Research Article

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Strong convergence inertial projection algorithm with self-adaptive step size rule for pseudomonotone variational inequalities in Hilbert spaces

Abstract: In this paper, we introduce a new algorithm for solving pseudomonotone variational inequalities with a Lipschitz-type condition in a real Hilbert space. The algorithm is constructed around two algorithms: the subgradient extragradient algorithm and the inertial algorithm. The proposed algorithm uses a new step size rule based on local operator information rather than its Lipschitz constant or any other line search scheme and functions without any knowledge of the Lipschitz constant of an operator. The strong convergence of the algorithm is provided. To determine the computational performance of our algorithm, some numerical results are presented.

1 Introduction

This paper studies the problem of classic variational inequalities [1,2]. The *variational inequality problem* (VIP) for a mapping $\mathcal{A} : \mathcal{E} \to \mathcal{E}$, which is formulated in the following way:

$$\text{Find } q^* \in \mathcal{K} \text{ such that } \langle \mathcal{A}(q^*), v - q^* \rangle \geq 0, \quad \forall v \in \mathcal{K}, \tag{VIP}$$

where $\mathcal{K}$ is a nonempty, convex and closed subset of a real Hilbert space $\mathcal{E}$ and $\langle . , . \rangle$ and $\|.\|$ represent an inner product and the induced norm in $\mathcal{E}$, respectively. Moreover, $\mathbb{R}$, $\mathbb{N}$ are the sets of real numbers and natural numbers, respectively. It is important to note that the problem (VIP) is equivalent to solve the following problem:

$$\text{Find } q^* \in \mathcal{K} \text{ such that } q^* = P_{\mathcal{K}}[q^* - \zeta \mathcal{A}(q^*)].$$

The concept of variational inequalities has been used as an important tool for covering a large number of topics, i.e., physics, engineering, economics and optimization theory. This was introduced by Stampacchia [1]

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in 1964. This is a significant mathematical design that unifies several key topics of applied mathematics, such as the network equilibrium problem, the necessary optimality conditions, the complementarity problems and the systems of nonlinear equations (for more details [3–12]). On the other hand, the projection algorithms are important to find the numerical solution of variational inequalities. Many experts have introduced and considered many projection algorithms to study the variational inequality problems (see for more details [13–25]) and others in [26–28]. Korpelevich [13] and Antipin [29] introduced the following extragradient algorithm:

$$\begin{cases} u_0 \in \mathcal{K}, \\ v_n = P_{\mathcal{K}}[u_n - \zeta\mathcal{A}(u_n)], \\ u_{n+1} = P_{\mathcal{K}}[u_n - \zeta\mathcal{A}(v_n)]. \end{cases} \tag{1}$$

Recently, the subgradient extragradient algorithm was provided by Censor et al. [15] for solving the problem (VIP) in a real Hilbert space. Their algorithm takes the form

$$\begin{cases} u_0 \in \mathcal{K}, \\ v_n = P_{\mathcal{K}}[u_n - \zeta\mathcal{A}(u_n)], \\ u_{n+1} = P_{\mathcal{E}_n}[u_n - \zeta\mathcal{A}(v_n)], \end{cases} \tag{2}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta\mathcal{A}(u_n) - v_n, z - v_n \rangle \leq 0\}.$$

It is important to note that the above well-established algorithm carries two serious drawbacks, the first is the fixed constant step size that requires the knowledge or approximation of the Lipschitz constant of the relevant operator and it only converges weakly in Hilbert spaces. From the computational point of view, it might be problematic to use fixed step size, and hence the convergence rate and usefulness of the algorithm could be affected.

Yang et al. [30] proposed two explicit subgradient extragradient methods to solve monotone variational inequalities. An iterative sequence $\{u_n\}$ was generated in the following way:

Algorithm A.
 (i) Let $u_0 \in \mathcal{K}$, $\mu \in (0, 1)$ and $\zeta_0 > 0$.
 (ii) Compute iterative sequence $\{u_n\}$ for $n \geq 1$ as follows:

$$\begin{cases} v_n = P_{\mathcal{K}}[u_n - \zeta_n\mathcal{A}(u_n)], \\ u_{n+1} = P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{A}(v_n)], \end{cases} \tag{3}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta_n\mathcal{A}(u_n) - v_n, z - v_n \rangle \leq 0\}.$$

(iii) Update the step size rule in the following way:

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \dfrac{\mu\|u_n - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{\langle\mathcal{A}(u_n) - \mathcal{A}(v_n), u_{n+1} - v_n\rangle}\right\} & \text{if } \langle\mathcal{A}(u_n) - \mathcal{A}(v_n), u_{n+1} - v_n\rangle > 0, \\ \zeta_n & \text{otherwise}. \end{cases}$$

If $u_n = v_n$, then stop. Otherwise, set $n := n + 1$ and return to Step (ii).

Algorithm B.
 (i) Let $u_0 \in \mathcal{K}$, $\mu \in (0, 1)$, $\zeta_0 > 0$ and a sequence $\phi_n \subset (0, 1)$ with $\phi_n \to 0$ and $\sum_{n=1}^{\infty}\phi_n = +\infty$.
 (ii) Compute iterative sequence $\{u_n\}$ for $n \geq 1$ as follows:

$$\begin{cases} v_n = P_{\mathcal{K}}[u_n - \zeta_n\mathcal{A}(u_n)], \\ z_n = P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{A}(v_n)], \\ u_{n+1} = \phi_n u_0 + (1 - \phi_n)z_n, \end{cases} \tag{4}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta_n \mathcal{A}(u_n) - v_n, z - v_n \rangle \le 0\}.$$

(iii) Update the step size rule in the following way:

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \dfrac{\mu\|u_n - v_n\|^2 + \mu\|z_n - v_n\|^2}{\langle \mathcal{A}(u_n) - \mathcal{A}(v_n), z_n - v_n \rangle}\right\} & \text{if } \langle \mathcal{A}(u_n) - \mathcal{A}(v_n), z_n - v_n \rangle > 0, \\ \zeta_n & \text{otherwise.} \end{cases}$$

If $u_n = v_n$, then stop. Otherwise, set $n := n + 1$ and return to Step (ii).

The main objective of this paper is to set up an inertial-type algorithm that is used to improve the convergence rate of the iterative sequence. Such algorithms have been previously established due to the oscillator equation with a damping and conservative force restoration. This second-order dynamical system is called a heavy friction ball, which was originally studied by Polyak in [31]. The main feature of inertial-type algorithms is that they can use the two previous iterations to obtain the next iteration. Numerical results confirm that inertial terms normally improve the performance of the algorithm in terms of the number of iterations and elapsed time in this context.

So there is an important question:

"*Is it possible to establish a new inertial-like strongly convergent extragradient-type algorithm with a monotone variable step size rule?*"

In this research, we provide a positive answer to the above question, i.e., the gradient algorithm indeed establishes a strong convergence sequence by maintaining variable step size rule for solving problem (VIP) combined with pseudomonotone mappings. Motivated by the works of Censor et al. [15] and Polyak [31], we introduce a new inertial extragradient-type algorithm to figure out the problem (VIP) in the situation of an infinite-dimensional real Hilbert space.

Specifically, our key contributions to this paper are as follows:
⊙ We introduce an inertial subgradient extragradient algorithm with the use of a variable monotone step size rule independent of the Lipschitz constant to figure out pseudo-monotone VIPs.
⊙ We also provide numerical experiments corresponding to the proposed algorithms for the verification of theoretical findings and compare them with the results in Algorithm 1 in [30] and Algorithm 2 in [30]. Our numerical data have shown that the proposed algorithms are useful and performed better compared to the existing ones in most situations.

