Research Article

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Graded I-second submodules

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Abstract: Let G be a group with identity e, R be a G-graded commutative ring with a nonzero unity 1, I be a graded ideal of R, and M be a G-graded R-module. In this article, we introduce the concept of graded I-second submodules of M as a generalization of graded second submodules of M and achieve some relevant outcomes.

Keywords: graded second submodules, graded prime submodules, graded weakly prime submodules

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1 Introduction

A proper graded ideal P of R is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. Graded prime ideals have been admirably introduced and studied in [1]. Graded prime submodules have been introduced by Atani in [2]. A proper graded R-submodule N of M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$. Graded prime submodules have been widely studied by several authors, for more details one can look in [3–6]. Atani introduced in [7] the concept of graded weakly prime ideals. A proper graded ideal P of R is said to be a graded weakly prime ideal of R if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$, then $x \in P$ or $y \in P$. Also, Atani extended the concept of graded weakly prime ideals into graded weakly prime submodules in [8]. A proper graded submodule N of M is called graded weakly prime if for $r \in h(R)$ and $m \in h(M)$ with $0 \neq rm \in N$, either $m \in N$ or $r \in (N :_R M)$.

Let M and S be two G-graded R-modules. An R-homomorphism $f : M \to S$ is said to be graded R-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ (see [9]). Graded second submodules have been introduced by Ansari-Toroghy and Farshadifar in [10]. A nonzero graded R-submodule N of M is said to be graded second if for each $a \in h(R)$, the graded R-homomorphism $f : N \to N$ defined by $f(x) = ax$ is either surjective or zero. In this case, $\mathrm{Ann}_R(N)$ is a graded prime ideal of R. Graded second submodules have been wonderfully studied by Çeken and Alkan in [11]. On the other hand, graded secondary modules have been introduced by Atani and Farzalipour in [12]. A nonzero graded R-module M is said to be graded secondary if for each $a \in h(R)$, the graded R-homomorphism $f : M \to M$ defined by $f(x) = ax$ is either surjective or nilpotent.

The main purpose of this article is to follow [13] in order to introduce and study the concept of graded I-second submodules of a graded R-module M as a generalization of graded second submodules of M and achieve some relevant outcomes. Among several results, we show that a graded second submodule is a graded I-second submodule for every graded ideal I of R, but we prove that the converse is not true in general (Examples 2.5, 2.6, and 2.7). We follow [14] to introduce the concept of graded I-prime ideals of a graded ring R, we show that a graded prime ideal is a graded I-prime ideal for every graded ideal I of R, but

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we prove that the converse is not true in general (Example 2.16). We prove that if N is a graded I-second R-submodule of M such that $\mathrm{Ann}_R((N :_M I)) \subseteq I\,\mathrm{Ann}_R(N)$, then $\mathrm{Ann}_R(N)$ is a graded I-prime ideal of R (Proposition 2.21). We show that if M is a graded comultiplication R-module and N is a graded R-submodule of M such that $\mathrm{Ann}_R(N)$ is an I-prime ideal of R, then N is a graded I-second R-submodule of M (Proposition 2.23). We prove that if M is primary, then every proper graded $\{0\}$-second R-submodule of M is a graded primary R-submodule of M (Proposition 2.27). In Proposition 2.28, we study graded I-second submodules under graded homomorphism. Finally, in Proposition 2.29, we study the relation between graded I-second submodules of M and I_e-second submodules of M_e when $|G| = 2$.

1.1 Preliminaries

Throughout this article, G will be a group with identity e and R will be a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. Consider $\mathrm{supp}(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, $h(R) = \bigcup_{g\in G} R_g$. Moreover, it has been proved in [9] that R_e is a subring of R and $1 \in R_e$.

Let I be an ideal of a graded ring R. Then I is said to be a graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. Let R be a G-graded ring and I be a graded ideal of R. Then R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$.

Assume that M is a left R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $\mathrm{supp}(M, G) = \{g \in G : M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$, where $x_g \in N$ for all $g \in G$. Let M be a G-graded R-module and N be a graded R-submodule of M. Then M/N is a graded R-module by $(M/N)_g = (M_g + N)/N$ for all $g \in G$.

