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Research Article

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Applications of some operators on supra topological spaces

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Abstract: In this paper, the notion of an operator γ on a supra topological space (X, μ) is studied and then utilized to analyze supra γ-open sets. The notions of μ_γ-g.closed sets on the subspace are introduced and investigated. Furthermore, some new μ_γ-separation axioms are formulated and the relationships between them are shown. Moreover, some characterizations of the new functions via operator γ on μ are presented and investigated. Finally, we give some properties of $S_{(\gamma,\beta)}$-closed graph and strongly $S_{(\gamma,\beta)}$-closed graph.

Keywords: supra topology, operator γ on μ, μ_γ-g.closed sets, operator on subspace ST, μ_γ-separation axioms, $S_{(\gamma,\beta)}$-continuous functions, $S_{(\gamma,\beta)}$-closed and strongly $S_{(\gamma,\beta)}$-closed graphs

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1 Introduction

Kasahara [1] defined an operator associated with a topology, namely, an α operator. He initiated some concepts that are equivalent to those given in topological spaces when the operator is the identity operator. Also, he studied α-closed graphs of α-continuous functions and α-compact spaces. Then, Jankovic [2] used α operator to introduce α-closure of a set and give some characterizations on α-closed graph of functions. Later, Ogata [3] defined the notion of γ-open sets to study operator-functions and operator-separation. Rosas and Vielma [4] investigated some features of operator-compact spaces and defined the concept of operator-connected spaces. Kalaivani and Krishnan [5] formulated the concept of α-γ-open sets in a topological space and studied their corresponding closure and interior operators. Quite recently, many notions of operators have been investigated on different classes of open sets and generalizations of open sets; see [6–15].

In 1983, Mashhour et al. [16] introduced supra topological spaces (STSs) by neglecting an intersection condition of topology. This makes supra topology (ST) more flexible to describe some real-life problems (see, [17]) and construct easily some examples that show the relationships between certain topological concepts. Al-shami [18] investigated the classical topological notions such as limit points of a set, compactness, and separation axioms on the STSs. Investigation of several types of compactness and Lindelöfness was the goal of some papers such as [19–22]. Al-shami [23] introduced the concept of paracompactness on STSs and explored main properties. Recently, the authors of [24–27] have employed some generalizations of supra open sets given in [28–31] to study limit points and separation axioms on STSs. They have provided

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various interesting examples to explain the given relationships and results. In [32–34], some new concepts and notions were introduced using supra b-open sets and supra D-open sets.

This paper is organized as follows: after this introduction, we recall some basic definitions that are necessary to understand this work. In Section 3, an operator γ depending on supra open sets is studied and then employed to analyze supra γ-open sets. In Section 4, we introduce and discuss μ_γ-g.closed sets on subspace ST. In Section 5, some new μ_γ-separation axioms are formulated using the operator γ on μ and the relationships between them are elucidated. In Section 6, some new classes of functions are defined and some characterizations of these functions are given. In Section 7, two new classes of closed graphs are studied and some relations and properties are obtained. Section 8 concludes the paper with summary.

2 Preliminaries

Let X be a non-empty set and $P(X)$ be the power set of X.

Definition 2.1. [16] A subfamily μ of $P(X)$ is called an ST if it is closed under arbitrary union and X is a member of μ.

Then the pair (X, μ) is called an STS. Terminologically, a member of μ is called a supra open set and its complement is called a supra closed set.

Definition 2.2. [16] For $A \subseteq (X, \mu)$, $\text{int}^\mu(A)$ is the union of all supra open sets that are contained in A and $cl^\mu(A)$ is the intersection of all supra closed sets containing A.

Remark 2.3. μ is called an associated ST with a topology τ if $\tau \subseteq \mu$.

Definition 2.4. [16,18] An STS (X, μ) is said to be:
- (i) supra T_0 (briefly, S-T_0) if $\forall x \neq y \in X$, $\exists U \in \mu$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$;
- (ii) supra T_1 (briefly, S-T_1) if $\forall x \neq y \in X$, $\exists U, V \in \mu$ with $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$;
- (iii) supra T_2 (briefly, S-T_2) if $\forall x \neq y \in X$, $\exists U, V \in \mu$ with $x \in U$, $y \in V$ and $U \cap V = \phi$;
- (iv) supra regular if for every supra closed set F and every $a \notin F$, there exist disjoint supra open sets U and V containing F and a, respectively;
- (v) supra normal if for every disjoint supra closed sets F and H, there exist disjoint supra open sets U and V containing F and H, respectively;
- (vi) S-T_3 (resp. S-T_4) if it is both supra regular (resp. supra normal) and S-T_1.

Definition 2.5. [16] The graph $G(f)$ of a function $f : X \to Y$ is called S-closed if $\forall (x, y) \in (X \times Y)\backslash G(f)$, there exist two supra open sets U in X and V in Y with $x \in U$, $y \in V$ and $(U \times V) \cap G(f) = \phi$.

