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Research Article

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On the non-hypercyclicity of scalar type spectral operators and collections of their exponentials

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Abstract: Generalizing the case of a normal operator in a complex Hilbert space, we give a straightforward proof of the non-hypercyclicity of a *scalar type spectral operator* A in a complex Banach space as well as of the collection $\{e^{tA}\}_{t\geq 0}$ of its exponentials, which, under a certain condition on the spectrum of the operator A, coincides with the C_0-semigroup generated by A. The spectrum of A lying on the imaginary axis, we also show that non-hypercyclic is the strongly continuous group $\{e^{tA}\}_{t\in\mathbb{R}}$ of bounded linear operators generated by A. From the general results, we infer that, in the complex Hilbert space $L_2(\mathbb{R})$, the anti-self-adjoint differentiation operator $A := \frac{d}{dx}$ with the domain $D(A) := W_2^1(\mathbb{R})$ is non-hypercyclic and so is the left-translation strongly continuous unitary operator group generated by A.

Keywords: hypercyclicity, scalar type spectral operator, normal operator, C_0-semigroup, strongly continuous operator group

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1 Introduction

The concept of *hypercyclicity*, underlying the theory of linear chaos, traditionally considered for *continuous* linear operators on Fréchet spaces, in particular for *bounded* linear operators on Banach spaces, and known to be a purely infinite-dimensional phenomenon (see, e.g., [1–3]), is extended in [4,5] to *unbounded* linear operators in Banach spaces, where also found are sufficient conditions for unbounded hypercyclicity and certain examples of hypercyclic unbounded linear differential operators.

Definition 1.1. (Hypercyclicity)
Let

$$A : X \supseteq \quad D(A) \to X$$

be a (bounded or unbounded) linear operator in a (real or complex) Banach space X with a domain $D(A)$.

A nonzero vector

$$f \in C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)$$

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($A^0 := I$, I is the *identity operator* on X) is called *hypercyclic* if its *orbit* under A

$$\operatorname{orb}(f, A) := \{A^n f\}_{n \in \mathbb{Z}_+}$$

($\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of nonnegative integers) is dense in X.

Linear operators possessing hypercyclic vectors are said to be *hypercyclic*.

More generally, a collection $\{T(t)\}_{t \in J}$ (J is a nonempty indexing set) of linear operators in X is called *hypercyclic* if it possesses *hypercyclic vectors*, i.e., such nonzero vectors $f \in \bigcap_{t \in J} D(T(t))$, whose *orbit*

$$\{T(t)f\}_{t \in J}$$

is dense in X.
Cf. [6,7].

As is easily seen, in the definition of hypercyclicity for a linear operator, the underlying space must necessarily be *separable*.

It is noteworthy that, for a hypercyclic linear operator A, the set $\operatorname{HC}(A)$ of all its hypercyclic vectors, containing the dense orbit of any vector hypercyclic under A, is dense in $(X, \|\cdot\|)$, and hence, the more so, is the subspace $C^\infty(A) \supseteq \operatorname{HC}(A)$.

Bounded *normal operators* on a complex Hilbert space are known to be non-hypercyclic [1, Corollary 5.31]. In [8], non-hypercyclicity is shown to hold for arbitrary normal operators (bounded or unbounded), certain collections of their exponentials, and symmetric operators.

Here, abandoning the comforts of a Hilbert space setting with its inherent orthogonality and self-duality, while generalizing non-hypercyclicity from normal to scalar type spectral operators, we furnish a straightforward proof of the non-hypercyclicity of an arbitrary *scalar type spectral operator* A (bounded or unbounded) in a complex Banach space as well as of the collection $\{e^{tA}\}_{t \geq 0}$ of its exponentials (see, e.g., [9–11]), which, provided the spectrum $\sigma(A)$ of the operator A is located in a left half plane