Lemma 1.1. [15] *Let R be a G-graded ring and M be a G-graded R-module.*
1. *If I and J are graded ideals of R, then $I + J$ and $I \cap J$ are graded ideals of R.*
2. *If N and K are graded R-submodules of M, then $N + K$ and $N \cap K$ are graded R-submodules of M.*
3. *If N is a graded R-submodule of M, $r \in h(R)$, $x \in h(M)$, and I is a graded ideal of R, then Rx, IN, and rN are graded R-submodules of M. Moreover, $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*

Similarly, if M is a graded R-module, N a graded R-submodule of M, and $m \in h(M)$, then $(N :_R m)$ is a graded ideal of R. Also, it has been proved in [16] that if N is a graded R-submodule of M, then $\mathrm{Ann}_R(N) = \{r \in R : rN = \{0\}\}$ is a graded ideal of R.

In [17], a proper $\mathbb{Z}$-graded R-submodule N of M is said to be graded completely irreducible if whenever $N = \bigcap_{k\in\Delta} N_k$, where $\{N_k\}_{k\in\Delta}$ is a family of $\mathbb{Z}$-graded R-submodules of M, then $N = N_k$ for some $k \in \Delta$. In [16], the concept of graded completely irreducible submodules has been extended into G-graded case, for any group G. It has been proved that every graded R-submodule of M is an intersection of graded completely irreducible R-submodules of M. In many instances, we use the following basic fact without further discussion.

Remark 1.2. Let N and L be two graded R-submodules of M. To prove that $N \subseteq L$, it is enough to prove that if K is a graded completely irreducible R-submodule of M such that $L \subseteq K$, then $N \subseteq K$.

2 Graded I-second submodules

In this section, we introduce and study the concept of graded I-second submodules.

Let $\Omega(M)$ be the set of all graded completely irreducible R-submodules of M. Assume that P is a graded prime ideal of R and N is a graded R-submodule of M. Then we define $I_P^M(N) = \bigcap_{K\in\Omega(M)}\{K : rN \subseteq K$ for some $r \in h(R) - P\}$. The following lemma gives some characterizations for graded second R-submodules.

Lemma 2.1. *Let N be a graded R-submodule of a graded R-module M. Then the following are equivalent.*
1. *If $N \neq \{0\}$, K is a graded completely irreducible R-submodule of M and $r \in h(R)$ such that $rN \subseteq K$, then either $rN = \{0\}$ or $N \subseteq K$.*
2. *N is a graded second R-submodule of M.*
3. *$P = \mathrm{Ann}_R(N)$ is a graded prime ideal of R and $I_P^M(N) = N$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $r \in h(R)$ and $rN \neq \{0\}$. If $rN \subseteq K$ for some graded completely irreducible R-submodule K of M, then by assumption, $N \subseteq K$. Hence, $N \subseteq rN$.

(2) $\Rightarrow$ (3): By [10], $P = \mathrm{Ann}_R(N)$ is a graded prime ideal of R. Now, let K be a graded completely irreducible R-submodule of M and $r \in h(R) - P$ such that $rN \subseteq K$. Then $N \subseteq K$ by assumption. Therefore, $N \subseteq I_P^M(N)$. The reverse inclusion is clear.

(3) $\Rightarrow$ (1): Since $\mathrm{Ann}_R(N)$ is a graded prime ideal of R, $N \neq \{0\}$. Let K be a graded completely irreducible R-submodule of M and $r \in h(R)$ such that $rN \subseteq K$. Suppose that $rN \neq \{0\}$. Then $r \in h(R) - P$. Therefore, $I_P^M(N) \subseteq K$. But $I_P^M(N) = N$ by assumption. Hence, $N \subseteq K$, as desired. □

Lemma 2.2. *Let M be a G-graded R-module and N a graded R-submodule of M. If $r \in h(R)$, then $(N :_M r) = \{m \in M : rm \in N\}$ is a graded R-submodule of M.*

Proof. Clearly, $(N :_M r)$ is an R-submodule of M. Let $m \in (N :_M r)$. Then $rm \in N$. Now, $m = \sum_{g\in G} m_g$, where $m_g \in M_g$ for all $g \in G$. Since $r \in h(R)$, $r \in R_h$ for some $h \in G$ and then $rm_g \in M_{hg} \subseteq h(M)$ for all $g \in G$ such that $\sum_{g\in G} m_g = r\left(\sum_{g\in G} m_g\right) = rm \in N$. Since N is graded, $rm_g \in N$ for all $g \in G$, which implies that $m_g \in (N :_M r)$ for all $g \in G$. Hence, $(N :_M r)$ is a graded R-submodule of M. □

Similarly, if N is a graded R-submodule of M and I is a graded ideal of R, then $(N :_M I)$ is a graded R-submodule of M.