3 Supra γ-open sets and operators

In this section, we introduce and study the concept of γ operator on an ST. Then, we define supra γ-regular and supra open operators and investigate main properties. We construct some examples to show the obtained results.

Definition 3.1. Let (X, μ) be an STS. An operator γ on an ST μ is a mapping from μ to $P(X)$ such that $U \subseteq \gamma(U)$ $\forall U \in \mu$, where $\gamma(U)$ denotes the value of γ at U. This operator will be denoted by $\gamma : \mu \to P(X)$.

Definition 3.2. Let (X, μ) be an STS and $\gamma : \mu \to P(X)$ be an operator on μ. A non-empty set A of X is called supra γ-open if $\forall x \in A$, $\exists U \in \mu$ with $x \in U \subseteq \gamma(U) \subseteq A$.

Suppose that the empty set ϕ is also supra γ-open set for any operator $\gamma : \mu \to P(X)$.

We denote the class of all supra γ-open subsets of an STS (X, μ) by μ_γ.

The identity operator id on μ is a mapping $id : \mu \to P(X)$ such that $V^{id} = V$ for every $V \in \mu$. This leads to that a subset A is μ_{id}-open of X iff A is supra open in X. In other words, $\mu_{id} = \mu$.

Remark 3.3. In fact, if $A \in \mu_\gamma$, then $\forall x \in A$, $\exists U \in \mu$ such that $x \in U \subseteq \gamma(U) \subseteq A$. Thus, $x \in \text{int}^\mu(A)$. This implies that $A \in \mu$. That is, every supra γ-open set is supra open. Hence, $\mu_\gamma \subseteq \mu$. But the converse of this relation is not true as illustrated in the following example.

Example 3.4. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{a, b\}, \{a, c\}\}$. Then (X, μ) is an STS. Define an operator $\gamma : \mu \to P(X)$ as follows: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\}; \\ cl^\mu(A) & \text{if } A \neq \{a, b\}. \end{cases}$$

Clearly, $\mu_\gamma = \{\phi, X, \{a, b\}\}$. Then it can be easy to check that the set $\{a, c\}$ is supra open, but not supra γ-open. So, $\mu \nsubseteq \mu_\gamma$.

Lemma 3.5. *Arbitrary union of supra γ-open sets are also supra γ-open.*

Proof. Suppose $\{A_\lambda : \lambda \in \Lambda\}$ is a class of supra γ-open sets in X. We have to show that $\bigcup_{\lambda\in\Lambda}\{A_\lambda\} \in \mu_\gamma$. For this, let $x \in \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$. Then $x \in A_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Hence, $\exists U \in \mu$ such that $x \in U$ and $\gamma(U) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$. Therefore, $\bigcup_{\lambda\in\Lambda}\{A_\lambda\} \in \mu_\gamma$. ☐

Example 3.6. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$. Define an operator $\gamma : \mu \to P(X)$ by $\gamma(A) = A$, $\forall A \in \mu$. Thus, $\mu_\gamma = \mu$. So, $\{a, b\} \in \mu_\gamma$ and $\{a, c\} \in \mu_\gamma$, but $\{a, b\} \cap \{a, c\} = \{a\} \notin \mu_\gamma$.

Remark 3.7. Lemma 3.5 demonstrates that μ_γ is an ST on X, and Example 3.6 shows that μ_γ is not always a topology.

Definition 3.8. Let (X, μ) be any STS. An operator γ on μ is said to be supra regular if $\forall x \in X$ and $\forall U, V \in \mu$ both containing x, $\exists W \in \mu$ containing x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Example 3.9. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{a\}, \{b, c\}\}$. Let $\gamma : \mu \to P(X)$ be the mapping defined by: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} A & \textit{if } a \in A; \\ X & \textit{if } a \notin A. \end{cases}$$

Thus, it can easily check that $\gamma : \mu \to P(X)$ is a supra regular operator.

Theorem 3.10. *Let $\gamma : \mu \to P(X)$ be supra regular operator on μ. If $A, B \in \mu_\gamma$, then $A \cap B \in \mu_\gamma$.*

Proof. Assume that $A, B \in \mu_\gamma$ and let $x \in A \cap B$. Then $x \in A$ and $x \in B$. So, $\exists U, V \in \mu$ such that $U \subseteq A$ and $V \subseteq B$. Since $\gamma : \mu \to P(X)$ is a supra regular operator on μ, then $\exists W \in \mu$ containing x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B$. Hence, $A \cap B \in \mu_\gamma$. ☐

Remark 3.11. If γ is a supra regular operator on μ, then μ_γ is a topology on X.

Example 3.12. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$. Let $\gamma : \mu \to P(X)$ be the mapping defined by: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{a\}; \\ A & \text{if } A \neq \{a\}. \end{cases}$$

Clearly, γ is not a supra regular operator on μ. Thus, $\mu_\gamma = \{\phi, X, \{a, c\}, \{b, c\}\}$ is not a topology on X.

