

Research Article

Mohamed M. A. Metwali*

On perturbed quadratic integral equations and initial value problem with nonlocal conditions in Orlicz spaces

<https://doi.org/10.1515/dema-2020-0052>

received February 8, 2020; accepted May 3, 2020

Abstract: The existence of a.e. monotonic solutions for functional quadratic Hammerstein integral equations with the perturbation term is discussed in Orlicz spaces. We utilize the strategy of measure of noncompactness related to the Darbo fixed point principle. As an application, we discuss the presence of solution of the initial value problem with nonlocal conditions.

Keywords: quadratic integral equation, Orlicz spaces, initial value problem, nonlocal conditions, Darbo fixed point theorem

MSC 2010: 46G10, 47H30, 47N20

1 Introduction

This article is dedicated to examine the presence of solutions for the functional quadratic integral equation:

$$x(t) = h(t) + f_3(t, x(t)) + f_2 \left(t, f_1(t, x(t)) \cdot \int_a^b K(t, s)g(s, x(s))ds \right), \quad t \in [a, b]. \quad (1)$$

It is helpful to find solutions in Orlicz spaces, when we deal with some problems involving strong nonlinearities in which either the growth of the functions f_i or the kernel K is not polynomial (of exponential growth, for instance), then discontinuous solutions are expected. This is motivated by some mathematical models in physics and statistical physics [1–3]. The considered thermodynamic problem leads to the integral equation with exponential nonlinearities of the form

$$x(t) + \int_I k(t, s) \exp x(s) ds = 0.$$

Additionally, the continuous solutions for the quadratic integral equation of Chandrasekhar type [4,5] seem to be inadequate for integral problems and lead to several restrictions on the considered functions, then discontinuous solutions are imperative (see also some comments in [6]). Let us also note that the solutions in Orlicz spaces are also studied in the case of partial differential equations (see [7,8] for instance).

Recall that, we started in [6] to solve quadratic integral equations in Banach-Orlicz algebras. It means that we have some additional properties of solutions, but with conditions stronger than that in this article.

* Corresponding author: Mohamed M. A. Metwali, Department of Mathematics, Faculty of Sciences, Damanhour University, Damanhour, Egypt, e-mail: metwali@sci.dmu.edu.eg

In [9,10], the authors extend the results in [6] to arbitrary Orlicz spaces which are not necessary Banach algebras using different sets of assumptions controlling the intermediate spaces. The authors in [11] discussed the quadratic functional-integral equations in Orlicz spaces L_φ for φ satisfying Δ_2 -condition. The solutions for the Volterra integral equation in generalized Orlicz spaces (Musielak-Orlicz spaces) were studied in [12], see also [13].

The presented class of Orlicz spaces permits us to include the case of Lebesgue spaces L_p for $p \geq 1$ as a special case. The key point is to dominate optimally the acting, continuity, and monotonicity conditions for the considered operators between the target spaces, which was not sufficiently utilized in previous studies such as [14].

As an application of our results, we will discuss the solvability of the initial value problem (IVP)

$$\frac{dy(t)}{dt} = h(t) + f_3\left(t, \frac{dy(t)}{dt}\right) + f_2\left(t, f_1\left(t, \frac{dy(t)}{dt}\right) \cdot \int_a^b K(t, s)g\left(s, \frac{dy(s)}{dt}\right)ds\right) \quad (2)$$

with nonlocal condition

$$y(\tau) = \beta y(\zeta), \quad \tau \in [0, 1], \quad \zeta \in (0, 1], \quad \beta \neq 1. \quad (3)$$

The IVP for ordinary differential equations has applications in various regions, for example, in physics and different areas of applied mathematics such as theory of elasticity (see [15–17]) and has the preferable effect with nonlocal conditions than the initial or Darboux conditions.

The results displayed in this article are motivated by unifying some known results for particular cases of equation (1) in one proof and will extend them to general functional quadratic Hammerstein integral equations with linear perturbation of second kind in Orlicz spaces on the bounded interval. We use the strategy of measure of noncompactness with the Darbo fixed point principle to prove the existence of a.e. monotonic solution of the considered problems. The solution of IVP (2) with nonlocal condition (3) is also examined.

2 Preliminaries

Let \mathbb{R} be the field of real numbers and I be an interval $[a, b] \subset \mathbb{R}$. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ . The symbol B_r stands for the closed ball centered at θ and with radius r and we will recall the space by the notation $B_r(E)$. If X is a subset of E , then \bar{X} and $\text{conv } X$ denote the closure and convex closure of X , respectively.