Proposition 2.3. *Let M be a graded R-module, I be a graded ideal of R, and N be a nonzero graded R-submodule of M. Then the following statements are equivalent:*
1. *For each $r \in h(R)$, a graded R-submodule K of M, $r \in (K :_R N) - (K :_R (N :_M I))$ implies that $N \subseteq K$ or $r \in \mathrm{Ann}_R(N)$;*
2. *For each $r \notin (rN :_R (N :_M I)) \cap h(R)$, we have $rN = N$ or $rN = \{0\}$;*
3. *$(K :_R N) = \mathrm{Ann}_R(N) \cup (K :_R (N :_M I))$, for any graded R-submodule K of M with $N \nsubseteq K$;*
4. *$(K :_R N) = \mathrm{Ann}_R(N)$ or $(K :_R N) = (K :_R (N :_M I))$, for any graded R-submodule K of M with $N \nsubseteq K$.*

Proof. (1) $\Rightarrow$ (2): Let $r \notin (rN :_R (N :_M I)) \cap h(R)$. Then as $rN \subseteq rN$, we have $N \subseteq rN$ or $rN = \{0\}$ by part (1). Thus, $rN = N$ or $rN = \{0\}$.

(2) $\Rightarrow$ (1): Let $r \in h(R)$ and K be a graded R-submodule of M such that $r \in (K :_R N) - (K :_R (N :_M I))$. Then if $r \in (rN :_R (N :_M I))$, then $r \in (K :_R (N :_M I))$, which is a contradiction. Thus, $r \notin (rN :_R (N :_M I))$. Now, by part (2), $rN = N$ or $rN = \{0\}$. So, $N \subseteq K$ or $rN = \{0\}$, as desired.

(1) $\Rightarrow$ (3): Let $r \in (K :_R N)$ and $N \nsubseteq K$. If $r \notin (K :_R (N :_M I))$, then $r \in \mathrm{Ann}_R(N)$ by part (1). Hence, $(K :_R N) \subseteq \mathrm{Ann}_R(N)$. If $r \in (K :_R (N :_M I))$, then $(K :_R N) \subseteq (K :_R (N :_M I))$. Therefore, $(K :_R N) \subseteq \mathrm{Ann}_R(N) \cup (K :_R (N :_M I))$. The other inclusion is clear.

(3) $\Rightarrow$ (4): If a graded ideal is a union of two graded ideals, then it is equal to one of them.

(4) $\Rightarrow$ (1): Obvious. □

Definition 2.4. Let M be a graded R-module, I be a graded ideal of R, and N be a nonzero graded R-submodule of M. Then N is said to be a graded I-second R-submodule of M if N satisfies the equivalent conditions of Proposition 2.3.

Clearly, every graded second submodule is a graded I-second submodule for every graded ideal I of R. However, the following examples prove that the converse is not true in general.

Example 2.5. Every graded R-module M is a graded $I = \{0\}$-second R-submodule of itself, but not every graded R-module is a graded second R-submodule of itself.

Example 2.6. Consider $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$, and $G = \mathbb{Z}_2$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Now, $N = \mathbb{Z}$ is a graded R-submodule of M. If $I = R$, then N is a graded I-second R-submodule of M that is not a graded second R-submodule of M.

Example 2.7. Consider $R = \mathbb{Z}$, $M = \mathbb{Z}_{12}[i]$, and $G = \mathbb{Z}_4$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = R_2 = R_3 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}_{12}$, $M_2 = i\mathbb{Z}_{12}$, and $M_1 = M_3 = \{0\}$. Now, $N = \bar{3}\mathbb{Z}_{12}$ is a graded R-submodule of M. If $I = 4\mathbb{Z}$, then N is a graded I-second R-submodule of M that is not a graded second R-submodule of M.

Remark 2.8.
1. If $I = R$, then every graded R-submodule of M is a graded I-second R-submodule of M. So in the rest of our article, we can assume that $I \neq R$.
2. If Condition (1) in Proposition 2.3 holds for graded completely irreducible submodules, that is, if for each $r \in h(R)$, and a graded completely irreducible R-submodule L of M, $r \in (L :_R N) - (L :_R (N :_M I))$ implies that $N \subseteq L$ or $r \in \mathrm{Ann}_R(N)$, we cannot achieve that N is a graded I-second R-submodule of M (as in Lemma 2.1 for graded second submodules), see the following example:

Example 2.9. Consider $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$, and $G = \mathbb{Z}_2$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Now, $N = 2\mathbb{Z}$ is a graded R-submodule of M. If $I = 4\mathbb{Z}$, then N is not a graded I-second R-submodule of M, but Condition (1) in Proposition 2.3 holds for graded completely irreducible R-submodules of M.