Definition 3.13. An STS (X, μ) with an operator γ on μ is said to be supra γ-regular if $\forall x \in X$ and $\forall U \in \mu$ *with* $x \in U$, $\exists W \in \mu$ with $x \in W$ and $\gamma(W) \subseteq U$.

Theorem 3.14. *Let (X, μ) be an STS and $\gamma : \mu \to P(X)$ be an operator on μ. Then the following statements are equivalent:*
1. $\mu = \mu_\gamma$.
2. *(X, μ) is supra γ-regular.*
3. *$\forall x \in X$ and $\forall U \in \mu$ with $x \in U$, $\exists W \in \mu_\gamma$ with $x \in W$ and $W \subseteq U$.*

Proof. (1) ⇒ (2) Let $x \in X$ and $U \in \mu$ with $x \in U$. It follows from assumption that $U \in \mu_\gamma$. This implies that $\exists W \in \mu$ with $x \in W$ and $\gamma(W) \subseteq U$. Thus, (X, μ) is a supra γ-regular space.

(2) ⇒ (3) Let $x \in X$ and $U \in \mu$ with $x \in U$. Then by (2), $\exists W \in \mu$ such that $x \in W \subseteq \gamma(W) \subseteq U$. Again, by using (2) for the set W, we obtain $W \in \mu_\gamma$ such that $x \in W$ and $W \subseteq U$.

(3) ⇒ (1) By using (3) and Lemma 3.5, we obtain $U \in \mu_\gamma$. That is, $\mu \subseteq \mu_\gamma$. Since $\mu_\gamma \subseteq \mu$ in general. Thus, $\mu = \mu_\gamma$. □

Definition 3.15. A subset B of an STS (X, μ) is called supra γ-closed if $X \backslash B$ is supra γ-open in (X, μ).

Definition 3.16. Let A be any subset of an STS (X, μ) and γ be an operator on μ. Then
1. $\forall x \in X$, $x \in cl_\gamma^\mu(A)$ if $\gamma(U) \cap A \neq \phi$ $\forall U \in \mu$ with $x \in U$.
2. The supra γ-closure of A is denoted by $\mu_\gamma\text{-}cl^\mu(A)$ and is defined as

$$\mu_\gamma\text{-}cl^\mu(A) = \bigcap\{F : F \text{ is } a \text{ supra } \gamma\text{-closed set in } X \text{ and } A \subseteq F\}.$$

Theorem 3.17. *Let A be any subset of an STS (X, μ) and γ be an operator on μ. Then $x \in \mu_\gamma\text{-}cl^\mu(A)$ iff $A \cap U \neq \phi$ $\forall U \in \mu_\gamma$ with $x \in U$.*

Proof. Let $x \in \mu_\gamma\text{-}cl^\mu(A)$ and $A \cap U = \phi$ for some $U \in \mu_\gamma$ with $x \in U$. Then $A \subseteq X \backslash U$ and $X \backslash U$ is a supra γ-closed set in X. Hence, $\mu_\gamma\text{-}cl^\mu(A) \subseteq X \backslash U$. Thus, $x \in X \backslash U$. This is a contradiction. Hence, the proof is complete.

Conversely, let $x \notin \mu_\gamma\text{-}cl^\mu(A)$. So $\exists$ a supra γ-closed set F containing A with $x \notin F$. Thus, $X \backslash F \in \mu_\gamma$ with $x \in X \backslash F$ and $(X \backslash F) \cap A = \phi$. This is a contradiction. Therefore, $x \in \mu_\gamma\text{-}cl^\mu(A)$. □

Lemma 3.18. *Let (X, μ) be an STS and γ be an operator on μ. Then the following statements are true for any subsets $A, B \subseteq X$:*
1. *$\mu_\gamma\text{-}cl^\mu(A)$ is supra γ-closed set in X and $cl_\gamma^\mu(A)$ is supra closed set in X.*
2. $A \subseteq cl_\gamma^\mu(A) \subseteq \mu_\gamma\text{-}cl^\mu(A)$.
3. *(a) A is supra γ-closed iff $\mu_\gamma\text{-}cl^\mu(A) = A$ and*
   (b) A is supra γ-closed iff $cl_\gamma^\mu(A) = A$.
4. *If $A \subseteq B$, then $\mu_\gamma\text{-}cl^\mu(A) \subseteq \mu_\gamma\text{-}cl^\mu(B)$ and $cl_\gamma^\mu(A) \subseteq cl_\gamma^\mu(B)$.*
5. *(a) $\mu_\gamma\text{-}cl^\mu(A \cap B) \subseteq \mu_\gamma\text{-}cl^\mu(A) \cap \mu_\gamma\text{-}cl^\mu(B)$ and*
   (b) $cl_\gamma^\mu(A \cap B) \subseteq cl_\gamma^\mu(A) \cap cl_\gamma^\mu(B)$.