Next, we give some lemmas and theorems in the Orlicz spaces theory (cf. also [2,18]). Let M and N be complementary N -functions, i.e., $N(x) = \sup_{y \geq 0} (x|y| - M(x))$, where $M: [0, +\infty) \rightarrow [0, +\infty)$ is continuous, convex and even with $\lim_{x \rightarrow 0} \frac{M(x)}{x} = 0$, $\lim_{x \rightarrow \infty} \frac{M(x)}{x} = \infty$ and $M(x) > 0$ if $x > 0$ ($M(u) = 0 \Leftrightarrow u = 0$) ([2, p. 9]). The Orlicz class, denoted by \mathcal{O}_M , consists of measurable functions $x: I \rightarrow \mathbb{R}$ for which $\rho(x; M) = \int_I M(x(t))dt < \infty$. We shall denote by $L_M(I)$ the Orlicz spaces of all measurable functions $x: I \rightarrow \mathbb{R}$ for which

$$\|x\|_M = \inf_{\varepsilon > 0} \left\{ \int_I M\left(\frac{x(s)}{\varepsilon}\right)ds \leq 1 \right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Note that $E_M \subseteq L_M \subseteq \mathcal{O}_M$.

If M satisfies the Δ_2 -condition, i.e., there exist $\omega, t_0 \geq 0$ such that for $t \geq t_0$, we have $M(2t) \leq \omega M(t)$, then we have $E_M = L_M = \mathcal{O}_M$.

Now, we present the definition of a regular measure of noncompactness: we indicate by \mathcal{M}_E the family of all nonempty and bounded subsets of E and \mathcal{N}_E its subfamily consisting of all relatively compact subsets.

Definition 2.1. [19] A mapping $\mu: \mathcal{M}_E \rightarrow [0, \infty)$ is called a measure of noncompactness in E if it satisfies the following conditions:

- (i) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E$.
- (ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (iii) $\mu(\bar{X}) = \mu(\text{conv } X) = \mu(X)$.
- (iv) $\mu(\lambda X) = |\lambda| \mu(X)$, for $\lambda \in \mathbb{R}$.
- (v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
- (vi) $\mu(X \cup Y) = \max \{\mu(X), \mu(Y)\}$.
- (vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \dots$, and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

It is well-known that the Hausdorff measure of noncompactness [19] is defined by:

$$\beta_H(X) = \inf\{r > 0: \text{there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_r\}$$

for nonempty and bounded subsets of $X \subset E$.

For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X in $L_M(I)$ (cf. Definition 3.9 in [20] or [21]):

$$c(X) = \lim_{\varepsilon \rightarrow 0} \left\{ \sup_{\text{mes } D \leq \varepsilon} \left[\sup_{x \in X} \{ \|x \cdot \chi_D\|_{L_M(I)} \} \right] \right\},$$

where χ_D denotes the characteristic function of a measurable subset $D \subset I$.

It forms a regular measure of noncompactness if restricted to the family of subsets being compact in measure in a class of regular ideal (or Orlicz) spaces (cf. [21]).

Lemma 2.1. [21, Theorem 1] *Let X be a nonempty, bounded and compact in measure subset of an ideal regular space Y . Then,*

$$\beta_H(X) = c(X).$$

Theorem 2.1. [19] *Let Q be a nonempty, bounded, closed and convex subset of E and let $V: Q \rightarrow Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ , i.e., there exists $k \in [0, 1)$ such that*

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty subset X of E . Then, V has at least one fixed point in the set Q and the set of all fixed points for V is compact in E .

Definition 2.2. [22] Assume that a function $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions, i.e., it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every function $x(t)$ being measurable on I we may assign the function

$$F_f(x)(t) = f(t, x(t)), \quad t \in I.$$

The operator F_f is said to be the superposition (Nemytskii) operator generated by the function f .

Lemma 2.2. [2, Theorem 17.5] *Assume that a function $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. Then,*

$$M_2(f(s, x)) \leq a(s) + bM_1(x),$$

where $b \geq 0$ and $a \in L_1(I)$, if and only if the superposition operator F_f acts from $L_{M_1}(I) \rightarrow L_{M_2}(I)$.