Proposition 2.10. *Let M be a graded R-module and I_1, I_2 be graded ideals of R such that $I_1 \subseteq I_2$. If N is a graded I_1-second R-submodule of M, then N is a graded I_2-second R-submodule of M.*

Proof. Since $I_1 \subseteq I_2$, we conclude that $(rN :_R N) - (rN :_R (N :_M I_2)) \subseteq (rN :_R N) - (rN :_R (N :_M I_1))$ for each $r \in h(R)$. So, the result holds. □

Corollary 2.11. *Let M be a graded R-module. Then every graded $\{0\}$-second R-submodule of M is a graded I-second R-submodule of M for each graded ideal I of R.*

Definition 2.12. Let M be a G-graded R-module, I be a graded ideal of R, N be a nonzero graded R-submodule of M, and $g \in G$. Then N is said to be a g-I-second R-submodule of M if for each $r \in R_g$, and a graded R-submodule K of M, $r \in (K :_{R_g} N) - (K :_{R_g} (N :_{M_g} I))$ implies that $N \subseteq K$ or $r \in \mathrm{Ann}_{R_g}(N)$.

Definition 2.13. Let M be a G-graded R-module and $g \in G$. A nonzero graded R-submodule N of M is said to be a g-second R-submodule of M if K is a graded R-submodule of M and $r \in R_g$ such that $rN \subseteq K$, then either $rN = \{0\}$ or $N \subseteq K$.

Proposition 2.14. *Let M be a G-graded R-module and $g \in G$. If N is a g-I-second R-submodule of M which is not graded g-second, then $\mathrm{Ann}_{R_g}(N)(N :_{M_g} I) \subseteq N$.*

Proof. Suppose that $\mathrm{Ann}_{R_g}(N)(N :_{M_g} I) \nsubseteq N$. We show that N is a g-second R-submodule of M. Let $rN \subseteq K$ for some $r \in R_g$ and a graded R-submodule K of M. If $r \notin (K :_{R_g} (N :_{M_g} I))$, then N is a graded g-I-second R-submodule implies that $N \subseteq K$ or $r \in Ann_{R_g}(N)$ as required. Assume that $r \in (K :_{R_g} (N :_{M_g} I))$. Suppose that $r(N :_{M_g} I) \nsubseteq N$. Then there exists a graded R-submodule L of M such that $N \subseteq L$ with $r(N :_{M_g} I) \nsubseteq L$, and then $r \in (K \cap L :_{R_g} N) - (K \cap L :_{R_g} (N :_{M_g} I))$. So, $N \subseteq K \cap L$ or $r \in \mathrm{Ann}_{R_g}(N)$ and hence $N \subseteq K$ or $r \in \mathrm{Ann}_{R_g}(N)$. Assume that $r(N :_{M_g} I) \subseteq N$. If $\mathrm{Ann}_{R_g}(N)(N :_{M_g} I) \nsubseteq K$, then there exists $t \in \mathrm{Ann}_{R_g}(N)$ such that $t \notin (K :_{R_g} (N :_{M_g} I))$. Then $t + r \in (K :_{R_g} N) - (K :_{R_g} (N :_{M_g} I))$. Thus, $N \subseteq K$ or $t + r \in \mathrm{Ann}_{R_g}(N)$ and hence, $N \subseteq K$ or $r \in \mathrm{Ann}_{R_g}(N)$. Assume that $\mathrm{Ann}_{R_g}(N)(N :_{M_g} I) \subseteq K$. Since $\mathrm{Ann}_{R_g}(N)(N :_{M_g} I) \nsubseteq N$, there exist $t \in \mathrm{Ann}_{R_g}(N)$, and a graded R-submodule T of M such that $N \subseteq T$ and $t(N :_{M_g} I) \nsubseteq T$. Now we have $r + t \in (K \cap T :_{R_g} N) - (K \cap T :_{R_g} (N :_{M_g} I))$. So, N is a g-I-second R-submodule of M gives $N \subseteq K \cap T$ or $r + t \in \mathrm{Ann}_{R_g}(N)$. Hence, $N \subseteq K$ or $r \in \mathrm{Ann}_{R_g}(N)$, as needed. □

In the following definition, we follow [14] to introduce the concept of graded I-prime ideals of a graded ring R.