6. (*a*) $\mu_\gamma\text{-}cl^\mu(A) \cup \mu_\gamma\text{-}cl^\mu(B) \subseteq \mu_\gamma\text{-}cl^\mu(A \cup B)$ *and*
   (*b*) $cl_\gamma^\mu(A) \cup cl_\gamma^\mu(B) \subseteq cl_\gamma^\mu(A \cup B)$.
7. $\mu_\gamma\text{-}cl^\mu(\mu_\gamma\text{-}cl^\mu(A)) = \mu_\gamma\text{-}cl^\mu(A)$.

Proof. Straightforward. □

Lemma 3.19. *Let A, B be subsets of an STS (X, μ) and γ be a supra regular operator on μ. Then*
1. $\mu_\gamma\text{-}cl^\mu(A) \cup \mu_\gamma\text{-}cl^\mu(B) = \mu_\gamma\text{-}cl^\mu(A \cup B)$.
2. $cl_\gamma^\mu(A) \cup cl_\gamma^\mu(B) = cl_\gamma^\mu(A \cup B)$.

Proof.
(1) It follows directly from Lemma 3.18 (6) that $\mu_\gamma\text{-}cl^\mu(A) \cup \mu_\gamma\text{-}cl^\mu(B) \subseteq \mu_\gamma\text{-}cl^\mu(A \cup B)$. Then it is enough to prove that $\mu_\gamma\text{-}cl^\mu(A \cup B) \subseteq \mu_\gamma\text{-}cl^\mu(A) \cup \mu_\gamma\text{-}cl^\mu(B)$. Let $x \notin \mu_\gamma\text{-}cl^\mu(A) \cup \mu_\gamma\text{-}cl^\mu(B)$. Then $\exists U, V \in \mu_\gamma$ with $x \in U$, $x \in V$, $A \cap U = \phi$ and $B \cap V = \phi$. Since γ is a supra regular operator on μ, then by Theorem 3.10, $U \cap V \in \mu_\gamma$ such that
$$(U \cap V) \cap (A \cup B) = \phi.$$
This means that $x \notin \mu_\gamma\text{-}cl^\mu(A \cup B)$. Hence,
$$\mu_\gamma\text{-}cl^\mu(A \cup B) \subseteq \mu_\gamma\text{-}cl^\mu(A) \cup \mu_\gamma\text{-}cl^\mu(B).$$
(2) Let $x \notin cl_\gamma^\mu(A) \cup cl_\gamma^\mu(B)$. Then $\exists U_1 \in \mu$ and $U_2 \in \mu$ with $x \in U_1$, $x \in U_2$, $A \cap \gamma(U_1) = \phi$ and $A \cap \gamma(U_2) = \phi$. Since γ is a supra regular operator on μ, then $\exists W \in \mu$ with $x \in W$ and $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$. Thus, we have
$$(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).$$
The disjoint of $(A \cup B)$ and $(\gamma(U_1) \cap \gamma(U_2))$ leads to $(A \cup B) \cap \gamma(W) = \phi$. This means that $x \notin cl_\gamma^\mu(A \cup B)$. Therefore, $cl_\gamma^\mu(A \cup B) \subseteq cl_\gamma^\mu(A) \cup cl_\gamma^\mu(B)$. From Lemma 3.18 (6), we obtain the equality. □

Lemma 3.20. *Let (X, μ) be an STS and γ be a supra regular operator on μ. Then $\mu_\gamma\text{-}cl^\mu(A) \cap U \subseteq \mu_\gamma\text{-}cl^\mu(A \cap U)$ holds $\forall U \in \mu_\gamma$ and $\forall A \subseteq X$.*

Proof. Suppose that $x \in \mu_\gamma\text{-}cl^\mu(A) \cap U \; \forall U \in \mu_\gamma$, then $x \in \mu_\gamma\text{-}cl^\mu(A)$ and $x \in U$. Let $V \in \mu_\gamma$ with $x \in V$. Since γ is supra regular on μ. So by Theorem 3.10, $U \cap V \in \mu_\gamma$ with $x \in U \cap V$. Since $x \in \mu_\gamma\text{-}cl^\mu(A)$, then by Theorem 3.17, we find that $A \cap (U \cap V) \neq \phi$. Therefore, $(A \cap U) \cap V \neq \phi$. Thus, by Theorem 3.17, we have that $x \in \mu_\gamma\text{-}cl^\mu(A \cap U)$. Hence, $\mu_\gamma\text{-}cl^\mu(A) \cap U \subseteq \mu_\gamma\text{-}cl^\mu(A \cap U)$. □

Theorem 3.21. *If $A \subseteq (X, \mu)$ and γ is an operator on μ, then the next four properties are equivalent:*
1. $A \in \mu_\gamma$.
2. $cl_\gamma^\mu(X \backslash A) = X \backslash A$.
3. $\mu_\gamma\text{-}cl^\mu(X \backslash A) = X \backslash A$.
4. *$X \backslash A$ is supra γ-closed.*

Definition 3.22. Let (X, μ) be any STS. An operator γ on μ is said to be supra open if $\forall x \in X$ and $\forall U \in \mu$ with $x \in U$, $\exists W \in \mu_\gamma$ with $x \in W$ and $W \subseteq \gamma(U)$.