The rest of the paper is arranged as follows: Section 2 consists of the necessary definitions and fundamental lemmas needed in the article. Section 3 consists of an inertial-type iterative scheme and convergence analysis theorem. Section 4 provides numerical results to explain the performance of the new algorithm and to compare them with other existing algorithms.

2 Preliminaries

In this section of the text, we have written a number of significant identities and related lemmas and definitions.

The *metric projection* $P_{\mathcal{K}}(v_1)$ of $v_1 \in \mathcal{E}$ is defined by

$$P_{\mathcal{K}}(v_1) = \arg\min\{\|v_1 - v_2\| : v_2 \in \mathcal{K}\}.$$

Next, we list some of the important properties of the projection mapping.

Lemma 2.1. [32] *Suppose that $P_{\mathcal{K}} : \mathcal{E} \to \mathcal{K}$ is a metric projection. Then, we have*
(i) $v_3 = P_{\mathcal{K}}(v_1)$ *if and only if*

$$\langle v_1 - v_3, v_2 - v_3 \rangle \leq 0, \quad \forall v_2 \in \mathcal{K},$$

(ii)

$$\|v_1 - P_{\mathcal{K}}(v_2)\|^2 + \|P_{\mathcal{K}}(v_2) - v_2\|^2 \leq \|v_1 - v_2\|^2, \quad v_1 \in \mathcal{K}, \ v_2 \in \mathcal{E},$$

(iii)

$$\|v_1 - P_{\mathcal{K}}(v_1)\| \leq \|v_1 - v_2\|, \quad v_2 \in \mathcal{K}, \ v_1 \in \mathcal{E}.$$

Lemma 2.2. [33] *Let $\{p_n\} \subset [0, +\infty)$ be a sequence satisfying the following inequality*:

$$p_{n+1} \leq (1 - q_n)p_n + q_n r_n, \quad \forall n \in \mathbb{N}.$$

Furthermore, let $\{q_n\} \subset (0, 1)$ and $\{r_n\} \subset \mathbb{R}$ be two sequences such that

$$\lim_{n \to +\infty} q_n = 0, \quad \sum_{n=1}^{+\infty} q_n = +\infty \quad and \quad \limsup_{n \to +\infty} r_n \leq 0.$$

Then, $\lim_{n \to +\infty} p_n = 0$.

Lemma 2.3. [34] *Suppose that $\{p_n\}$ is a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$p_{n_i} < p_{n_{i+1}}, \quad \forall i \in \mathbb{N}.$$

Then, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to +\infty$ as $k \to +\infty$, and satisfies the following conditions for numbers $k \in \mathbb{N}$:

$$p_{m_k} \leq p_{m_{k+1}} \quad and \quad p_k \leq p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : p_j \leq p_{j+1}\}$.

Next, we list some of the important identities that were used to prove the convergence analysis.

Lemma 2.4. [32] *For any $v_1, v_2 \in \mathcal{E}$ and $\ell \in \mathbb{R}$, the following inequalities hold*:
(i)

$$\|\ell v_1 + (1 - \ell)v_2\|^2 = \ell\|v_1\|^2 + (1 - \ell)\|v_2\|^2 - \ell(1 - \ell)\|v_1 - v_2\|^2,$$

(ii)

$$\|v_1 + v_2\|^2 \leq \|v_1\|^2 + 2\langle v_2, v_1 + v_2 \rangle.$$

Lemma 2.5. [35] *Assume that $\mathcal{A} : \mathcal{K} \to \mathcal{E}$ is a pseudomonotone and continuous mapping. Then, q^* is a solution of the problem (VIP) if and only if q^* is a solution of the following problem*:

$$Find \ u \in \mathcal{K} \ such \ that \ \langle \mathcal{A}(v), v - u \rangle \geq 0, \quad \forall v \in \mathcal{K}.$$

3 Inertial two-step proximal-like algorithm and convergence analysis

In this section, we introduce an inertial-type subgradient extragradient algorithm which incorporates the new step size rule and the inertial term as well as provides both strong convergence theorems. The following main result is outlined as follows:

Algorithm 1

Step 0: Let $u_{-1}, u_0 \in \mathcal{K}$, $\alpha > 0$, $\mu \in (0, 1)$, $\zeta_0 > 0$. Moreover, choose $\{\beta_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n \to +\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \beta_n = +\infty.$$

Step 1: Compute

$$\eta_n = (1 - \beta_n)[u_n + \alpha_n(u_n - u_{n-1})],$$

where α_n such that

$$0 \leq \alpha_n \leq \hat{\alpha}_n \quad \text{and} \quad \hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \dfrac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if } u_n \neq u_{n-1}, \\ \alpha & \text{else}, \end{cases} \tag{5}$$

while $\varepsilon_n = \circ(\beta_n)$ is a positive sequence, i.e., $\lim_{n \to +\infty} \frac{\varepsilon_n}{\beta_n} = 0$.

Step 2: Compute

$$v_n = P_{\mathcal{K}}(\eta_n - \zeta_n \mathcal{A}(\eta_n)).$$

If $\eta_n = v_n$, then STOP and v_n is a solution. Otherwise, go to **Step 3**.

Step 3: Compute

$$u_{n+1} = P_{\mathcal{E}_n}(\eta_n - \zeta_n \mathcal{A}(v_n)),$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle \eta_n - \zeta_n \mathcal{A}(\eta_n) - v_n, z - v_n \rangle \leq 0\}.$$

(iii) Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \dfrac{\mu\|\eta_n - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{2\langle \mathcal{A}(\eta_n) - \mathcal{A}(v_n), u_{n+1} - v_n \rangle}\right\} & \text{if } \langle \mathcal{A}(\eta_n) - \mathcal{A}(v_n), u_{n+1} - v_n \rangle > 0, \\ \zeta_n & \text{otherwise}. \end{cases}$$

Set $n = n + 1$ and go back to **Step 1**.

Lemma 3.1. *A step size sequence $\{\zeta_n\}$ is monotonically decreasing with a lower bound $\min\left\{\frac{\mu}{L}, \zeta_0\right\}$ and converges to $\zeta > 0$.*

Proof. Consider that $\langle \mathcal{A}(\eta_n) - \mathcal{A}(v_n), u_{n+1} - v_n \rangle > 0$ such that

$$\frac{\mu(\|\eta_n - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2\langle \mathcal{A}(\eta_n) - \mathcal{A}(v_n), u_{n+1} - v_n \rangle} \geq \frac{2\mu\|\eta_n - v_n\|\|u_{n+1} - v_n\|}{2\|\mathcal{A}(\eta_n) - \mathcal{A}(v_n)\|\|u_{n+1} - v_n\|} \geq \frac{2\mu\|\eta_n - v_n\|\|u_{n+1} - v_n\|}{2L\|\eta_n - v_n\|\|u_{n+1} - v_n\|} \geq \frac{\mu}{L}. \tag{6}$$

This implies that the sequence $\{\zeta_n\}$ has a lower bound $\min\left\{\frac{\mu}{L}, \zeta_0\right\}$. Furthermore, there exists a real number $\zeta > 0$, such that $\lim_{n\to\infty}\zeta_n = \zeta$.   □

In order to study the strong convergence, the following conditions are satisfied:

($\mathcal{A}1$) The solution set of problem (VIP) denoted by Ω is nonempty;

($\mathcal{A}2$) An operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is said to be pseudomonotone, i.e.,

$$\langle \mathcal{A}(v_1), v_2 - v_1 \rangle \geq 0 \Rightarrow \langle \mathcal{A}(v_2), v_1 - v_2 \rangle \leq 0, \quad \forall v_1, v_2 \in \mathcal{K};$$