$$\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \leq \omega\}$$

with some $\omega \in \mathbb{R}$, coincides with the C_0-*semigroup* generated by A [12] (see also [13,14]). The spectrum of A lying on the imaginary axis $i\mathbb{R}$ (i is the *imaginary unit*), we also show that non-hypercyclic is the strongly continuous group $\{e^{tA}\}_{t \in \mathbb{R}}$ of bounded linear operators generated by A. From the general results, we immediately infer that, in the complex Hilbert space $L_2(\mathbb{R})$, the *anti-self-adjoint* differentiation operator $A := \frac{d}{dx}$ with the domain

$$W_2^1(\mathbb{R}) := \{f \in L_2(\mathbb{R}) \,|\, f(\cdot) \text{ is absolutely continuous on } \mathbb{R} \text{ with } f' \in L_2(\mathbb{R})\}$$

is non-hypercyclic and so is the left-translation strongly continuous unitary operator group generated by it [15–17].

2 Preliminaries

More extensive preliminaries concerning the *scalar type spectral operators* in complex Banach spaces, which, in particular, encompass *normal operators* in complex Hilbert spaces [18] (see also [19,20]), can be found in the corresponding section of [21] (see also [9–11]). Here, we outline only a few facts indispensable for our subsequent discourse.

With a *scalar type spectral operator* A in a complex Banach space $(X, \|\cdot\|)$ associated are its *spectral measure* (the *resolution of the identity*) $E_A(\cdot)$, whose support is the spectrum $\sigma(A)$ of A, and the so-called *Borel operational calculus* assigning to any Borel measurable function $F : \sigma(A) \to \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda)$$

(see [10,11]).

In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+,$$

and

$$e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \quad t \in \mathbb{R}.$$

Provided

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le \omega\},$$

with some $\omega \in \mathbb{R}$, the collection of exponentials $\{e^{tA}\}_{t\ge 0}$ coincides with the C_0-*semigroup* generated by A [12, Proposition 3.1] (see also [13,14]), and hence, if

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid -\omega \le \operatorname{Re}\lambda \le \omega\},$$

with some $\omega \ge 0$, the collection of exponentials $\{e^{tA}\}_{t\in\mathbb{R}}$ coincides with the *strongly continuous group* of bounded linear operators generated by A.

The orbit maps

$$y(t) = e^{tA}f, \quad t \ge 0, \quad f \in \bigcap_{t\ge 0} D(e^{tA}), \tag{2.1}$$

describe all *weak/mild solutions* of the abstract evolution equation

$$y'(t) = Ay(t), \quad t \ge 0, \tag{2.2}$$

[22, Theorem 4.2], whereas the orbit maps

$$y(t) = e^{tA}f, \quad t \in \mathbb{R}, \quad f \in \bigcap_{t\in\mathbb{R}} D(e^{tA}),$$

describe all *weak/mild solutions* of the abstract evolution equation

$$y'(t) = Ay(t), \quad t \in \mathbb{R}, \tag{2.3}$$

[21, Theorem 7] (see also [23]). Such generalized solutions need not be differentiable in the strong sense and encompass the *classical* ones, strongly differentiable and satisfying the corresponding equations in the traditional plug-in sense (cf. [17, Ch. II, Definition 6.3], see also [24, Preliminaries]).

The subspaces

$$C^\infty(A), \quad \bigcap_{t\ge 0} D(e^{tA}), \text{ and } \bigcap_{t\in\mathbb{R}} D(e^{tA})$$

of all possible initial values for the orbits under A, $\{e^{tA}\}_{t\ge 0}$, and $\{e^{tA}\}_{t\in\mathbb{R}}$ are *dense* in $(X, \|\cdot\|)$ since they contain the subspace

$$\bigcup_{\alpha>0} E_A(\Delta_\alpha)X, \text{ where } \Delta_\alpha := \{\lambda \in \mathbb{C} \,\|\lambda| \le \alpha\}, \quad \alpha > 0,$$

which is dense in $(X, \|\cdot\|)$ and coincides with the class $\mathscr{E}^{\{0\}}(A)$ of the *exponential type entire* vectors of the operator A [25] (see also [26]).