Theorem 3.23. *Let A be any subset of an STS (X, μ). If γ is a supra open operator on μ, then*
1. $cl_\gamma^\mu(A) = \mu_\gamma\text{-}cl^\mu(A)$,
2. $cl_\gamma^\mu(cl_\gamma^\mu(A)) = cl_\gamma^\mu(A)$,
3. *$cl_\gamma^\mu(A)$ is supra γ-closed in X.*

Proof.

(1) First we need to show that μ_γ-$cl^\mu(A) \subseteq cl^\mu_\gamma(A)$. By Lemma 3.18 (2), we have $cl^\mu_\gamma(A) \subseteq \mu_\gamma$-$cl^\mu(A)$. Now let $x \notin cl^\mu_\gamma(A)$, then $\exists U \in \mu$ with $x \in U$ and $A \cap \gamma(U) = \phi$. Since γ is a supra open on μ, then $\exists W \in \mu$ with $x \in W$ and $W \subseteq \gamma(U)$. So $A \cap W = \phi$ and hence by Theorem 3.17, $x \notin \mu_\gamma$-$cl^\mu(A)$. Therefore, μ_γ-$cl^\mu(A) \subseteq cl^\mu_\gamma(A)$. Hence, $cl^\mu_\gamma(A) = \mu_\gamma$-$cl^\mu(A)$.

(2) By (1) and Lemma 3.18 (7), we have $cl^\mu_\gamma(cl^\mu_\gamma(A)) = cl^\mu_\gamma(A)$.

(3) By (2) and Lemma 3.18 (3b), we get $cl^\mu_\gamma(A)$ is supra γ-closed in X. □

4 μ_γ-g.closed sets and operator on subspace ST

Through this section, we present the concept of μ_γ-generalized closed and give some characterizations.

Definition 4.1. A subset A of an STS (X, μ) with an operator γ on μ is called μ_γ-generalized closed (briefly, μ_γ-g.closed) if $cl^\mu_\gamma(A) \subseteq U$ $\forall U \in \mu_\gamma$ satisfies that $A \subseteq U$.

Lemma 4.2. Let (X, μ) be an STS and γ be an operator on μ. A set A in (X, μ) is μ_γ-g.closed iff $A \cap \mu_\gamma$-$cl^\mu(\{x\}) \neq \phi$ $\forall$ $x \in cl^\mu_\gamma(A)$.

Proof. Let A be a μ_γ-g.closed set in X and suppose (if possible) that $\exists x \in cl^\mu_\gamma(A)$ such that $A \cap \mu_\gamma$-$cl^\mu(\{x\}) = \phi$. This follows that $A \subseteq X \backslash \mu_\gamma$-$cl^\mu(\{x\})$. Since μ_γ-$cl^\mu(\{x\})$ is supra γ-closed and hence $X \backslash \mu_\gamma$-$cl^\mu(\{x\}) \in \mu_\gamma$. Now, μ_γ-g.closedness of A in X implies that $cl^\mu_\gamma(A) \subseteq X \backslash \mu_\gamma$-$cl^\mu(\{x\})$. Therefore, $x \notin cl^\mu_\gamma(A)$, which is a contradiction. Thus, $A \cap \mu_\gamma$-$cl^\mu(\{x\}) \neq \phi$.

Conversely, let $U \in \mu_\gamma$ with $A \subseteq U$. To show that $cl^\mu_\gamma(A) \subseteq U$. Let $x \in cl^\mu_\gamma(A)$. Then by hypothesis, $A \cap \mu_\gamma$-$cl^\mu(\{x\}) \neq \phi$. So, $\exists y \in A \cap \mu_\gamma$-$cl^\mu(\{x\})$. Thus, $y \in A \subseteq U$ and $y \in \mu_\gamma$-$cl^\mu(\{x\})$. By Theorem 3.17, $\{x\} \cap U \neq \phi$. Therefore, $x \in U$. Thus, $cl^\mu_\gamma(A) \subseteq U$. Hence, A is μ_γ-g.closed. □

Theorem 4.3. *Let γ be an operator on μ. If A is μ_γ-g.closed subset of (X, μ), then $cl^\mu_\gamma(A) \backslash A$ does not contain any non-empty supra γ-closed set in (X, μ).*

Proof. Suppose that $F \neq \varnothing$ is a supra γ-closed set in X with $F \subseteq cl^\mu_\gamma(A) \backslash A$. Then $F \subseteq X \backslash A$. Obviously, $A \subseteq X \backslash F$. Since $X \backslash F \in \mu_\gamma$ and A is μ_γ-g.closed, then $cl^\mu_\gamma(A) \subseteq X \backslash F$. That is, $F \subseteq X \backslash cl^\mu_\gamma(A)$. Therefore, $F \subseteq X \backslash cl^\mu_\gamma(A) \cap cl^\mu_\gamma(A) \backslash A \subseteq X \backslash cl^\mu_\gamma(A) \cap cl^\mu_\gamma(A) = \phi$. Thus, $F = \phi$. But this is a contradiction. Hence, $F \nsubseteq cl^\mu_\gamma(A) \backslash A$. □