($\mathcal{A}3$) An operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is said to be *Lipschitz continuous* with constant $L > 0$, i.e., there exists $L > 0$ such that

$$\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\| \leq L\|v_1 - v_2\|, \quad \forall v_1, v_2 \in \mathcal{K};$$

($\mathcal{A}4$) An operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is said to be *sequentially weakly continuous*, i.e., $\{\mathcal{A}(u_n)\}$ converges weakly to $\mathcal{A}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

Lemma 3.2. *Suppose that an operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ satisfies the conditions ($\mathcal{A}1$)–($\mathcal{A}4$). For a given $q^* \in \Omega \neq \varnothing$, we have*

$$\|u_{n+1} - q^*\|^2 \leq \|\eta_n - q^*\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\eta_n - v_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - v_n\|^2.$$

Proof. It is given that

$$
\begin{aligned}
\|u_{n+1} - q^*\|^2 &= \|P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - q^*\|^2 \\
&= \|P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] + [\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)] - q^*\|^2 \\
&= \|[\eta_n - \zeta_n\mathcal{A}(v_n)] - q^*\|^2 + \|P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)]\|^2 \\
&\quad + 2\langle P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)], [\eta_n - \zeta_n\mathcal{A}(v_n)] - q^*\rangle.
\end{aligned}
\tag{7}
$$

Thus, we have

$$
\begin{aligned}
&\|P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)]\|^2 + \langle P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)], [\eta_n - \zeta_n\mathcal{A}(v_n)] - q^*\rangle \\
&= \langle [\eta_n - \zeta_n\mathcal{A}(v_n)] - P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)], q^* - P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)]\rangle \leq 0,
\end{aligned}
\tag{8}
$$

which implies that

$$\langle P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)], [\eta_n - \zeta_n\mathcal{A}(v_n)] - q^*\rangle \leq -\|P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)]\|^2. \tag{9}$$

Combining expressions (7) and (9), we obtain

$$
\begin{aligned}
\|u_{n+1} - q^*\|^2 &\leq \|\eta_n - \zeta_n\mathcal{A}(v_n) - q^*\|^2 - \|P_{\mathcal{E}_n}[\eta_n - \zeta_n\mathcal{A}(v_n)] - [\eta_n - \zeta_n\mathcal{A}(v_n)]\|^2 \\
&\leq \|\eta_n - q^*\|^2 - \|\eta_n - u_{n+1}\|^2 + 2\zeta_n\langle \mathcal{A}(v_n), q^* - u_{n+1}\rangle.
\end{aligned}
\tag{10}
$$

Since q^* is the solution of problem (VIP), we have

$$\langle \mathcal{A}(q^*), v - q^*\rangle \geq 0, \quad \text{for all } v \in \mathcal{K}.$$

Due to the pseudomonotonicity of $\mathcal{A}$ on $\mathcal{K}$, we get

$$\langle \mathcal{A}(v), v - q^*\rangle \geq 0, \quad \text{for all } v \in \mathcal{K}.$$

By substituting $v = v_n \in \mathcal{K}$, we obtain

$$\langle \mathcal{A}(v_n), v_n - q^*\rangle \geq 0.$$

Thus, we have

$$\langle \mathcal{A}(v_n), q^* - u_{n+1}\rangle = \langle \mathcal{A}(v_n), q^* - v_n\rangle + \langle \mathcal{A}(v_n), v_n - u_{n+1}\rangle \leq \langle \mathcal{A}(v_n), v_n - u_{n+1}\rangle. \tag{11}$$

Combining expressions (10) and (11), we get

$$
\begin{aligned}
\|u_{n+1} - q^*\|^2 &\le \|\eta_n - q^*\|^2 - \|\eta_n - u_{n+1}\|^2 + 2\zeta_n \langle \mathcal{A}(v_n), v_n - u_{n+1} \rangle \\
&\le \|\eta_n - q^*\|^2 - \|\eta_n - v_n + v_n - u_{n+1}\|^2 + 2\zeta_n \langle \mathcal{A}(v_n), v_n - u_{n+1} \rangle \\
&\le \|\eta_n - q^*\|^2 - \|\eta_n - v_n\|^2 - \|v_n - u_{n+1}\|^2 + 2\langle \eta_n - \zeta_n \mathcal{A}(v_n) - v_n, u_{n+1} - v_n \rangle.
\end{aligned}
\tag{12}
$$

By the use of $u_{n+1} = P_{\mathcal{E}_n}[\eta_n - \zeta_n \mathcal{A}(v_n)]$ and by the definition of ζ_{n+1}, we have

$$
\begin{aligned}
2\langle \eta_n - \zeta_n \mathcal{A}(v_n) - v_n, u_{n+1} - v_n \rangle &= 2\langle \eta_n - \zeta_n \mathcal{A}(\eta_n) - v_n, u_{n+1} - v_n \rangle + 2\zeta_n \langle \mathcal{A}(\eta_n) - \mathcal{A}(v_n), u_{n+1} - v_n \rangle \\
&= 2\frac{\zeta_n}{\zeta_{n+1}} \zeta_{n+1} \langle \mathcal{A}(\eta_n) - \mathcal{A}(v_n), u_{n+1} - v_n \rangle \\
&\le \frac{\mu\zeta_n}{\zeta_{n+1}} \|\eta_n - v_n\|^2 + \frac{\mu\zeta_n}{\zeta_{n+1}} \|u_{n+1} - v_n\|^2.
\end{aligned}
\tag{13}
$$

Combining expressions (12) and (13), we obtain

$$
\begin{aligned}
\|u_{n+1} - q^*\|^2 &\le \|\eta_n - q^*\|^2 - \|\eta_n - v_n\|^2 - \|v_n - u_{n+1}\|^2 + \frac{\zeta_n}{\zeta_{n+1}}[\mu\|\eta_n - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2] \\
&\le \|\eta_n - q^*\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\eta_n - v_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - v_n\|^2.
\end{aligned}
\tag{14}
$$

$\square$

Theorem 3.3. *Let* $\{u_n\}$ *be a sequence generated by* Algorithm 1 *and satisfy the conditions* $(\mathcal{A}1)$–$(\mathcal{A}4)$. *Then,* $\{u_n\}$ *strongly converges to* $q^* \in \Omega$. *Moreover,* $P_{\Omega}(0) = q^*$.

Proof. It is given that $\zeta_n \to \zeta$ such that $\varepsilon \in (0, 1 - \mu)$ and

$$
\lim_{n\to\infty}\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) = 1 - \mu > \varepsilon > 0.
$$

Thus, there exists a finite number $n_1 \in \mathbb{N}$ such that

$$
\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) > \varepsilon > 0, \quad \forall n \ge n_1.
\tag{15}
$$

This implies that

$$
\|u_{n+1} - q^*\|^2 \le \|\eta_n - q^*\|^2, \quad \forall n \ge n_1.
\tag{16}
$$

It is given in expression (16) that

$$
\lim_{n\to+\infty}\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \le \lim_{n\to+\infty}\frac{\varepsilon_n}{\beta_n}\|u_n - u_{n-1}\| = 0.
\tag{17}
$$

By the use of definition of $\{\eta_n\}$ and inequality (17), we obtain

$$
\begin{aligned}
\|\eta_n - q^*\| &= \|u_n + \alpha_n(u_n - u_{n-1}) - \beta_n u_n - \alpha_n\beta_n(u_n - u_{n-1}) - q^*\| \\
&= \|(1 - \beta_n)(u_n - q^*) + (1 - \beta_n)\alpha_n(u_n - u_{n-1}) - \beta_n q^*\|
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\le (1 - \beta_n)\|u_n - q^*\| + (1 - \beta_n)\alpha_n\|u_n - u_{n-1}\| + \beta_n\|q^*\| \\
&\le (1 - \beta_n)\|u_n - q^*\| + \beta_n M_1,
\end{aligned}
\tag{19}
$$

where

$$
(1 - \beta_n)\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| + \|q^*\| \le M_1.
$$