Lemma 2.3. Assume that a function $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F_f maps $E_\phi(I) \rightarrow E_\phi(I)$ is continuous and bounded if and only if

$$|f(s, x)| \leq m(s) + b|x|,$$

where $b \geq 0$ and $m \in L_\phi(I)$ in which the N -function $\phi(x)$ satisfies the Δ_2 -condition.

Proof. Putting the N -functions $M_1 = M_2 = \phi$ and $m(s) = \phi^{-1}(a(s))$, where $a \in L_1(I)$ in [2, Theorem 17.6]. \square

Let $S = S(I)$ stand for the set of measurable (in Lebesgue sense) functions on I and let meas denote the Lebesgue measure in \mathbb{R} . Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x, y) = \inf_{a>0} [a + \text{meas}\{s: |x(s) - y(s)| \geq a\}]$$

be a complete metric space. Moreover, the convergence in measure on I is equivalent to the convergence with respect to the metric d (cf. Proposition 2.14 in [20]). The compactness in such a space is called a “compactness in measure”.

Lemma 2.4. [6] Let X be a bounded subset of $L_M(I)$. Assume that there is a family of measurable subsets $(\Omega_l)_{0 \leq l \leq b-a}$ of the interval I such that $\text{meas } \Omega_l = l$ for every $l \in [0, b-a]$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_l, t_2 \notin \Omega_l)$. Then, X is compact in measure in $L_M(I)$.

3 Main results

Rewrite equation (1) as

$$x(t) = B(x)(t),$$

where

$$B(x) = h + F_{f_3}(x) + A(x), \quad A(x)(t) = F_{f_1}F_{f_2}KF_g(x)(t)$$

such that

$$F_g(x)(t) = g(t, x(t)), \quad F_{f_i} = f_i(s, x(s)), \quad i = 1, 2, 3 \quad \text{and} \quad K_0x(t) = \int_a^b K(t, s)x(s)ds.$$

Note that L_p -spaces can be treated as Orlicz spaces $L_p = E_{M_p}$, where the N -function $M_p = \frac{|x|^p}{p}$ satisfies the Δ_2 -condition, which will be useful in the next theorem.

First, equation (1) shall be examined under the following assumptions.

Let $\frac{1}{p} + \frac{1}{q} = 1$ and assume that φ is an N -function which satisfies Δ_2 -condition and that:

- (i) $h \in E_\phi(I)$ is nondecreasing a.e. on I ;
- (ii) $g, f_i: I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Carathéodory conditions and $g(t, x), f_i(t, x)$ are assumed to be nondecreasing with respect to both variables t and x separately, for $i = 1, 2, 3$;
- (iii) There exist constants $b_j \geq 0$, $j = 1, \dots, 4$ and positive functions $a_4 \in L_1(I)$, $a_i \in L_\varphi(I)$ such that

$$|f_i(t, x)| \leq a_i(t) + b_i|x|, \quad i = 1, 2, 3$$

and

$$|g(t, x)|^p \leq a_4(t) + b_4\varphi(|x|);$$

(iv) Assume that function K is measurable in (t, s) and assume that the linear integral operator K_0 with kernel $K(t, s)$ maps $L_{M_p}(I)$ into $L_\infty(I)$, $s \mapsto |K(\cdot, s)| \in L_{M_q}(I)$, $k(t) = |K(t, \cdot)| \in L_\infty(I)$ and K_0 is continuous with a norm

$$\|K_0\|_{L_\infty} := \text{ess sup}_{t \in I} \left(\int_I |K(t, s)|^q ds \right)^{1/q};$$

(v) $\int_I K(t_1, s) ds \geq \int_I K(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$;
 (vi) Let $r \leq 1$ be a positive solution of the equation

$$\|h\|_\varphi + \|a_2\|_\varphi + \|a_3\|_\varphi + b_3 \cdot r + b_2(\|a_1\|_\varphi + b_1 \cdot r) \cdot \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4 \cdot r)^{1/p} = r.$$

Proposition 3.1.