Theorem 4.4. *The converse of Theorem 4.3 is true when the operator $\gamma : \mu \to P(X)$ is supra open.*

Proof. Let $U \in \mu_\gamma$ with $A \subseteq U$. Since $\gamma : \mu \to P(X)$ is a supra open operator, then by Theorem 3.23 (3), $cl^\mu_\gamma(A)$ is supra γ-closed set in X. Hence, we have $cl^\mu_\gamma(A) \cap X \backslash U$ is a supra γ-closed set in (X, μ). Since $X \backslash U \subseteq X \backslash A$, $cl^\mu_\gamma(A) \cap X \backslash U \subseteq cl^\mu_\gamma(A) \backslash A$. By using the assumption of the converse of Theorem 4.3, $cl^\mu_\gamma(A) \subseteq U$. Thus, A is μ_γ-g.closed set in (X, μ). □

Corollary 4.5. *Let A be a μ_γ-g.closed subset of STS (X, μ) and let γ be an operator on μ. Then A is supra γ-closed iff $cl^\mu_\gamma(A) \backslash A$ is supra γ-closed set in (X, μ).*

Proof. (Necessity) Let A be a supra γ-closed set in (X, μ). It follows from Lemma 3.18 (3b) that $cl^\mu_\gamma(A) = A$ and hence $cl^\mu_\gamma(A) \backslash A = \phi$ which is supra γ-closed.

(Sufficiency) Suppose $cl^{\mu}_{\gamma}(A)\backslash A$ is supra γ-closed and A is μ_{γ}-g.closed. It follows from Theorem 4.3 that $cl^{\mu}_{\gamma}(A)\backslash A$ does not contain any non-empty supra γ-closed set in (X, μ). Since $cl^{\mu}_{\gamma}(A)\backslash A$ is supra γ-closed subset of itself, then $cl^{\mu}_{\gamma}(A)\backslash A = \phi$ implies $cl^{\mu}_{\gamma}(A) \cap X\backslash A = \phi$. Hence, $cl^{\mu}_{\gamma}(A) = A$. By Lemma 3.18 (3b), we obtain A is a supra γ-closed set in (X, μ). □

Theorem 4.6. *Let (X, μ) be an STS and γ be an operator on μ. If A is μ_{γ}-g.closed and supra γ-open subset of X, then A is supra γ-closed.*

Proof. Since A is μ_{γ}-g.closed and supra γ-open set in X, then $cl^{\mu}_{\gamma}(A) \subseteq A$ and hence by Lemma 3.18 (3b), A is supra γ-closed. □

Theorem 4.7. *For any STS (X, μ) with an operator γ on μ, $X\backslash\{x\}$ is μ_{γ}-g.closed or supra γ-open $\forall x \in X$.*

Proof. Let $X\backslash\{x\} \notin \mu_{\gamma}$. Then the only supra γ-open set containing $X\backslash\{x\}$ is X. Automatically, we have $cl^{\mu}_{\gamma}(X\backslash\{x\}) \subseteq X$. This ends the proof that $X\backslash\{x\}$ is a μ_{γ}-g.closed set in X. □

Corollary 4.8. *For any STS (X, μ) with an operator γ on μ, $\{x\}$ is a supra γ-closed set or $X\backslash\{x\}$ is a μ_{γ}-g.closed set $\forall x \in X$.*

Proof. Let $\{x\}$ be not supra γ-closed. Then $X\backslash\{x\}$ is not supra γ-open. Therefore, it follows from Theorem 4.7 that $X\backslash\{x\}$ is μ_{γ}-g.closed. □

Definition 4.9. Let $A \subseteq (X, \mu)$ and γ be an operator on μ. Then the μ_{γ}-kernel of A, denoted by μ_{γ}-ker(A), is defined as follows:

$$\mu_{\gamma}\text{-ker}(A) = \cap\{U : A \subseteq U \text{ and } U \in \mu_{\gamma}\},$$

i.e., μ_{γ}-ker(A) is the intersection of all supra γ-open sets of (X, μ) containing A.