Definition 2.15. Let R be a graded ring and I be a graded ideal of R. Then a proper graded ideal P of R is said to be graded I-prime if for $x, y \in h(R)$ such that $xy \in P - IP$, then either $x \in P$ or $y \in P$.

Clearly, every graded prime ideal is a graded I-prime ideal for every graded ideal I of R. However, the following example shows that the converse is not true in general.

Example 2.16. Consider $R = \mathbb{Z}_{12}[i]$ and $G = \mathbb{Z}_4$. Then R is G-graded by $R_0 = \mathbb{Z}_{12}$, $R_2 = i\mathbb{Z}_{12}$, and $R_1 = R_3 = \{0\}$. If we take $P = I = \langle \bar{4} \rangle$, then P is a graded I-prime ideal of R which is neither graded prime nor graded weakly prime.

Lemma 2.17. *Let R be a G-graded ring, I be an ideal of R, and J be a graded ideal of R such that $J \subseteq I$. Then I is a graded ideal of R if and only if I/J is a graded ideal of R/J.*

Proof. Suppose that I is a graded ideal of R. Clearly, I/J is an ideal of R/J. Let $x + J \in I/J$. Then $x \in I$ and since I is graded, $x = \sum_{g \in G} x_g$, where $x_g \in I$ for all $g \in G$ and then $(x + J)_g = x_g + J \in I/J$ for all $g \in G$. Hence, I/J is a graded ideal of R/J. Conversely, let $x \in I$. Then $x = \sum_{g \in G} x_g$, where $x_g \in R_g$ for all $g \in G$ and then $(x_g + J) \in (R_g + J)/J = (R/J)_g$ for all $g \in G$ such that

$$\sum_{g \in G} (x + J)_g = \sum_{g \in G} (x_g + J) = \left(\sum_{g \in G} x_g \right) + J = x + J \in I/J.$$

Since I/J is graded, $x_g + J \in I/J$ for all $g \in G$, which implies that $x_g \in I$ for all $g \in G$. Hence, I is a graded ideal of R. □

Proposition 2.18. *Let P be a proper graded ideal of R. Then P is a graded I-prime ideal of R if and only if P/IP is a graded weakly prime ideal of R/IP.*

Proof. Suppose that P is a graded I-prime ideal of R. By Lemma 2.17, P/IP is a graded ideal of R/IP. Let $x + IP, y + IP \in h(R/IP)$ such that $0 + IP \neq (x + IP)(y + IP) \in P/IP$. Then $x, y \in h(R)$ such that $xy \in P - IP$, and then $x \in P$ or $y \in P$. So, $x + IP \in P/IP$ or $y + IP \in P/IP$. Hence, P/IP is a graded weakly prime ideal of R/IP. Conversely, let $x, y \in h(R)$ such that $xy \in P - IP$. Then $x + IP, y + IP \in h(R/IP)$ such that $0 + IP \neq (x + IP)(y + IP) \in P/IP$, and then $x + IP \in P/IP$ or $y + IP \in P/IP$. So, $x \in P$ or $y \in P$. Hence, P is a graded I-prime ideal of R. □

Proposition 2.19. *Let I and J be two graded ideals of R such that $I \subseteq J$. Then every graded I-prime ideal of R is graded J-prime.*

Proof. Let P be a graded I-prime ideal of R. Then the result follows from the fact that $P - JP \subseteq P - IP$. □

The following example shows that if I and J are two graded ideals of R such that $I \subseteq J$ and P is a graded J-prime ideal of R, then P does not need to be graded I-prime.

Example 2.20. Consider $R = \mathbb{Z}_{12}[x]$ and $G = \mathbb{Z}$. Then R is G-graded by $R_j = \mathbb{Z}_{12}x^j$ for $j \geq 0$ and $R_j = \{0\}$ otherwise. Choose $I = \{\bar{0}\}$, $J = \langle \bar{4} \rangle$, and $P = \langle \bar{4}x \rangle$, then I, J, and P are graded ideals of R such that $I \subseteq J$, $P - IP = \langle \bar{4}x \rangle - \{\bar{0}\}$, and $P - JP = \emptyset$. Clearly, P is a graded J-prime ideal of R but not graded I-prime.