Combining expressions (16) and (19), we obtain

$$\|u_{n+1} - q^*\| \le (1 - \beta_n)\|u_n - q^*\| + \beta_n M_1$$
$$\le \max\{\|u_n - q^*\|, M_1\}$$
$$\vdots$$
$$\le \max\{\|u_0 - q^*\|, M_1\}. \tag{20}$$

Thus, we conclude that $\{u_n\}$ is a bounded sequence. Indeed, by expression (19) we have

$$\|\eta_n - q^*\|^2 \le (1 - \beta_n)^2\|u_n - q^*\|^2 + \beta_n^2 M_1^2 + 2M_1\beta_n(1 - \beta_n)\|u_n - q^*\|$$
$$\le \|u_n - q^*\|^2 + \beta_n[\beta_n M_1^2 + 2M_1(1 - \beta_n)\|u_n - q^*\|]$$
$$\le \|u_n - q^*\|^2 + \beta_n M_2, \tag{21}$$

where

$$\beta_n M_1^2 + 2M_1(1 - \beta_n)\|u_n - q^*\| \le M_2,$$

for some $M_2 > 0$. By using the expressions (14) with (21), we have

$$\|u_{n+1} - q^*\|^2 \le \|u_n - q^*\|^2 + \beta_n M_2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\eta_n - v_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - v_n\|^2. \tag{22}$$

The rest of the proof will be divided into the following two parts:

Case 1: Consider that a fixed number $n_2 \in \mathbb{N}$ ($n_2 \ge n_1$) such that

$$\|u_{n+1} - q^*\| \le \|u_n - q^*\|, \quad \forall n \ge n_2. \tag{23}$$

This implies that $\lim_{n\to+\infty}\|u_n - q^*\|$ exists and let $\lim_{n\to+\infty}\|u_n - q^*\| = l$ for some $l \ge 0$. From the expression (22), we have

$$\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\eta_n - v_n\|^2 + \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - v_n\|^2 \le \|u_n - q^*\|^2 + \beta_n M_2 - \|u_{n+1} - q^*\|^2. \tag{24}$$

Due to the existence of a limit of sequence $\|u_n - q^*\|$ and $\beta_n \to 0$, we deduce that

$$\|\eta_n - v_n\| \to 0 \quad \text{and} \quad \|u_{n+1} - v_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{25}$$

By the use of expression (25), we have

$$\lim_{n\to+\infty}\|\eta_n - u_{n+1}\| \le \lim_{n\to+\infty}\|\eta_n - v_n\| + \lim_{n\to+\infty}\|v_n - u_{n+1}\| = 0. \tag{26}$$

Next, we have compute

$$\|\eta_n - u_n\| = \|u_n + \alpha_n(u_n - u_{n-1}) - \beta_n[u_n + \alpha_n(u_n - u_{n-1})] - u_n\|$$
$$\le \alpha_n\|u_n - u_{n-1}\| + \beta_n\|u_n\| + \alpha_n\beta_n\|u_n - u_{n-1}\|$$
$$= \beta_n\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| + \beta_n\|u_n\| + \beta_n^2\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{27}$$

The following provides that

$$\lim_{n\to+\infty}\|u_n - u_{n+1}\| \le \lim_{n\to+\infty}\|u_n - \eta_n\| + \lim_{n\to+\infty}\|\eta_n - u_{n+1}\| = 0. \tag{28}$$

The above expression guarantees that the sequences $\{\eta_n\}$ and $\{v_n\}$ are also bounded. By the use of reflexivity of $\mathcal{E}$ and the boundedness of $\{u_n\}$ guarantees that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{u} \in \mathcal{E}$ as $k \to +\infty$. Next, we have to prove that $\hat{u} \in \Omega$.

It is given that

$$v_{n_k} = P_{\mathcal{K}}[\eta_{n_k} - \zeta_n\mathcal{A}(\eta_{n_k})],$$

that is equivalent to

$$\langle \eta_{n_k} - \zeta_n \mathcal{A}(\eta_{n_k}) - v_{n_k}, v - v_{n_k} \rangle \leq 0, \quad \forall v \in \mathcal{K}. \tag{29}$$

The inequality described above implies that

$$\langle \eta_{n_k} - v_{n_k}, v - v_{n_k} \rangle \leq \zeta_n \langle \mathcal{A}(\eta_{n_k}), v - v_{n_k} \rangle, \quad \forall v \in \mathcal{K}. \tag{30}$$

Furthermore, we obtain

$$\frac{1}{\zeta_n} \langle \eta_{n_k} - v_{n_k}, v - v_{n_k} \rangle + \langle \mathcal{A}(\eta_{n_k}), v_{n_k} - \eta_{n_k} \rangle \leq \langle \mathcal{A}(\eta_{n_k}), v - \eta_{n_k} \rangle, \quad \forall v \in \mathcal{K}. \tag{31}$$

The sequence $\{\mathcal{A}(\eta_{n_k})\}$ is bounded due to the boundedness of sequence $\{\eta_{n_k}\}$. By the use of $\lim_{k\to\infty}\|\eta_{n_k} - v_{n_k}\|$ = 0 and $k \to \infty$ in (31), we obtain

$$\liminf_{k\to\infty} \langle \mathcal{A}(\eta_{n_k}), v - \eta_{n_k} \rangle \geq 0, \quad \forall v \in \mathcal{K}. \tag{32}$$

Furthermore, we have

$$\langle \mathcal{A}(v_{n_k}), v - v_{n_k} \rangle = \langle \mathcal{A}(v_{n_k}) - \mathcal{A}(\eta_{n_k}), v - \eta_{n_k} \rangle + \langle \mathcal{A}(\eta_{n_k}), v - \eta_{n_k} \rangle + \langle \mathcal{A}(v_{n_k}), \eta_{n_k} - v_{n_k} \rangle. \tag{33}$$

Since $\lim_{k\to\infty}\|\eta_{n_k} - v_{n_k}\|$ = 0 and $\mathcal{A}$ is L-Lipschitz continuous. Thus, we have

$$\lim_{k\to\infty} \|\mathcal{A}(\eta_{n_k}) - \mathcal{A}(v_{n_k})\| = 0. \tag{34}$$

Combining expressions (33) and (34), we obtain

$$\liminf_{k\to\infty} \langle \mathcal{A}(v_{n_k}), v - v_{n_k} \rangle \geq 0, \quad \forall v \in \mathcal{K}. \tag{35}$$

Consider a sequence of positive numbers $\{\varepsilon_k\}$ that is decreasing and converges to zero. For each k, we denote m_k by the smallest positive integer such that

$$\langle \mathcal{A}(\eta_{n_i}), v - \eta_{n_i} \rangle + \varepsilon_k \geq 0, \quad \forall i \geq m_k. \tag{36}$$

It is obvious that $\{m_k\}$ is an increasing sequence because $\{\varepsilon_k\}$ is a decreasing sequence.

Case A: Let there exists a subsequence $\left\{\eta_{n_{m_{k_j}}}\right\}$ of sequence $\left\{\eta_{n_{m_k}}\right\}$ such that $\mathcal{A}\left(\eta_{n_{m_{k_j}}}\right)$ = 0 ($\forall j$). Consider that $j \to \infty$, we obtain

$$\langle \mathcal{A}(\hat{u}), v - \hat{u} \rangle = \lim_{j\to\infty} \left\langle \mathcal{A}(\eta_{n_{m_{k_j}}}), v - \hat{u} \right\rangle = 0. \tag{37}$$

Hence $\hat{u} \in \mathcal{K}$, therefore we obtain $\hat{u} \in \Omega$.