Due to its strong countable additivity, the spectral measure $E_A(\cdot)$ is bounded, i.e., there exists such an $M \ge 1$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \le M \tag{2.4}$$

[11,27], the notation $\|\cdot\|$ being used here to designate the norm in the space $L(X)$ of all bounded linear operators on X. Adhering to this rather conventional economy of symbols hereafter, we also adopt the same notation for the norm in the dual space X^*.

For arbitrary $f \in X$ and $g^* \in X^*$, the *total variation measure* $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the *pairing* between the space X and its dual X^*) is a *finite* positive Borel measure with

$$v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \le 4M\|f\|\|g^*\| \tag{2.5}$$

(see, e.g., [28,29]).

Also [28,29], for any Borel measurable function $F : \mathbb{C} \to \mathbb{C}$, arbitrary $f \in D(F(A))$ and $g^* \in X^*$, and each Borel set $\delta \subseteq \mathbb{C}$,

$$\int_{\delta} |F(\lambda)|\,dv(f, g^*, \lambda) \le 4M\|E_A(\delta)F(A)f\|\|g^*\|. \tag{2.6}$$

In particular, for $\delta = \sigma(A)$,

$$\int_{\sigma(A)} |F(\lambda)|\,dv(f, g^*, \lambda) \le 4M\|F(A)f\|\|g^*\|. \tag{2.7}$$

Observe that the constant $M \ge 1$ in (2.5)–(2.7) is from (2.4).

3 Main results

Theorem 3.1. *An arbitrary scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ is non-hypercyclic and so is the collection $\{e^{tA}\}_{t\ge0}$ of its exponentials, which, provided the spectrum of A is located in a left half plane*

$$\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \le \omega\}$$

with some $\omega \in \mathbb{R}$, coincides with the C_0-semigroup generated by A.

Proof. Let $f \in C^\infty(A)\setminus\{0\}$ be arbitrary.

There are two possibilities: either

$$E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| > 1\})f \ne 0$$

or

$$E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| > 1\})f = 0.$$

In the first case, as follows from the *Hahn-Banach theorem* (see, e.g., [27]), there exists a functional $g^* \in X^*\setminus\{0\}$ such that

$$\langle E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| > 1\})f, g^*\rangle \ne 0$$

and hence, for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}
&\|A^n f\| \\
&\qquad\text{by (2.7);} \\
&\ge [4M\|g^*\|]^{-1} \int_{\sigma(A)} |\lambda|^n\,dv(f, g^*, \lambda) \ge [4M\|g^*\|]^{-1} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|>1\}} |\lambda|^n\,dv(f, g^*, \lambda) \\
&\ge [4M\|g^*\|]^{-1} v(f, g^*, \{\lambda \in \sigma(A) \,|\, |\lambda| > 1\}) \\
&\ge [4M\|g^*\|]^{-1} |\langle E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| > 1\})f, g^*\rangle| > 0,
\end{aligned}$$

which implies that the orbit $\{A^n f\}_{n\in\mathbb{Z}_+}$ of f under A cannot approximate the zero vector, and hence, is not dense in $(X, \|\cdot\|)$.

In the second case, since

$$f = E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| > 1\})f + E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| \le 1\})f,$$

we infer that

$$f = E_A(\{\lambda \in \sigma(A) \,|\, |\lambda| \le 1\})f \ne 0$$

and hence, for any $n \in \mathbb{Z}_+$,

$$
\begin{aligned}
&\|A^n f\| \\
&\qquad\qquad \text{by the properties of the } \textit{operational calculus}; \\
&= \left\| \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\}} \lambda^n \, dE_A(\lambda) f \right\| \\
&\qquad\qquad \text{as follows from the } \textit{Hahn – Banach theorem}; \\
&= \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \left| \left\langle \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\}} \lambda^n \, dE_A(\lambda) f, g^* \right\rangle \right| \\
&\qquad\qquad \text{by the properties of the } \textit{operational calculus}; \\
&= \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \left| \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\}} \lambda^n \, d\langle E_A(\lambda) f, g^*\rangle \right| \\
&\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\}} |\lambda|^n \, dv(f, g^*, \lambda) \\
&\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\}} 1 \;\, dv(f, g^*, \lambda) \\
&\qquad\qquad \text{by (2.6) with } F(\lambda) \equiv 1; \\
&\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} 4M\|E_A(\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\})f\|\|g^*\| \\
&= 4M\|E_A(\{\lambda\in\sigma(A)\,|\,|\lambda|\le 1\})f\|,
\end{aligned}
$$

which also implies that the orbit $\{A^n f\}_{n\in\mathbb{Z}_+}$ of f under A, being bounded, is not dense in $(X, \|\cdot\|)$ and completes the proof for the case of the operator.