Proposition 2.21. *Let M be a graded R-module and N be a graded R-submodule of M. If N is a graded I-second R-submodule of M such that $\mathrm{Ann}_R((N :_M I)) \subseteq I\mathrm{Ann}_R(N)$, then $\mathrm{Ann}_R(N)$ is a graded I-prime ideal of R.*

Proof. By [16], $\mathrm{Ann}_R(N)$ is a graded ideal of R. Let $xy \in \mathrm{Ann}_R(N) - I\mathrm{Ann}_R(N)$ for some $x, y \in h(R)$. Then $xN \subseteq (0 :_M y)$. As $xy \notin I\mathrm{Ann}_R(N)$ and $\mathrm{Ann}_R((N :_M I)) \subseteq I\mathrm{Ann}_R(N)$, we have $xy \notin \mathrm{Ann}_R((N :_M I))$. This implies that $x \notin ((0 :_M y) :_R (N :_M I))$. Thus, $x \in \mathrm{Ann}_R(N)$ or $N \subseteq (0 :_M y)$. Hence, $x \in \mathrm{Ann}_R(N)$ or $y \in \mathrm{Ann}_R(N)$, as required. □

Corollary 2.22. *If M is a graded faithful R-module and N is a graded $\{0\}$-second R-submodule of M, then $\mathrm{Ann}_R(N)$ is a graded weakly prime ideal of R.*

Proof. Apply Proposition 2.21 with $I = \{0\}$. □

Graded comultiplication modules have been introduced by H. A. Toroghy and F. Farshadifar in [18]; a graded R-module M is said to be graded comultiplication if for every graded R-submodule N of M, $N = (0 :_M I)$ for some graded ideal I of R, or equivalently, $N = (0 :_M \mathrm{Ann}_R(N))$. The concept of graded comultiplication modules has been studied by several authors, for example, see [19,20].

Proposition 2.23. *Let M be a graded comultiplication R-module and N be a graded R-submodule of M. If $\mathrm{Ann}_R(N)$ is an I-prime ideal of R, then N is a graded I-second R-submodule of M.*

Proof. Let $r \in (K :_R N) - (K :_R (N :_M I))$ for some $r \in h(R)$ and a graded R-submodule K of M. As M is a graded comultiplication R-module, there exists a graded ideal J of R such that $K = (0 :_M J)$. Thus, $rJ \subseteq \mathrm{Ann}_R(N)$. Since $r \notin (K :_R (N :_M I))$, we have $Jr(N :_M I) \neq \{0\}$. This implies that $Jr \nsubseteq \mathrm{Ann}_R((N :_M I))$. Since clearly, $I\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R((N :_M I))$, we have $rJ \nsubseteq I\mathrm{Ann}_R(N)$. Thus, $r \in \mathrm{Ann}_R(N)$ or $J \subseteq \mathrm{Ann}_R(N)$ by ([14], Theorem 2.12), and so $N \subseteq (0 :_M J) = K$. □

Corollary 2.24. *Let M be a graded comultiplication R-module and N be a graded R-submodule of M. If $\mathrm{Ann}_R(N)$ is a weakly prime ideal of R, then N is a graded $\{0\}$-second R-submodule of M.*

Proof. Apply Proposition 2.23 with $I = \{0\}$. □

The next example shows that the condition "M is a graded comultiplication R-module" in Corollary 2.24 is necessary.

Example 2.25. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Consider the trivial graduation of R and M by any group G. Then M is not a graded comultiplication R-module. Now, $N = 2\mathbb{Z} \oplus \{0\}$ is a graded R-submodule of M such that $\mathrm{Ann}_R(N) = \{0\}$ is a weakly prime ideal of R, but N is not a graded $\{0\}$-second R-submodule of M.

Proposition 2.26. *Let I be a graded ideal of a graded ring R and M be a graded R-module. Let N be a graded I-second R-submodule of M. If L is a graded R-submodule of M with $L \subset N$, then N/L is a graded I-second R-submodule of M/L.*

Proof. Similar to the proof of Lemma 2.17, one can prove that N/L is a graded R-submodule of M. The result follows by $r \notin (r(N/L) :_R (N/L :_{M/L} I))$ implies that $r \notin (rN :_R (N :_M I))$. □

Graded primary ideals have been introduced and studied in [21]. A proper graded ideal P of R is said to be graded primary if for $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in \mathrm{Grad}(P)$, where $\mathrm{Grad}(P)$ is the graded radical of P, and is defined to be the set of all $r \in R$ such that for each $g \in G$, there exists a positive integer n_g that satisfies $r_g^{n_g} \in P$. One can see that if $r \in h(R)$, then $r \in \mathrm{Grad}(P)$ if and only if $r^n \in P$ for some positive integer n. In [22], a proper graded R-submodule N of M is said to be graded primary if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in \mathrm{Grad}((N :_R M))$. An R-module M is said to be a primary R-module if $\{0\}$ is a primary R-submodule of M.