Case B: If there exists $n_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq n_0$, $\mathcal{A}(\eta_{n_{m_k}}) \neq 0$. Suppose that

$$\Delta_{n_{m_k}} = \frac{\mathcal{A}(\eta_{n_{m_k}})}{\|\mathcal{A}(\eta_{n_{m_k}})\|^2}, \quad \forall n_{m_k} \geq n_0. \tag{38}$$

On the basis of the above definition, we obtain

$$\left\langle \mathcal{A}(\eta_{n_{m_k}}), \Delta_{n_{m_k}} \right\rangle = 1, \quad \forall n_{m_k} \geq n_0. \tag{39}$$

By using expressions (36) and (39) for all $n_{m_k} \geq n_0$, we have

$$\left\langle \mathcal{A}(\eta_{n_{m_k}}), \ v + \varepsilon_k \Delta_{n_{m_k}} - \eta_{n_{m_k}} \right\rangle \geq 0. \tag{40}$$

Due to the pseudomonotonicity of $\mathcal{A}$ for $n_{m_k} \geq n_0$, we have

$$\left\langle \mathcal{A}(v + \varepsilon_k \Delta_{n_{m_k}}), \; v + \varepsilon_k \Delta_{n_{m_k}} - \eta_{n_{m_k}} \right\rangle \geq 0. \tag{41}$$

For all $n_{m_k} \geq n_0$, we have

$$\left\langle \mathcal{A}(v), v - \eta_{n_{m_k}} \right\rangle \geq \left\langle \mathcal{A}(v) - \mathcal{A}(v + \varepsilon_k \Delta_{n_{m_k}}), \; v + \varepsilon_k \Delta_{n_{m_k}} - \eta_{n_{m_k}} \right\rangle - \varepsilon_k \left\langle \mathcal{A}(v), \Delta_{n_{m_k}} \right\rangle. \tag{42}$$

Since $\{\eta_{n_k}\}$ weakly converges to $\hat{u} \in \mathcal{K}$ and $\mathcal{A}$ is sequentially weakly continuous, it implies that $\{\mathcal{A}(\eta_{n_k})\}$ weakly converges to $\mathcal{A}(\hat{u})$. Suppose that $\mathcal{A}(\hat{u}) \neq 0$, we have

$$\|\mathcal{A}(\hat{u})\| \leq \liminf_{k \to \infty} \left\| \mathcal{A}(\eta_{n_k}) \right\|. \tag{43}$$

Since $\{\eta_{n_{m_k}}\} \subset \{\eta_{n_k}\}$ and $\lim_{k \to \infty} \varepsilon_k = 0$, we have

$$0 \leq \lim_{k \to \infty} \|\varepsilon_k \Delta_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\varepsilon_k}{\|\mathcal{A}(\eta_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{A}(\hat{u})\|} = 0. \tag{44}$$

Next, considering $k \to \infty$ in (42), we obtain

$$\langle \mathcal{A}(v), v - \hat{u} \rangle \geq 0, \quad \forall v \in \mathcal{K}. \tag{45}$$

By the use of Minty Lemma 2.5, we infer $\hat{u} \in \Omega$. It is given that $q^* = P_\Omega(0)$ and by using Lemma 2.1(ii), we have

$$\langle 0 - q^*, v - q^* \rangle \leq 0, \quad \forall v \in \Omega. \tag{46}$$

Next, we have to

$$\limsup_{n \to +\infty} \langle q^*, q^* - u_n \rangle = \lim_{k \to +\infty} \langle q^*, q^* - u_{n_k} \rangle = \langle q^*, q^* - \hat{u} \rangle \leq 0. \tag{47}$$

Since $\lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0$, it gives that

$$\limsup_{n \to +\infty} \langle q^*, q^* - u_{n+1} \rangle \leq \limsup_{n \to +\infty} \langle q^*, q^* - u_n \rangle + \limsup_{n \to +\infty} \langle q^*, u_n - u_{n+1} \rangle \leq 0. \tag{48}$$

Taking into account expression (18), we have

$$\begin{aligned}
\|\eta_n - q^*\|^2 &= \|u_n + \alpha_n(u_n - u_{n-1}) - \beta_n u_n - \alpha_n \beta_n(u_n - u_{n-1}) - q^*\|^2 \\
&= \|(1 - \beta_n)(u_n - q^*) + (1 - \beta_n)\alpha_n(u_n - u_{n-1}) - \beta_n q^*\|^2 \\
&\leq \|(1 - \beta_n)(u_n - q^*) + (1 - \beta_n)\alpha_n(u_n - u_{n-1})\|^2 + 2\beta_n \langle -q^*, \eta_n - q^* \rangle \\
&= (1 - \beta_n)^2 \|u_n - q^*\|^2 + (1 - \beta_n)^2 \alpha_n^2 \|u_n - u_{n-1}\|^2 \\
&\quad + 2\alpha_n(1 - \beta_n)^2 \|u_n - q^*\| \|u_n - u_{n-1}\| + 2\beta_n \langle -q^*, \eta_n - u_{n+1} \rangle + 2\beta_n \langle -q^*, u_{n+1} - q^* \rangle \\
&\leq (1 - \beta_n)\|u_n - q^*\|^2 + \alpha_n^2 \|u_n - u_{n-1}\|^2 + 2\alpha_n(1 - \beta_n)\|u_n - q^*\| \|u_n - u_{n-1}\| \\
&\quad + 2\beta_n \|q^*\| \|\eta_n - u_{n+1}\| + 2\beta_n \langle -q^*, u_{n+1} - q^* \rangle \\
&= (1 - \beta_n)\|u_n - q^*\|^2 + \beta_n \Bigg[\alpha_n \|u_n - u_{n-1}\| \frac{\alpha_n}{\beta_n} \|u_n - u_{n-1}\| \\
&\quad + 2(1 - \beta_n)\|u_n - q^*\| \frac{\alpha_n}{\beta_n} \|u_n - u_{n-1}\| + 2\|q^*\| \|\eta_n - u_{n+1}\| + 2\langle q^*, q^* - u_{n+1} \rangle \Bigg].
\end{aligned} \tag{49}$$

From expressions (16) and (49), we obtain

$$\begin{aligned}
\|u_{n+1} - q^*\|^2 &\leq (1 - \beta_n)\|u_n - q^*\|^2 + \beta_n \Bigg[\alpha_n \|u_n - u_{n-1}\| \frac{\alpha_n}{\beta_n} \|u_n - u_{n-1}\| \\
&\quad + 2(1 - \beta_n)\|u_n - q^*\| \frac{\alpha_n}{\beta_n} \|u_n - u_{n-1}\| + 2\|q^*\| \|\eta_n - u_{n+1}\| + 2\langle q^*, q^* - u_{n+1} \rangle \Bigg].
\end{aligned} \tag{50}$$

By using expressions (26), (48), (50) and Lemma 2.2 together implies that $\lim_{n \to +\infty} \|u_n - q^*\| = 0$.