Now, let us consider the case of the exponential collection $\{e^{tA}\}_{t\ge 0}$ assuming that $f \in \bigcap_{t\ge 0} D(e^{tA})\setminus\{0\}$ is arbitrary.

There are two possibilities: either

$$E_A(\{\lambda \in \sigma(A)\,|\,\operatorname{Re}\lambda > 0\})f \neq 0$$

or

$$E_A(\{\lambda \in \sigma(A)\,|\,\operatorname{Re}\lambda > 0\})f = 0.$$

In the first case, as follows from the *Hahn-Banach theorem*, there exists a functional $g^* \in X^*\setminus\{0\}$ such that

$$\langle E_A(\{\lambda \in \sigma(A)\,|\,\operatorname{Re}\lambda > 0\})f, g^*\rangle \neq 0$$

and hence, for any $t \ge 0$,

$$
\begin{aligned}
&\|e^{tA} f\| \\
&\qquad\qquad \text{by (2.7)}; \\
&\ge [4M\|g^*\|]^{-1} \int_{\sigma(A)} |e^{t\lambda}| \, dv(f, g^*, \lambda) \\
&\ge [4M\|g^*\|]^{-1} \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda>0\}} e^{t\operatorname{Re}\lambda} \, dv(f, g^*, \lambda) \\
&\qquad\qquad \text{since for } t \ge 0 \text{ and } \lambda \in \sigma(A) \text{ with } \operatorname{Re}\lambda > 0,\ e^{t\operatorname{Re}\lambda} \ge 1; \\
&\ge [4M\|g^*\|]^{-1} v(f, g^*, \{\lambda \in \sigma(A)\,|\,\operatorname{Re}\lambda > 0\}) \\
&\ge [4M\|g^*\|]^{-1} |\langle E_A(\{\lambda \in \sigma(A)\,|\,\operatorname{Re}\lambda > 0\})f, g^*\rangle| > 0,
\end{aligned}
$$

which implies that the orbit $\{e^{tA} f\}_{t\ge 0}$ of f cannot approximate the zero vector, and hence, is not dense in $(X, \|\cdot\|)$.

In the second case, since

$$f = E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > 0\})f + E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \le 0\})f,$$

we infer that

$$f = E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \le 0\})f \neq 0,$$

and hence, for any $t \ge 0$,

$$\begin{aligned}
&\|e^{tA}f\| \\
&\qquad\qquad \text{by the properties of the } \textit{operational calculus}; \\
&= \left\| \int\limits_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le 0\}} e^{t\lambda}\,dE_A(\lambda)f \right\| \\
&\qquad\qquad \text{as follows from the } \textit{Hahn-Banach theorem}; \\
&= \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \left| \left\langle \int\limits_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le 0\}} e^{t\lambda}\,dE_A(\lambda)f, g^* \right\rangle \right| \\
&\qquad\qquad \text{by the properties of the } \textit{operational calculus}; \\
&= \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \left| \int\limits_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le 0\}} e^{t\lambda}\,d\langle E_A(\lambda)f, g^*\rangle \right| \\
&\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le 0\}} |e^{t\lambda}|\,dv(f, g^*, \lambda) \\
&= \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le 0\}} e^{t\operatorname{Re}\lambda}\,dv(f, g^*, \lambda) \\
&\qquad\qquad \text{since for } t \ge 0 \text{ and } \lambda \in \sigma(A) \text{ with } \operatorname{Re}\lambda \le 0,\ e^{t\operatorname{Re}\lambda} \le 1; \\
&\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le 0\}} 1\ dv(f, g^*, \lambda) \\
&\qquad\qquad \text{by (2.6) with } F(\lambda) \equiv 1; \\
&\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} 4M\,\|E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \le 0\})f\|\,\|g^*\| \\
&= 4M\,\|E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \le 0\})f\|,
\end{aligned}$$

which also implies that the orbit $\{e^{tA}f\}_{t\ge 0}$ of f, being bounded, is not dense in $(X, \|\cdot\|)$ and completes the entire proof. □