Proposition 2.27. *Let M be a graded R-module. If M is primary, then every proper graded $\{0\}$-second R-submodule of M is a graded primary R-submodule of M.*

Proof. Let N be a proper graded $\{0\}$-second R-submodule of M and $rm \in N$ for some $r \in h(R)$ and $m \in h(M)$. If $r \notin (rN :_R M)$, then $rN = \{0\}$ or $rN = N$ since N is a graded $\{0\}$-second R-submodule of M. If $rN = \{0\}$, then $r^2m \in rN = \{0\}$. Now as M is primary, $m = 0$ or $r \in \mathrm{Grad}((0 :_R M))$. This implies that $m \in N$ or $r \in \mathrm{Grad}((N :_R M))$, as required. If $rN = N$, then $rm = ra$ for some $a \in N$. This implies that $m = a \in N$ or $r \in \mathrm{Grad}((0 :_R M)) \subseteq \mathrm{Grad}(N :_R M))$ since M is primary. Suppose that $r \in (rN :_R M)$. Then $rm \in rM \subseteq rN$. Therefore, similar to the pervious case we are done. □

Let M and S be two G-graded R-modules. An R-homomorphism $f : M \to S$ is said to be graded R-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ (see [9]).

Proposition 2.28. *Let I be a graded ideal of a graded ring R, M and S be graded R-modules, and let $f : M \to S$ be an injective graded R-monomorphism. If K is a graded I-second R-submodule of S such that $K \subseteq Im(f)$, then $f^{-1}(K)$ is a graded I-second R-submodule of M.*

Proof. Since $K \neq \{0\}$ and $K \subseteq Im(f)$, we conclude that $f^{-1}(K) \neq \{0\}$. Let $r \notin (rf^{-1}(K) :_R (f^{-1}(K) :_M I))$ for some $r \in h(R)$. Then $r \notin (rK :_R (K :_S I))$. Thus, $rK = \{0\}$ or $rK = K$. This implies that $rf^{-1}(K) = \{0\}$ or $rf^{-1}(K) = f^{-1}(K)$, as needed. □

Proposition 2.29. *Let $G = \{e, g\}$, where $g \neq e$. Suppose that R is a nontrivially G-graded ring with $R = R_e \oplus R_g$, I is a graded ideal of R, and M is a nontrivially G-graded R-module by $M = M_e \oplus M_g$. Assume that N is an R_e-submodule of M_e. Then $N \oplus \{0\}$ is a graded I-second R-submodule of M if and only if N is an I_e-second R_e-submodule of M_e and for $r \in (rN :_{R_e} (N :_{M_e} I_e))$ with $rN \neq \{0\}$ and $rN \neq N$, we have $r \in Ann_{R_e}((0 :_{M_g} I_e))$.*

Proof. Suppose that $N \oplus \{0\}$ is a graded I-second R-submodule of M. Let $r \notin (rN :_{R_e} (N :_{M_e} I_e))$. Then $r \notin (r(N \oplus \{0\}) :_R (N \oplus \{0\} :_M I))$. Since $N \oplus \{0\}$ is graded I-second, either $r(N \oplus \{0\}) = N \oplus \{0\}$ or $r(N \oplus \{0\}) = \{0\} \oplus \{0\}$. Thus, either $rN = N$ or $rN = \{0\}$, so N is I_e-second. Assume that $r \in (rN :_{R_e} (N :_{M_e} I_e))$ with $rN \neq \{0\}$ and $rN \neq N$. Suppose that $r \notin \mathrm{Ann}_{R_e}((0 :_{M_g} I_e))$. Then there exists $x \in M_g$ such that $Ix = \{0\}$ and $rx \neq 0$. This implies that $r(0, x) \in r(N \oplus \{0\} :_M I) - r(N \oplus \{0\})$. So, since $N \oplus \{0\}$ is graded I-second, either $r(N \oplus \{0\}) = N \oplus \{0\}$ or $r(N \oplus \{0\}) = \{0\} \oplus \{0\}$. Thus, either $rN = N$ or $rN = \{0\}$, which is a contradiction. So, $r \in \mathrm{Ann}_{R_e}((0 :_{M_g} I_e))$. Conversely, let $r \notin (r(N \oplus \{0\}) :_R (N \oplus \{0\} :_M I))$. Then if $rN = N$ or $rN = \{0\}$, the result is clear. So, suppose that $rN \neq$ N and $rN \neq \{0\}$. We show that $r \notin (rN :_{R_e} (N :_{M_e} I_e))$, and this contradiction proves the result because N is an I_e-second R_e-submodule of M_e. Assume on the contrary that $r \in (rN :_{R_e} (N :_{M_e} I_e))$. Then by assumption,