Theorem 4.10. *Let $A \subseteq (X, \mu)$ and γ be an operator on μ. Then A is μ_{γ}-g.closed iff $cl^{\mu}_{\gamma}(A) \subseteq \mu_{\gamma}$-ker$(A)$.*

Proof. Suppose that A is μ_{γ}-g.closed. Then $cl^{\mu}_{\gamma}(A) \subseteq U$, whenever $A \subseteq U$ and $U \in \mu_{\gamma}$. Let $x \in cl^{\mu}_{\gamma}(A)$. Then by Lemma 4.2, $A \cap \mu_{\gamma}$-$cl^{\mu}(\{x\}) \neq \phi$. So $\exists$ a point $z \in X$ such that $z \in A \cap \mu_{\gamma}$-$cl^{\mu}(\{x\})$ implies that $z \in A \subseteq U$ and $z \in \mu_{\gamma}$-$cl^{\mu}(\{x\})$. By Theorem 3.17, $\{x\} \cap U \neq \phi$. Hence, we show that $x \in \mu_{\gamma}$-ker(A). Thus, $cl^{\mu}_{\gamma}(A) \subseteq \mu_{\gamma}$-ker$(A)$.

Conversely, let $cl^{\mu}_{\gamma}(A) \subseteq \mu_{\gamma}$-ker$(A)$. Let $U \in \mu_{\gamma}$ with $A \subseteq U$. Let x be a point in X such that $x \in cl^{\mu}_{\gamma}(A)$. Then $x \in \mu_{\gamma}$-ker(A). We have $x \in U$, because $A \subseteq U$ and $U \in \mu_{\gamma}$. That is, $cl^{\mu}_{\gamma}(A) \subseteq \mu_{\gamma}$-ker$(A) \subseteq U$. Thus, A is μ_{γ}-g.closed in X. □

Now we define an operator on subspace ST as follows:

Definition 4.11. Let (A, μ_A) be a subspace of an STS (X, μ) and $\gamma : \mu \to P(X)$ be an operator on μ. We define the restriction of γ to μ_A, denoted by γ_A, to be the mapping from μ_A into $P(X)$ such that $\forall U \in \mu_A$, $\gamma_A(U) = \gamma(V) \cap A$ for some $V \in \mu$ with $U = V \cap A$.

Lemma 4.12. *Let (A, μ_A) be a subspace of an STS (X, μ) and μ_A be the restriction of γ to μ_A. If $B \in \mu_{\gamma}$ in X, then $B \cap A$ is supra γ_A-open in A.*

Proof. Let $x \in B \cap A$. Since $B \in \mu_{\gamma}$ in X, then $\exists U \in \mu$ with $x \in U$ and $\gamma(U) \subseteq B$. So, $U \cap A$ is supra γ_A-open set with $x \in U \cap A$ and

$$\gamma_A(U \cap A) = \gamma(U) \cap A \subseteq B \cap A.$$

Thus, $B \cap A$ is supra γ_A-open in A. □

Lemma 4.13. *Let (A, μ_A) be a subspace of an STS (X, μ) and μ_A be the restriction of γ to μ_A. If the mapping γ is supra open and the set B is supra γ_A-open in A, then $\exists C \in \mu_\gamma$ with $B = C \cap A$.*

Proof. Since B is supra γ_A-open in A, then $\forall x \in B$, $\exists U_x \in \mu$ with $x \in U_x$ and

$$\gamma_A(U_x \cap A) = \gamma(U_x) \cap A \subseteq B.$$

Since γ is supra open, then $\exists W_x \in \mu_\gamma$ with $x \in W_x$ and $W_x \subseteq \gamma(U_x)$. Put $C = \bigcup_{x \in B} W_x$. So, $C \in \mu_\gamma$ in X and

$$B \subseteq C \cap A = \left(\bigcup_{x\in B} W_x\right) \cap A \subseteq \left(\bigcup_{x\in B} \gamma(U_x)\right) \cap A \subseteq \left(\bigcup_{x\in B} \gamma_A(U_x \cap A)\right) \subseteq B.$$

This completes the proof. □

Theorem 4.14. *Let (A, μ_A) be a subspace of an STS (X, μ) and $B \subseteq A \subseteq X$. If μ_A is the restriction of γ to μ_A, then $cl^{\mu}_{\gamma_A}(B) = cl^{\mu}_{\gamma}(B) \cap A$.*

Proof. Let $x \in cl^{\mu}_{\gamma_A}(B)$ and $U \in \mu$ with $x \in U$. Then $\gamma_A(U \cap A) \cap B = \gamma(U) \cap B \neq \phi$ and hence $x \in (cl^{\mu}_{\gamma}(B) \cap A)$. On the other hand, let $x \in (cl^{\mu}_{\gamma}(B) \cap A)$ and $V \in \mu_A$ with $x \in V$. Then $V = U \cap A$ for some $U \in \mu$ with $x \in U$. Since $x \in cl^{\mu}_{\gamma}(B)$,

$$\gamma_A(V) \cap B = (\gamma(U) \cap A) \cap B = \gamma(U) \cap B \neq \phi.$$

Thus, $x \in cl^{\mu}_{\gamma_A}(B)$. □

Theorem 4.15. *Let (A, μ_A) be a subspace of an STS (X, μ) and μ_A be the restriction of γ to μ_A. If the mapping γ is supra open, the set B is μ_{γ_A}-g closed in A and A is μ_γ-g closed in X, then B is μ_γ-g closed in X.*