Case 2: Consider that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - q^*\| \leq \|u_{n_{i+1}} - q^*\|, \quad \forall i \in \mathbb{N}.$$

By using Lemma 2.3, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to +\infty$ such that

$$\|u_{m_k} - q^*\| \leq \|u_{m_{k+1}} - q^*\| \text{ and } \|u_k - q^*\| \leq \|u_{m_{k+1}} - q^*\|, \quad \text{for all } k \in \mathbb{N}. \tag{51}$$

Similar to Case 1, the relation (24) implies that

$$\left(1 - \frac{\mu\zeta_{m_k}}{\zeta_{m_k+1}}\right)\|\eta_{m_k} - v_{m_k}\|^2 + \left(1 - \frac{\mu\zeta_{m_k}}{\zeta_{m_k+1}}\right)\|u_{m_k+1} - v_{m_k}\|^2 \leq \|u_{m_k} - q^*\|^2 + \beta_{m_k}M_2 - \|u_{m_k+1} - q^*\|^2. \tag{52}$$

Due to $\beta_{m_k} \to 0$, we can conclude the following:

$$\lim_{k\to+\infty}\|\eta_{m_k} - v_{m_k}\| = \lim_{k\to+\infty}\|u_{m_k+1} - v_{m_k}\| = 0. \tag{53}$$

The above implies that

$$\lim_{k\to+\infty}\|u_{m_k+1} - \eta_{m_k}\| \leq \lim_{k\to+\infty}\|u_{m_k+1} - v_{m_k}\| + \lim_{k\to+\infty}\|v_{m_k} - \eta_{m_k}\| = 0. \tag{54}$$

Next, we have to compute

$$\begin{aligned}
\|\eta_{m_k} - u_{m_k}\| &= \|u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1}) - \beta_{m_k}[u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1})] - u_{m_k}\| \\
&\leq \alpha_{m_k}\|u_{m_k} - u_{m_k-1}\| + \beta_{m_k}\|u_{m_k}\| + \alpha_{m_k}\beta_{m_k}\|u_{m_k} - u_{m_k-1}\| \\
&= \beta_{m_k}\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + \beta_{m_k}\|u_{m_k}\| + \beta_{m_k}^2\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| \longrightarrow 0.
\end{aligned} \tag{55}$$

This leads that

$$\lim_{k\to+\infty}\|u_{m_k} - u_{m_k+1}\| \leq \lim_{k\to+\infty}\|u_{m_k} - \eta_{m_k}\| + \lim_{k\to+\infty}\|\eta_{m_k} - u_{m_k+1}\| = 0. \tag{56}$$

By using the same argument as in Case 1, we have

$$\limsup_{k\to+\infty}\langle q^*, q^* - u_{m_k+1}\rangle \leq 0. \tag{57}$$

Combining expressions (50) and (51), we obtain

$$\begin{aligned}
\|u_{m_k+1} - q^*\|^2 &\leq (1 - \beta_{m_k})\|u_{m_k} - q^*\|^2 + \beta_{m_k}\left[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\|\right. \\
&\quad + 2(1 - \beta_{m_k})\|u_{m_k} - q^*\|\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|q^*\|\|\eta_{m_k} - u_{m_k+1}\| + 2\langle q^*, q^* - u_{m_k+1}\rangle\right] \\
&\leq (1 - \beta_{m_k})\|u_{m_k+1} - q^*\|^2 + \beta_{m_k}\left[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\|\right. \\
&\quad + 2(1 - \beta_{m_k})\|u_{m_k} - q^*\|\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|q^*\|\|\eta_{m_k} - u_{m_k+1}\| + 2\langle q^*, q^* - u_{m_k+1}\rangle\right].
\end{aligned} \tag{58}$$

This implies that

$$\begin{aligned}
\|u_{m_k+1} - q^*\|^2 &\leq \left[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\|\right. \\
&\quad + 2(1 - \beta_{m_k})\|u_{m_k} - q^*\|\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|q^*\|\|\eta_{m_k} - u_{m_k+1}\| + 2\langle q^*, q^* - u_{m_k+1}\rangle\right].
\end{aligned} \tag{59}$$

Since $\beta_{m_k} \to 0$ and $\|u_{m_k} - q^*\|$ is a bounded, (57) and (59) imply that

$$\|u_{m_k+1} - q^*\|^2 \to 0, \quad \text{as } k \to +\infty. \tag{60}$$

This means that

$$\lim_{n \to +\infty} \|u_k - q^*\|^2 \le \lim_{n \to +\infty} \|u_{m_k+1} - q^*\|^2 \le 0. \tag{61}$$

As a consequence $u_n \to q^*$. This completes the proof of the theorem. □

4 Numerical illustrations

This section examines four numerical experiments to show the efficacy of the proposed algorithms. Any of these numerical experiments provide a detailed understanding of how better control parameters can be chosen. Some of them show the advantages of the proposed algorithms compared to the existing ones in the literature.

Example 4.1. First consider the HpHard problem that is considered from [36]. Let $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$ be an operator which is defined by

$$\mathcal{A}(u) = Mu + q,$$

where $q \in \mathbb{R}^N$ and

$$M = AA^T + B + D,$$

where A is an $N \times N$ matrix, B is an $N \times N$ skew-symmetric matrix and D is an $N \times N$ positive definite diagonal matrix. The set $\mathcal{K}$ is taken in the following way:

$$\mathcal{K} = \{u \in \mathbb{R}^N : -10 \le u_i \le 10\}.$$

It is clear that $\mathcal{A}$ is monotone and Lipschitz continuous through $L = \|M\|$. The control condition is taken as follows:

  (i) Algorithm 1 in [30] (shortly, Alg.1): $\zeta_0 = 0.20$, $\mu = 0.55$.
 (ii) Algorithm 2 in [30] (shortly, Alg.2): $\zeta_0 = 0.20$, $\mu = 0.55$, $\phi_n = \frac{1}{100(n+2)}$.
(iii) Algorithm 1 (shortly, M.Alg.1): $\zeta_0 = 0.20$, $\mu = 0.55$, $\alpha = 0.66$, $\varepsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{100(n+2)}$.

During this experiment, the initial point is $u_0 = u_1 = (1, 1, \dots, 1)$ and $D_n = \|\eta_n - v_n\| \le 10^{-4}$. The numerical results of these algorithms are shown in Figures 1–4 (Table 1).

Table 1: Numerical data for Figures 1–4

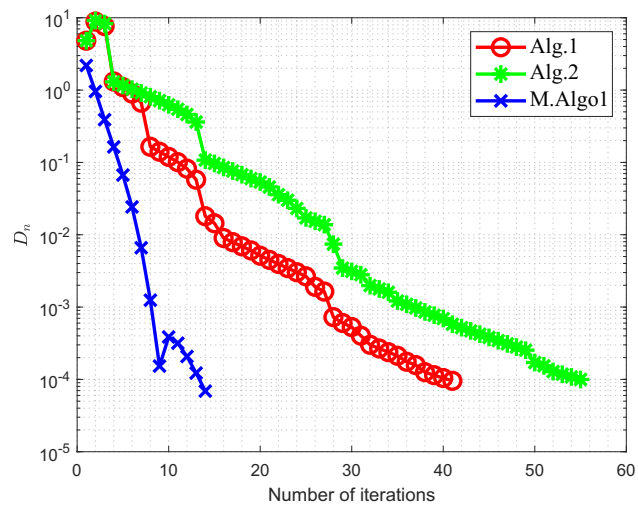| Algo.name	Algorithm 1 in [30]		Algorithm 2 in [30]		Algorithm 1	
N	iter.	time	iter.	time	iter.	time
5	41	0.223457	55	0.279106	14	0.079261
10	56	0.251080	60	0.236725	23	0.094484
20	390	2.016970	260	1.426439	64	0.516909
50	539	3.373166	607	5.546712	149	0.660777