(a) Suppose the N -function φ satisfies the Δ_2 -condition, then F_g , F_{f_i} , $i = 1, 2, 3$ are bounded in $E_\varphi(I)$, and for $x \in E_\varphi$ we have

$$\|(F_{f_i})(x)\|_\varphi \leq \|a_i + b_i|x|\|_\varphi \leq \|a_i\|_\varphi + b_i\|x\|_\varphi, \quad i = 1, 2, 3$$

and

$$\int_a^b |g(t, x(t))|^p dt \leq \int_a^b a_4(t) dt + b_4 \int_a^b \varphi(|x(t)|) dt \Rightarrow \|(F_g)(x)\|_{M_p} \leq (\|a_4\|_{L_1} + b_4\|x\|_\varphi)^{1/p}.$$

(b) By assumption (iv), the operator K_0 is continuous and the norm of $K_0(x)$ is estimated by (cf. [10])

$$\|K(x)\|_\infty \leq \|K_0\|_{L_\infty} \|x\|_{M_p}.$$

Now, we are ready to proclaim our main results.

Theorem 3.1. Let assumptions (i)–(vi) be satisfied. If $(b_3 + b_1 b_2 \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4 \cdot r)^{1/p}) < 1$, then equation (1) has at least one solution $x \in E_\varphi(I)$ which is a.e. nondecreasing on I .

Proof.

I. Assumption (ii), assumption (iii) and Lemma 2.3 imply that each F_{f_i} maps $E_\varphi(I)$ into itself and is continuous and by Lemma 2.2 the operator F_g maps $E_\varphi(I)$ into $E_{M_p}(I)$. By assumption (iv), the operator KF_g maps $E_\varphi(I)$ into $L_\infty(I)$ and is continuous. Thus, the operator A is a continuous mapping from $E_\varphi(I)$ into itself. Finally, by assumption (i), we can deduce that the operator $B = h + F_{f_3} + A$ maps $E_\varphi(I)$ into itself and is continuous.

II. Now, we will prove the operator B is bounded in $E_\varphi(I)$. For $x \in E_\varphi(I)$ with $\|x\|_\varphi \leq r \leq 1$, and using Proposition 3.1, we have

$$\begin{aligned} \|B(x)\|_\varphi &\leq \|h\|_\varphi + \|F_{f_3}(x)\|_\varphi + \|A(x)\|_\varphi \\ &\leq \|h\|_\varphi + \|a_3\|_\varphi + b_3\|x\|_\varphi + \|a_2\|_\varphi + b_2\|F_{f_1}(x) \cdot KF_g(x)\|_\varphi \\ &\leq \|h\|_\varphi + \|a_2\|_\varphi + \|a_3\|_\varphi + b_3\|x\|_\varphi + b_2\|F_{f_1}(x)\|_\varphi \cdot \|KF_g(x)\|_{L_\infty} \\ &\leq \|h\|_\varphi + \|a_2\|_\varphi + \|a_3\|_\varphi + b_3\|x\|_\varphi + b_2(\|a_1\|_\varphi + b_1\|x\|_\varphi) \cdot \|K_0\|_{L_\infty} \|F_g(x)\|_{M_p} \\ &\leq \|h\|_\varphi + \|a_2\|_\varphi + \|a_3\|_\varphi + b_3\|x\|_\varphi + b_2(\|a_1\|_\varphi + b_1\|x\|_\varphi) \cdot \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4\|x\|_\varphi)^{1/p}. \end{aligned}$$

Thus, $B: E_\varphi(I) \rightarrow E_\varphi(I)$.

By our assumption (vi), it follows that there exists a positive solution $r \leq 1$ of the equation

$$\|h\|_\varphi + \|a_2\|_\varphi + \|a_3\|_\varphi + b_3 \cdot r + b_2(\|a_1\|_\varphi + b_1 \cdot r) \cdot \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4 \cdot r)^{1/p} = r,$$

which implies that $B: B_r(E_\varphi(I)) \rightarrow B_r(E_\varphi(I))$ is continuous.

III. Let Q_r stand for the subset of $B_r(E_\varphi(I))$ consisting of all functions which are a.e. nondecreasing on I .

Similarly, as claimed in [10] this set is nonempty, bounded, closed and convex in $L_\varphi(I)$. The set Q_r is compact in measure in view of Lemma 2.4.

IV. Now, we will show that B preserves the monotonicity of functions. Take $x \in Q_r$, then x is a.e. nondecreasing on I and consequently F_g and F_{f_i} , $i = 1, 2, 3$, are also of the same type in virtue of assumption (ii). Furthermore, $K_0(x)$ is a.e. nondecreasing on I (thanks for assumption (v)). Since the pointwise product of a.e. monotone functions is still of the same type, the operator A is a.e. nondecreasing on I . Then by assumption (i), we deduce that $B(x)(t) = h + F_{f_3}(x)(t) + A(x)(t)$ is also a.e. nondecreasing on I . This gives us that $B: Q_r \rightarrow Q_r$ and is continuous.