Remark 3.1. Now, [8, Theorem 1] is the important particular case of Theorem 3.1 for a (bounded or unbounded) *normal operator* in a complex Hilbert space.

If further for a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, we have the inclusion:

$$\sigma(A) \subseteq i\mathbb{R},$$

by [11, Theorem XVIII.2.11 (c)], for any $t \in \mathbb{R}$,

$$\|e^{tA}\| = \left\| \int\limits_{\sigma(A)} e^{t\lambda}\,dE_A(\lambda) \right\| \le 4M \sup_{\lambda\in\sigma(A)} |e^{t\lambda}| = 4M \sup_{\lambda\in\sigma(A)} e^{t\operatorname{Re}\lambda} = 4M,$$

where the constant $M \ge 1$ is from (2.4). Therefore, the strongly continuous group $\{e^{tA}\}_{t\in\mathbb{R}}$ of bounded linear operators generated by A is *bounded* (cf. [13]), which implies that every orbit $\{e^{tA}f\}_{t\in\mathbb{R}}$, $f \in X$, is bounded, and hence, is not dense in $(X, \|\cdot\|)$. Thus, we arrive at the following.

Proposition 3.1. *For a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$ with $\sigma(A) \subseteq i\mathbb{R}$, the strongly continuous group $\{e^{tA}\}_{t\in\mathbb{R}}$ of bounded linear operators generated by A is bounded, and hence, non-hypercyclic.*

As is known [15], for an *anti-self-adjoint operator A* in a complex Hilbert space, $\sigma(A) \subseteq i\mathbb{R}$ and the generated by A strongly continuous operator group $\{e^{tA}\}_{t\in\mathbb{R}}$ is *unitary*, which, in particular, implies that

$$\|e^{tA}\| = 1, \quad t \in \mathbb{R}.$$

Thus, from Theorem 3.1 (see also [8, Theorem 1]) and Proposition 3.1, we derive the following corollary.

Corollary 3.1. (The Case of an Anti-Self-Adjoint Operator)
An anti-self-adjoint operator A in a complex Hilbert space is non-hypercyclic and so is the generated by A strongly continuous unitary operator group $\{e^{tA}\}_{t\in\mathbb{R}}$.

4 An application

Since, in the complex Hilbert space $L_2(\mathbb{R})$, the differentiation operator $A := \frac{d}{dx}$ with the domain

$$W_2^1(\mathbb{R}) := \{f \in L_2(\mathbb{R}) \,|\, f(\cdot) \text{ is absolutely continuous on } \mathbb{R} \text{ with } f' \in L_2(\mathbb{R})\}$$

is *anti-self-adjoint* (see, e.g., [30]), by Corollary 3.1, we obtain:

Corollary 4.1. (The Case of Differentiation Operator)
In the complex Hilbert space $L_2(\mathbb{R})$, the differentiation operator $A := \frac{d}{dx}$ with the domain $D(A) := W_2^1(\mathbb{R})$ is non-hypercyclic and so is the left-translation strongly continuous unitary operator group generated by A.

Remark 4.1. In a different setting, the situation with the differentiation operator can be vastly different (cf. [1, Example 2.21], [4, Corollary 2.3], [31, Corollary 4.1], and [32, Theorem 3.1]).

5 Concluding remark

The exponentials given by (2.1) describing all *weak/mild solutions* of evolution equation (2.2) (see section Preliminaries), Theorem 3.1, in particular, implies that such an equation is void of chaos (see [1]). By Proposition 3.1 (see also Preliminaries), the same is true for evolution equation (2.3) provided $\sigma(A) \subseteq i\mathbb{R}$.

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