Figure 1: Numerical comparison of Algorithm 1 with Algorithm 1 in [30] and Algorithm 2 in [30] taking $N = 5$.
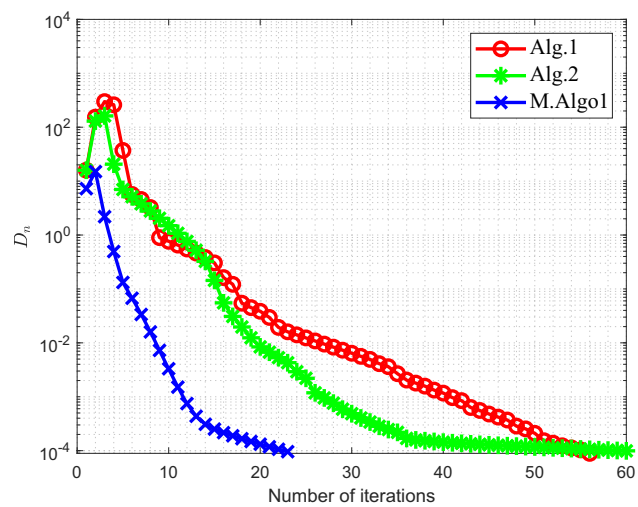

Figure 2: Numerical comparison of Algorithm 1 with Algorithm 1 in [30] and Algorithm 2 in [30] taking $N = 10$.
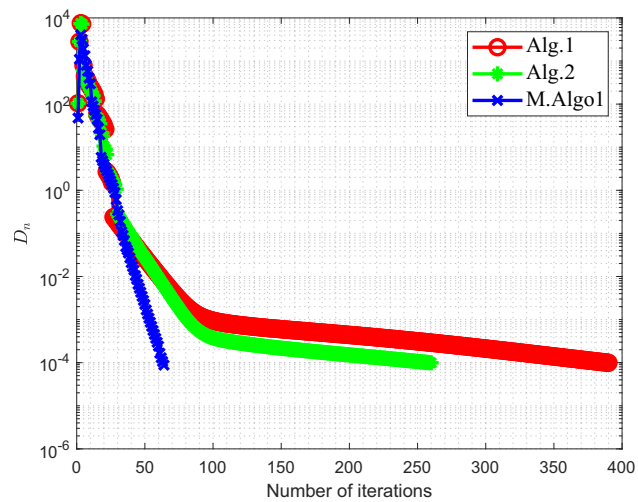

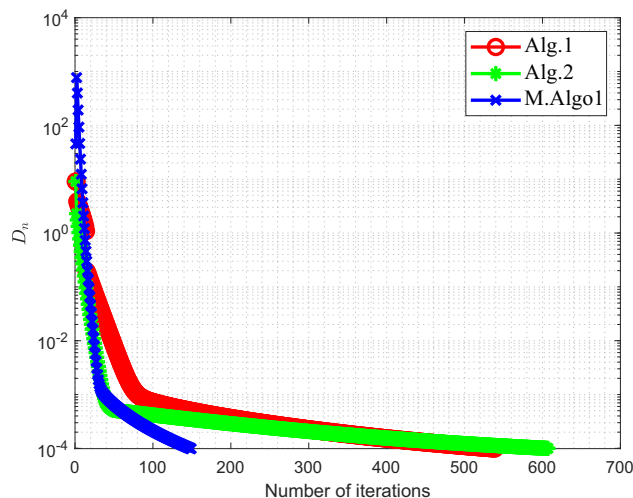**Figure 3:** Numerical comparison of Algorithm 1 with Algorithm 1 in [30] and Algorithm 2 in [30] taking $N = 20$.

Figure 4: Numerical comparison of Algorithm 1 with Algorithm 1 in [30] and Algorithm 2 in [30] taking $N = 50$.

Example 4.2. For second example, consider the quadratic fractional programming problem in the following form [37]:

$$\begin{cases} \min f(u) = \dfrac{u^T Q u + a^T u + a_0}{b^T u + b_0}, \\ \text{subject to } u \in \mathcal{K} = \{u \in \mathbb{R}^4 : b^T u + b_0 > 0\}, \end{cases}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad a_0 = -2, \quad \text{and} \quad b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite on $\mathbb{R}^4$ and consequently f is pseudo-convex on $\mathcal{K}$. Hence, ∇f is pseudo-monotone. Using the quotient rule, we obtain

$$\nabla f(u) = \frac{(b^T u + b_0)(2Qu + a) - b(u^T Q + a^T u + a_0)}{(b^T u + b_0)^2}. \tag{62}$$

In this point of view, we can set $\mathcal{A} = \nabla f$ in Theorem 3.3. We minimize f over $\mathcal{K} = \{u \in \mathbb{R}^4 : 1 \le u_i \le 10, i = 1, 2, 3, 4\}$. This problem has a unique solution $q^* = (1, 1, 1, 1)^T \in \mathcal{K}$. The control condition is taken as follows:

(i) Algorithm 1 in [30] (shortly, Alg.1): $\zeta_0 = 0.25$, $\mu = 0.35$.

(ii) Algorithm 2 in [30] (shortly, Alg.2): $\zeta_0 = 0.25$, $\mu = 0.35$, $\phi_n = \frac{1}{100(n+2)}$.

(iii) Algorithm 1 (shortly, M.Alg.1): $\zeta_0 = 0.25$, $\mu = 0.35$, $\alpha = 0.66$, $\varepsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{(n+2)}$.

During this experiment, the initial points are different and $D_n = \|\eta_n - v_n\| \le 10^{-4}$. The numerical results of these algorithms are shown in Tables 2–10.

Table 2: Example 4.2: Numerical study of Algorithm 1 in [30] and $u_0 = u_1 = [10, 10, 10, 10]^T$

Iter (*n*)	u_1	u_2	u_3	u_4
1	9.98361147989035	9.92819343243907	9.71188534311147	9.81908863135204
2	9.96559296019223	9.85721648952338	9.94292447196205	9.64014655402360
3	9.94598894074751	9.78705715319703	9.91314087916256	9.46314531337088
4	9.92484233950876	9.71770355613636	9.88255724745141	9.28805739204770
5	9.90219457402178	9.64914397858768	9.85119549140577	9.11485616807417
⋮	⋮	⋮	⋮	⋮
201	1.00047864580961	0.999581568887345	1.05015863755837	0.999952167633982
202	1.00047026644202	0.999595088981525	1.02743350011189	0.999953578214968
203	1.00046172864065	0.999608271436251	1.00540907952986	0.999954965358929
204	1.00012975635753	0.999877171237029	1.00019733298933	0.999962898727956
205	1.00001546608645	0.999987811938773	1.00001254139761	0.999995569409186
CPU time in seconds	0.855695			

Table 3: Example 4.2: Numerical study of Algorithm 2 in [30] and $u_0 = u_1 = [10, 10, 10, 10]^T$

Iter (*n*)	u_1	u_2	u_3	u_4
1	9.98722793812391	9.94270273066136	9.77776276401056	9.85566851062606
2	9.97338838960988	9.88584013385139	9.95494507925562	9.71235360979276
3	9.95851156075490	9.82945581779816	9.93157589425467	9.57016568769825
4	9.94262261197368	9.77356365065526	9.90767370103466	9.42914060818516
5	9.92574466957898	9.71816759658748	9.88325332674730	9.28928956648529
⋮	⋮	⋮	⋮	⋮
251	1.00030003847792	0.999736275445436	1.05436872725549	0.999970209316682
252	1.00029584155592	0.999743150158623	1.03596626487418	0.999970912238813
253	1.00029157082422	0.999749876898552	1.01801890409748	0.999971605820891
254	1.00028691694860	0.999756697601778	1.00053744537069	0.999972241456658
255	1.00002587972667	0.999978288670545	1.00002339132901	0.999992867605767
CPU time in seconds	1.249835			