V. We will prove that B is a contraction with respect to a measure of strong noncompactness.

Assume that $X \subset Q_r$ is a nonempty and let the fixed constant $\varepsilon > 0$ be arbitrary. Then, for an arbitrary $x \in X$ and for an arbitrary measurable set $D \subset I$, $\text{mes } D \leq \varepsilon$ and $t \in D$ we have

$$\begin{aligned} \|B(x)\chi_D\|_\varphi &\leq \|h\chi_D\|_\varphi + \|F_{f_3}(x)\chi_D\|_\varphi + \|A(x)\chi_D\|_\varphi \\ &\leq \|h\chi_D\|_\varphi + \|(a_3 + b_3|x|)\chi_D\|_\varphi + \|(a_2 + b_2 F_{f_1}(x) \cdot KF_g(x))\chi_D\|_\varphi \\ &\leq \|h\chi_D\|_\varphi + \|a_3\chi_D\|_\varphi + b_3\|x\chi_D\|_\varphi + \|a_2\chi_D\|_\varphi + b_2\|F_{f_1}(x)\chi_D\|_\varphi \|KF_g(x)\|_\infty \\ &\leq \|h\chi_D\|_\varphi + \|a_3\chi_D\|_\varphi + b_3\|x\chi_D\|_\varphi + \|a_2\chi_D\|_\varphi \\ &\quad + b_2(\|a_1\chi_D\|_\varphi + b_1\|x\chi_D\|_\varphi) \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4\|x\|_\varphi)^{1/p} \\ &\leq \|h\chi_D\|_\varphi + \|a_3\chi_D\|_\varphi + b_3\|x\chi_D\|_\varphi + \|a_2\chi_D\|_\varphi \\ &\quad + b_2(\|a_1\chi_D\|_\varphi + b_1\|x\chi_D\|_\varphi) \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4\cdot r)^{1/p}. \end{aligned}$$

Hence, taking into account that $h, a_1, a_2, a_3 \in E_\varphi(I)$, we have

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{\text{mes } D \leq \varepsilon} \left[\sup_{x \in X} \{ \|h\chi_D\|_\varphi \} \right] \right\} = 0, \quad \lim_{\varepsilon \rightarrow 0} \left\{ \sup_{\text{mes } D \leq \varepsilon} \left[\sup_{x \in X} \{ \|a_i\chi_D\|_\varphi \} \right] \right\} = 0, \quad i = 1, 2, 3.$$

Thus, by definition of $c(x)$, we get

$$c(B(X)) \leq (b_3 + b_1 b_2 \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4 \cdot r)^{1/p}) \cdot c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space $E_\varphi(I)$, we can use Lemma 2.1 and get

$$\beta_H(B(X)) \leq (b_3 + b_1 b_2 \|K_0\|_{L_\infty} (\|a_4\|_{L_1} + b_4 \cdot r)^{1/p}) \cdot \beta_H(X).$$

Thus, we can use Darbo fixed point theorem 2.1, which completes the proof. Moreover, the set of solutions is compact in $E_\varphi(I)$. \square

Next, we present L_p -solutions for equation (1), which is still a more general result than the earlier ones.

Corollary 3.1. Assume that $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

- (i) $h \in L_p(I)$ is nondecreasing a.e. on I .
- (ii) $g, f_i: I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Carathéodory conditions and $g(t, x)$, $f_i(t, x)$ are assumed to be nondecreasing with respect to both variables t and x separately, for $i = 1, 2, 3$.
- (iii) There exist constants $b_j \geq 0$, $j = 1, \dots, 4$ and positive functions $a_4 \in L_q(I)$, $a_i \in L_p(I)$ such that

$$|f_i(t, x)| \leq a_i(t) + b_i|x|, \quad i = 1, 2, 3$$

and

$$|g(t, x)| \leq a_4(t) + b_4|x|^{p/q}.$$

(iv) Assume that the function K is measurable in (t, s) and the linear integral operator K_0 with kernel $K(\cdot, \cdot)$ maps $L^q(I)$ into $L_\infty(I)$ and is continuous, such that

$$\|K'_0\|_{L_\infty} = \text{ess sup}_{t \in [a, b]} \left(\int_a^b |K(t, s)|^{q'} ds \right)^{1/q'} < \infty.$$

(v) $\int_I K(t_1, s) ds \geq \int_I K(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$.

