\ell) \leq -\zeta(\ell) h(\ell) (1 - \rho(g(\ell))) \left(\frac{g(\ell)}{\ell} \right)^{3/\eta} + \frac{1}{2} \frac{(\zeta'(\ell))^2}{\zeta(\ell)} \frac{a(\ell)}{\lambda_2 \ell^2}. \quad (33)$$

Integrating (33) from ℓ_1 to ℓ , we have

$$\int_{\ell_1}^{\ell} \left(\zeta(s)h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{3/\eta} - \frac{1}{2} \frac{(\zeta'(s))^2}{\zeta(s)} \frac{a(s)}{\lambda_2 s^2} \right) ds \leq \phi(\ell_1),$$

which contradicts (26).

Assume that case (ii) holds. Define the function $\varphi(\ell)$ by

$$\varphi(\ell) = g(\ell) \frac{v'(\ell)}{v(\ell)}.$$

Then, clearly $\varphi(\ell)$ is positive for $\ell \geq \ell_1$ and satisfies

$$\varphi'(\ell) = \frac{g'(\ell)}{g(\ell)} \varphi(\ell) + g(\ell) \left(\frac{v''(\ell)}{v(\ell)} - \frac{(v'(\ell))^2}{v^2(\ell)} \right).$$

From the definition of $\varphi(\ell)$, we obtain

$$\varphi'(\ell) = \frac{g'(\ell)}{g(\ell)} \varphi(\ell) + g(\ell) \frac{v''(\ell)}{v(\ell)} - \frac{\varphi^2(\ell)}{g(\ell)}. \quad (34)$$

Integrating (1) from ℓ to ∞ , we have

$$-a(\ell)v'''(\ell) = - \int_{\ell}^{\infty} h(s)u(g(s))ds. \quad (35)$$

Using (11) and (35), we get

$$-a(\ell)v'''(\ell) = - \int_{\ell}^{\infty} h(s)(1 - \rho(g(s)))v(g(s))ds. \quad (36)$$

From Lemma 1.2, we get

$$v(\ell) \geq \eta \ell v'(\ell),$$

that is,

$$\frac{v(g(\ell))}{v(\ell)} \geq \left(\frac{g(\ell)}{\ell} \right)^{1/\eta}. \quad (37)$$

Combining (37) and (36), we get

$$-a(\ell)v'''(\ell) \leq -v(\ell) \int_{\ell}^{\infty} h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{1/\eta} ds,$$

that is,

$$-v'''(\ell) \leq -\frac{v(\ell)}{a(\ell)} \int_{\ell}^{\infty} h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{1/\eta} ds.$$

Integrating the above inequality from ℓ to ∞ , we have

$$v''(\ell) \leq -v(\ell) \int_{\ell}^{\infty} \left(\frac{1}{a(v)} \int_v^{\infty} h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{1/\eta} ds \right) dv.$$

Combining the above inequality with (34), we obtain

$$\varphi'(\ell) \leq -\vartheta(\ell) \int_{\ell}^{\infty} \left(\frac{1}{a(v)} \int_v^{\infty} h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{1/\eta} ds \right) dv + \frac{\vartheta'(\ell)}{\vartheta(\ell)} \varphi(\ell) - \frac{\varphi^2(\ell)}{\vartheta(\ell)}.$$

Thus, we have

$$\varphi'(\ell) \leq -\vartheta(\ell) \int_{\ell}^{\infty} \left(\frac{1}{a(v)} \int_v^{\infty} h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{1/\eta} ds \right) dv + \frac{(\vartheta'(\ell))^2}{4\vartheta(\ell)}. \quad (38)$$

Integrating (38) from ℓ_1 to ℓ , we have

$$\int_{\ell_1}^{\ell} \left(\vartheta(s) \int_s^{\infty} \left(\frac{1}{a(v)} \int_v^{\infty} h(\varsigma)(1 - \rho(g(\varsigma))) \left(\frac{g(\varsigma)}{\varsigma} \right)^{1/\eta} d\varsigma \right) dv - \frac{(\vartheta'(s))^2}{4\vartheta(s)} \right) ds \leq \varphi(\ell_1),$$

which contradicts (27).

The proof of the case where (iii) or (iv) holds is the same as that of Theorem 2.1.

This completes the proof. $\square$

Example 2.3. Consider the equation

$$(\ell^2(u(\ell) + \rho_0 u(\theta\ell))''')' + \frac{h_0}{\ell^2} u(\ell) = 0, \quad (39)$$

here $h_0 > 0$ and $\theta \in (0, 1]$. It is easy to see that $\delta(\ell) = 1/\ell$ and (4) holds. Let $\zeta(\ell) = \vartheta(\ell) = \ell$.

Next, using Theorem 2.2, we find

$$\limsup_{\ell \rightarrow \infty} \int_{\ell_0}^{\ell} \left(\frac{\lambda_1 h(s)(1 - \rho(g(s))) g^2(s)}{2!} \delta(s) - \frac{1}{4a(s)\delta(s)} \right) ds = \limsup_{\ell \rightarrow \infty} \int_{\ell_0}^{\ell} \left(\frac{h_0 \lambda_1 (1 - \rho_0) s^2}{s^2} \frac{1}{2!} \frac{1}{s} - \frac{1}{4s^2(1/s)} \right) ds = \infty,$$

which is verified if

$$h_0(1 - \rho_0) > \frac{1}{2}$$

Moreover,

$$\int_{\ell_0}^{\infty} \left(\zeta(s) h(s)(1 - \rho(g(s))) \left(\frac{g(s)}{s} \right)^{3/\eta} - \frac{1}{2} \frac{(\zeta'(s))^2 a(s)}{\zeta(s) \lambda_2 s^2} \right) ds = \int_{\ell_0}^{\infty} \left(s \frac{h_0}{s^2} (1 - \rho_0) \left(\frac{s}{s} \right)^{3/\eta} - \frac{1}{2} \frac{1}{s} \frac{s^2}{\lambda_2 s^2} \right) ds = \infty,$$

which is verified if

$$h_0(1 - \rho_0) > \frac{1}{2}$$

and

$$\begin{aligned} & \int_{\ell_0}^{\infty} \left(\vartheta(s) \int_s^{\infty} \left(\frac{1}{a(v)} \int_v^{\infty} h(\varsigma)(1 - \rho(g(\varsigma))) \left(\frac{g(\varsigma)}{\varsigma} \right)^{1/\eta} d\varsigma \right) dv - \frac{(\vartheta'(s))^2}{4\vartheta(s)} \right) ds \\ &= \int_{\ell_0}^{\infty} \left(s \int_s^{\infty} \left(\frac{1}{v^2} \int_v^{\infty} \frac{h_0}{\varsigma^2} (1 - \rho_0) \left(\frac{\varsigma}{\varsigma} \right)^{1/\eta} d\varsigma \right) dv - \frac{1}{4s} \right) ds = \infty. \end{aligned}$$

Thus, every solution of (39) is oscillatory or tends to zero if $h_0(1 - \rho_0) > \frac{1}{2}$.

3 Conclusion

In this paper, we have presented new theorems for studying the asymptotic behavior and oscillation of (1). By using comparison principle and Riccati transformation technique, we obtained new criteria which ensure that every solution of the studied equation is either oscillatory or converges to zero. Suitable illustrative examples have also been provided. It will be of interest to investigate the odd-order equations.

Conflict of interest: The authors state no conflict of interest.

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