Table 4: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [10, 10, 10, 10]^T$

Iter (*n*)	u_1	u_2	u_3	u_4
1	9.37670552615082	9.10443831204158	9.36964477485132	7.69617180087272
2	8.52638444520784	8.23128879399038	8.59942980024287	5.49342285882117
3	7.58413493042808	7.48686319918374	7.80372663027284	3.69657294857490
4	6.57691856280682	6.81862432209299	6.98748408521933	2.20234830742012
5	5.57239744678987	6.15234000819178	6.18334105148049	1.00000000023325
⋮	⋮	⋮	⋮	⋮
61	1.00001337267023	0.999994469763627	1.00000204591750	0.999995725911049
62	1.00001315800345	0.999994559118552	1.00000201212000	0.999995794483464
63	1.00001295008998	0.999994645650194	1.00000197941728	0.999995860897291
64	1.00001274863514	0.999994729478648	1.00000194775792	0.999995925248847
65	1.00001255335012	0.999994810724267	1.00000191709497	0.999995987630650
CPU time in seconds	0.314926			

Table 5: Example 4.2: Numerical study of Algorithm 1 in [30] and $u_0 = u_1 = [10, 20, 30, 40]^T$

Iter (n)	u_1	u_2	u_3	u_4
1	9.78664139597796	9.98228168073093	9.87885904329583	10.0892362484029
2	9.77332934751711	9.90702272489978	9.85225062712764	9.90395699577937
3	9.75825697187861	9.83416650992724	9.82474676153713	9.72192297304630
4	9.74150081629604	9.76222498224926	9.79640746406505	9.54191824999558
5	9.72310030216692	9.69112689206027	9.76721402065162	9.36390890485064
⋮	⋮	⋮	⋮	⋮
197	1.00048025798169	0.999578895451456	1.05467043404065	0.999951889957855
198	1.00047191338477	0.999592479755323	1.03180677371631	0.999953305260853
199	1.00046340300415	0.999605728706010	1.00964711683154	0.999954696790168
200	1.00020595114443	0.999806645776126	1.00033555706235	0.999949570767266
201	1.00002447915305	0.999980397285792	1.00002042478806	0.999993060869452
CPU time in seconds	0.881941			

Table 6: Example 4.2: Numerical study of Algorithm 2 in [30] and $u_0 = u_1 = [10, 20, 30, 40]^T$

Iter (n)	u_1	u_2	u_3	u_4
1	9.83025060597279	10.0358052147562	10.0034760237284	10.2207136435240
2	9.81992710006697	9.97710497209799	9.98219090107187	10.0007357121574
3	9.80848143600201	9.91890607188431	9.96030964532451	9.85485114455071
4	9.78233470337380	9.80406450245295	9.91483504597404	9.56675568183706
5	9.76768277826138	9.74742537160893	9.89127177156160	9.42455480338020
⋮	⋮	⋮	⋮	⋮
242	1.00029928444065	0.999737529401567	1.05100492547889	0.999970337183295
243	1.00029507461118	0.999744379644890	1.03268544459699	0.999971038422410
244	1.00029079252463	0.999751081711158	1.01481953661443	0.999971730317870
245	1.00023582350263	0.999792488973834	1.00039177564360	0.999966689624691
246	1.00002121894907	0.999982547402636	1.00001822632328	0.999994139100956
CPU time in seconds	1.189327			

Table 7: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [10, 20, 30, 40]^T$

Iter (n)	u_1	u_2	u_3	u_4
1	6.67509751978401	9.87241104722208	8.35640921356371	11.8247271582227
2	6.43630940705222	8.66032301948886	7.77319817387846	8.88400040091734
3	5.94458706274812	7.60686837201340	7.17867219777068	6.42404535927948
4	5.24960461466410	6.68896087204211	6.48897590605557	4.37736060600273
5	4.41658766198459	5.87872668632781	5.73342763899655	2.70108842826606
⋮	⋮	⋮	⋮	⋮
55	1.00001098233484	0.999995701995743	1.00000127186919	0.999996467970852
56	1.00001080643637	0.999995771227658	1.00000125097444	0.999996524518845
57	1.00001063608454	0.999995838264122	1.00000123076045	0.999996579284439
58	1.00001047102102	0.999995903207959	1.00000121119448	0.999996632350608
59	1.00001031100328	0.999995966155667	1.00000119224589	0.999996683795258
CPU time in seconds	0.263761			

Table 8: Example 4.2: Numerical study of Algorithm 1 in [30] and $u_0 = u_1 = [20, -20, 20, -20]^T$

Iter (n)	u_1	u_2	u_3	u_4
1	9.43637888611869	0.618641745108960	9.69030238747761	0.941343723156194
2	9.33347176555781	1.00048608755193	9.60042497443019	0.998236042635461
3	9.23071813735838	1.08107393627969	9.51088663606224	1.00000006665870
4	9.12812589346156	1.15965756600495	9.42168479550455	1.00000007856569
5	9.02570718544946	1.23625759975003	9.33282259887709	1.00000006494615
⋮	⋮	⋮	⋮	⋮
142	1.00048363745252	0.999573210882307	1.06428375964232	0.999951300959709
143	1.00047536747374	0.999586931862881	1.04112574964123	0.999952725950025
144	1.00046692302671	0.999600317222453	1.01867841478562	0.999954127429914
145	1.00037962363168	0.999667326588165	1.00061920024033	0.999946401991037
146	1.00004244986050	0.999965200722495	1.00003621655791	0.999988355488326
CPU time in seconds	0.539990			

Table 9: Example 4.2: Numerical study of Algorithm 2 in [30] and $u_0 = u_1 = [20, -20, 20, -20]^T$

Iter (n)	u_1	u_2	u_3	u_4
1	9.60181164686455	0.594020576572148	9.80334231814436	0.849825795779310
2	9.51962929953982	0.998933043283026	9.73167079074873	0.998926526320852
3	9.43746967852944	1.06410902526139	9.66014764334785	0.999997316486094
4	9.35536521846522	1.12804558949596	9.58879942367075	0.999999994833832
5	9.27333589349051	1.19074084404862	9.51763991225471	1.00000000015374
⋮	⋮	⋮	⋮	⋮
181	1.00029611652815	0.999742707382947	1.03714858016492	0.999970866828849
182	1.00029185000921	0.999749442874438	1.01917216369700	0.999971561005902
183	1.00028752889062	0.999756054384634	1.00164302685500	0.999972245807354
184	1.00004073447076	0.999962515806045	1.00005288902746	0.999987696986777
185	1.00000375548836	0.999997097825945	1.00000290174109	0.999998920572749
CPU time in seconds	0.846270			

Table 10: Example 4.2: Numerical study of Algorithm 1 and $u_0 = u_1 = [20, -20, 20, -20]^T$

Iter (n)	u_1	u_2	u_3	u_4
1	5.91302584242504	− 2.53013202319150	7.88340509066295	0.0494017361391670
2	5.11515304855251	1.05167987250724	7.12678420126625	0.828266207803134
3	4.44175286252904	1.44560459254135	6.47786278982089	1.00000000063044
4	3.78267512857499	1.65777433889948	5.83990979629541	1.00000000152068
5	3.17322385256513	1.76144714332220	5.24349607931747	1.00000000031145
⋮	⋮	⋮	⋮	⋮
49	1.00001367166987	0.999994310572133	1.00000215132574	0.999995633661616
50	1.00001345209473	0.999994402418620	1.00000211630632	0.999995703696399
51	1.00001323958262	0.999994491393242	1.00000208195699	0.999995771535743
52	1.00001303361761	0.999994577637339	1.00000204866350	0.999995837280664
53	1.00001283392546	0.999994661245350	1.00000201641107	0.999995901021640
CPU time in seconds	0.315846			

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