(vi) Assume that there exists a positive number $r \leq 1$ such that

$$\|h\|_p + \|a_2\|_p + \|a_3\|_p + b_3 \cdot r + b_2 \cdot \|K_0\|_{L_\infty} \cdot (\|a_1\|_p + b_1 \cdot r) (\|a_4\|_q + b_4 \cdot r^{p/q}) = r.$$

If $(b_3 + b_1 b_2 \|K_0\|_{L_\infty} (\|a_4\|_q + b_4 \cdot r^{p/q})) < 1$, then equation (1) has at least one solution $x \in L_p(I)$ which is a.e. nondecreasing on I .

4 IVP

Next, we will discuss the existence of some special class of solutions for IVP (2) with nonlocal condition (3). As a consequence of our main result solutions are not absolutely continuous, but they are in a narrower space.

Definition 4.1. A function $y = \int_0^t x(s) ds$ is called a solution of problem (2) with nonlocal condition (3), if $x \in E_\varphi(I)$ is a solution of equation (1), i.e., belongs to an Orlicz-Sobolev space $W^1 L_\varphi(I)$.

Theorem 4.1. Let assumptions of Theorem 3.1 be satisfied, then there exists a function

$$y(t) = \frac{\beta}{1-\beta} \int_0^\zeta x(s) ds - \frac{1}{1-\beta} \int_0^\tau x(s) ds + \int_0^t x(s) ds, \quad \text{for } x \in E_\varphi(I),$$

which satisfies IVP (2) with nonlocal condition (3).

Proof. Let x be a solution of integral equation (1). Put

$$Dy(t) = x(t), \quad D = \frac{d}{dt},$$

then by integrating both sides, we have

$$\int_0^t Dy(s) ds = \int_0^t x(s) ds,$$

which yields

$$y(t) = y(0) + \int_0^t x(s) ds. \tag{4}$$

Using condition (3)

$$y(\tau) = y(0) + \int_0^\tau x(s) ds \quad \text{and} \quad y(\zeta) = y(0) + \int_0^\zeta x(s) ds.$$

By omitting $y(0)$ from the aforementioned equations and substitute in (4), we have

$$y(t) = \frac{\beta}{1-\beta} \int_0^\zeta x(s)ds - \frac{1}{1-\beta} \int_0^\tau x(s)ds + \int_0^t x(s)ds.$$

Since $x \in E_\varphi$ (thanks to Theorem 3.1), then we deduce that y is a solution for IVP (2) with nonlocal condition (3), which completes the proof. \square

5 Remarks

Remark 5.1. The quadratic equations have many applications in astrophysics, neutron transport, radiative transfer theory and in the kinetic theory of gases [4,5,23].

Remark 5.2. We obtain the same results, if we assume that $F_{f_1}: L_\varphi(I) \rightarrow L_\infty(I)$ and the Hammerstein operator maps $L_\varphi(I)$ into itself (see [10]), where $f_2(t, x) = x$. This is the standard non-quadratic case which is reduced to the classical integral equation (cf. [12,24,25]).

Remark 5.3. Shragin [26] proved that the Nemytskii operators are bounded on “small” balls and in [27] the authors apply these results for Hammerstein integral equations in Orlicz spaces.

Remark 5.4. The continuity of the linear integral operator of the form K_0 is depending on the kernel K (cf. assumption (iv)). For example, the fractional integral operator

$$J_\alpha x(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s-t)^{\alpha-1} x(t)dt, \quad s \in [a, b]$$

maps $L_{M_p} \rightarrow L_\infty(I)$ and is continuous for $p < \frac{1}{\alpha}$, but is not continuous at $p = \frac{1}{\alpha}$ (cf. [28, Remark 4.1.2]).

Acknowledgments: The authors are grateful to the referee for his/her helpful suggestions and comments on this article which greatly improved this article.