$r \in \mathrm{Ann}_{R_e}((0 :_{M_g} I_e))$. This implies that if $(x, y) \in N \oplus (0 :_M I)$, then $r(x, y) \in r(N \oplus \{0\})$. Therefore, $r \in (r(N \oplus \{0\}) :_R (N \oplus \{0\} :_M I))$, which is a desired contradiction. □

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References

[1] M. Refai, M. Hailat and S. Obiedat, *Graded radicals and graded prime spectra*, Far East J. Math. Sci. **1** (2000), 59–73.
[2] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci. **33** (2006), no. 1, 3–7.
[3] R. Abu-Dawwas and K. Al-Zoubi, *On graded weakly classical prime submodules*, Iran. J. Math. Sci. Inform. **12** (2017), no. 1, 153–161.
[4] R. Abu-Dawwas, K. Al-Zoubi and M. Bataineh, *Prime submodules of graded modules*, Proyecciones **31** (2012), no. 4, 355–361.
[5] K. Al-Zoubi and R. Abu-Dawwas, *On graded quasi-prime submodules*, Kyungpook Math. J. **55** (2015), 259–266.
[6] K. Al-Zoubi, M. Jaradat and R. Abu-Dawwas, *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc. **41** (2015), no. 1, 205–2013.
[7] S. E. Atani, *On graded weakly prime ideals*, Turkish J. Math. **30** (2006), 351–358.
[8] S. E. Atani, *On graded weakly prime submodules*, Int. Math. Forum **1** (2006), no. 2, 61–66.
[9] C. Nastasescu and F. van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.
[10] H. Ansari-Toroghy and F. Farshadifar, *On graded second modules*, Tamkang J. Math. **43** (2012), no. 4, 499–505.
[11] S. Çeken and M. Alkan, *On graded second and coprimary modules and graded second representations*, Bull. Malaysian Math. Sci. Soc. Ser. 2 **38** (2015), no. 4, 1317–1330.
[12] S. E. Atani and F. Farzalipour, *On graded secondary modules*, Turkish J. Math. **31** (2007), 371–378.
[13] F. Farshadifar and H. Ansari-Toroghy, *I-second submodules of a module*, Matematicki Vesnik **72** (2020), no. 1, 58–65.
[14] I. Akray, *I-prime ideals*, J. Algebra Relat. Topics **4** (2016), no. 2, 41–47.
[15] F. Farzalipour and P. Ghiasvand, *On the union of graded prime submodules*, Thai J. Math. **9** (2011), no. 1, 49–55.
[16] D. Northcott, *Lessons on Rings, Modules, and Multiplicities*, Cambridge University Press, Cambridge, 1968.
[17] J. Chen and Y. Kim, *Graded irreducible modules are irreducible*, Comm. Algebra **45** (2017), no. 5, 1907–1913.
[18] H. Ansari-Toroghy and F. Farshadifar, *Graded comultiplication modules*, Chiang Mai J. Sci. **38** (2011), no. 3, 311–320.
[19] R. Abu-Dawwas and M. Ali, *Comultiplication modules over strongly graded rings*, Int. J. Pure Appl. Math. **81** (2012), no. 5, 693–699.
[20] R. Abu-Dawwas, M. Bataineh and A. Dakeek, *Graded weak comultiplication modules*, Hokkaido Math. J. **48** (2019), 253–261.
[21] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math. **28** (2004), no. 3, 217–229.
[22] K. H. Oral, Ü. Tekir and A. G. Agargün, *On graded prime and primary submodules*, Turkish J. Math. **35** (2011), 159–167.