References

- [1] I.-Y. S. Cheng and J. J. Kozak, *Application of the theory of Orlicz spaces to statistical mechanics, I. Integral equations*, J. Math. Phys. **13** (1972), no. 51, 51–58, DOI: 10.1063/1.1665850.
- [2] M. A. Krasnosel'skii and Yu. Rutitskii, *Convex Functions and Orlicz Spaces*, Noordhoff, Gröningen, 1961.
- [3] W. A. Majewski and L. E. Labuschagne, *On applications of Orlicz spaces to statistical physics*, Ann. Henri Poincaré **15** (2014), 1197–1221, DOI: 10.1007/s00023-013-0267-3.
- [4] J. Caballero, A. B. Mingarelli, and K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer*, Electron. J. Differ. Equ. **57** (2006), 1–11.
- [5] S. Chandrasekhar, *Radiative Transfer*, Dover Publications, New York, 1960.
- [6] M. Cichoń and M. Metwali, *On quadratic integral equations in Orlicz spaces*, J. Math. Anal. Appl. **387** (2012), no. 1, 419–432, DOI: 10.1016/j.jmaa.2011.09.013.
- [7] A. Benkirane and A. Elmahi, *An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces*, Nonlinear Anal. **36** (1999), no. 1, 11–24.
- [8] J. Berger and J. Robert, *Strongly nonlinear equations of Hammerstein type*, J. Lond. Math. Soc. **15** (1977), no. 2, 277–287.
- [9] M. Cichoń and M. Metwali, *On solutions of quadratic integral equations in Orlicz spaces*, Mediterr. J. Math. **12** (2015), no. 3, 901–920, DOI: 10.1007/s00009-014-0450-x.

- [10] M. Cichoń and M. Metwali, *On the existence of solutions for quadratic integral equations in Orlicz space*, Math. Slovaca **66** (2016), no. 6, 1413–1426, DOI: 10.1515/ms-2016-0233.
- [11] M. Cichoń and M. Metwali, *Existence of monotonic L_ϕ -solutions for quadratic Volterra functional-integral equations*, Electron. J. Qual. Theory Differ. Equ. **13** (2015), 1–16.
- [12] R. Płuciennik and S. Szufla, *Nonlinear Volterra integral equations in Orlicz spaces*, Demonstr. Math. **17** (1984), no. 2, 515–532.
- [13] C. Bardaro, J. Musielak and G. Vinti, *Nonlinear Integral Operators and Applications*, Walter de Gruyter, Berlin, New York, 2003.
- [14] D. O'Regan, *Solutions in Orlicz spaces to Urysohn integral equations*, Proc. R. Irish Acad., Sect. A **96** (1996), 67–78.
- [15] S. K. Ntouyas, *Nonlocal initial and boundary value problems: a survey*, in: A. Cañada, P. Drábek, and A. Fonda (Eds.), *Handbook of Differential Equations: Ordinary Differential Equations*, Elsevier, Amsterdam, Vol. II, 2005, pp. 461–557.
- [16] Y. Raffoul, *Positive solutions of three-point nonlinear second order boundary value problem*, Electron. J. Qual. Theory Differ. Equ. **15** (2002), 1–11, DOI: 10.14232/ejqtde.2002.1.15.
- [17] S. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.
- [18] M. M. Rao and Z. D. Ren, *Applications of Orlicz Spaces*, Marcel Dekker, New York, 2002.
- [19] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Notes Math. 60, M. Dekker, New York-Basel, 1980.
- [20] M. Väth, *Volterra and Integral Equations of Vector Functions*, Marcel Dekker, New York-Basel, 2000.
- [21] N. Erzakova, *Compactness in measure and measure of noncompactness*, Sib. Math. J. **38** (1997), no. 5, 926–928, DOI: 10.1007/BF02673034.
- [22] J. Appell and P. P. Zabreiko, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1990.
- [23] I. K. Argyros, *On a class of quadratic integral equations with perturbations*, Funct. Approx. Comment. Math. **20** (1992), 51–63.
- [24] W. Orlicz and S. Szufla, *On some classes of nonlinear Volterra integral equations in Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (1982), 239–250.
- [25] A. Sołtysiak and S. Szufla, *Existence theorems for L_ϕ -solutions of the Hammerstein integral equation in Banach spaces*, Comment. Math. Prace Mat. **30** (1990), 177–190.
- [26] I. V. Shrigin, *On the boundedness of the Nemytskii operator in Orlicz spaces*, Kišinev. Gos. Univ. Učen. Zap. **50** (1962), 119–122.
- [27] M. M. Vainberg and I. V. Shrigin, *Nonlinear operators and the Hammerstein equation in Orlicz spaces*, Dokl. Akad. Nauk SSSR, **128** (1959), no. 1, 9–12.
- [28] R. Gorenflo and S. Vessela, *Abel Integral Equations*, Lect. Notes Math. 1461, Springer, Berlin-Heidelberg, 1991.