Proof. Let $U \in \mu_\gamma$ in X with $B \subseteq U$. Then by Lemma 4.12, $U \cap A$ is supra γ_A-open in A and $B \subseteq U \cap A$. By hypothesis, $cl^{\mu}_{\gamma_A}(B) = cl^{\mu}_{\gamma}(B) \cap A \subseteq U \cap A$. Hence, $A \subseteq U \bigcup (X \backslash cl^{\mu}_{\gamma}(B))$. Since A is μ_γ-g.closed in X, γ is supra open and $X \backslash cl^{\mu}_{\gamma}(B) \in \mu_\gamma$. So $cl^{\mu}_{\gamma}(A) \subseteq U \bigcup (X \backslash cl^{\mu}_{\gamma}(B))$. Hence,

$$cl^{\mu}_{\gamma}(B) \subseteq cl^{\mu}_{\gamma}(A) \subseteq U \bigcup (X \backslash cl^{\mu}_{\gamma}(B)).$$

Thus, $cl^{\mu}_{\gamma}(B) \subseteq U$. Therefore, B is μ_γ-g.closed in X. □

Corollary 4.16. *If γ is supra open, A is μ_γ-g.closed in X and F is supra γ-closed in X, then $A \cap F$ is μ_γ-g.closed in X.*

Proof. By Lemma 4.12, $A \cap F$ is supra γ_A-closed in A and hence $A \cap F$ is μ_γ-g.closed in A. Thus, by Theorem 4.15, $A \cap F$ is μ_γ-g.closed in X. □

Theorem 4.17. *If γ is supra open, A is μ_γ-g.closed in X and $A \subseteq B \subseteq cl^{\mu}_{\gamma}(A)$, then B is μ_γ-g.closed in X.*

Proof. Since $cl^{\mu}_{\gamma}(B) \backslash B \subseteq cl^{\mu}_{\gamma}(A) \backslash A$ and $cl^{\mu}_{\gamma}(A) \backslash A$ has no non-empty supra γ-closed set in X, neither does $cl^{\mu}_{\gamma}(B) \backslash B$. So, by Theorem 4.4, B is μ_γ-g closed in X. □

Theorem 4.18. *Let (A, μ_A) be a subspace of an STS (X, μ), μ_A be the restriction of γ to μ_A and $B \subseteq A \subseteq X$. If γ is supra open, B is μ_γ-g.closed in X, then B is μ_{γ_A}-g.closed in A.*

Proof. Let U be supra γ_A-open in A and $B \subseteq U$. Then $\exists V \in \mu_\gamma$ in X with $U = A \cap V$ and hence $B \subseteq V$. Thus, by hypothesis, $cl^{\mu}_{\gamma}(B) \subseteq V$. By Theorem 4.14, $cl^{\mu}_{\gamma_A}(B) = cl^{\mu}_{\gamma}(B) \cap A \subseteq V \cap A = U$. Therefore, B is μ_{γ_A}-g closed in A. □

5 μ_γ-Separation axioms

In this section, we investigate some types of μ_γ-separation axioms. Some results and examples of these spaces are studied.

Definition 5.1. An STS (X, μ) with an operator γ on μ is said to be:

(i) μ_γ-T_0^* if $\forall x, y \in X$ with $x \neq y$, $\exists U \in \mu$ such that either $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.

(ii) μ_γ-T_0 if $\forall x, y \in X$ with $x \neq y$, $\exists U \in \mu_\gamma$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

(iii) μ_γ-T_1^* if $\forall x, y \in X$ with $x \neq y$, $\exists U, V \in \mu$ with $x \in U$ but $y \notin \gamma(U)$ and $y \in V$ but $x \notin \gamma(V)$.

(iv) μ_γ-T_1 if $\forall x, y \in X$ with $x \neq y$, $\exists U, V \in \mu_\gamma$ with $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

(v) μ_γ-T_2^* if $\forall x, y \in X$ with $x \neq y$, $\exists U, V \in \mu$ with $x \in U$, $y \in V$ and $\gamma(U) \cap \gamma(V) = \phi$.

(vi) μ_γ-T_2 if $\forall x, y \in X$ with $x \neq y$, $\exists U, V \in \mu_\gamma$ with $x \in U$, $y \in V$ and $U \cap V = \phi$.

(vii) μ_γ-$T_{\frac{1}{2}}^*$ if every μ_γ-g.closed set in X is supra γ-closed.

Theorem 5.2. *An STS (X, μ) with an operator γ on μ is μ_γ-$T_{\frac{1}{2}}^*$ iff the set $\{x\}$ is supra γ-closed or supra γ-open $\forall x \in X$.*

Proof. Suppose that $\{x\}$ is not supra γ-closed set in a μ_γ-$T_{\frac{1}{2}}^*$ space (X, μ). Then Corollary 4.8 implies that $X \backslash \{x\}$ is a μ_γ-g.closed set. Since (X, μ) is μ_γ-$T_{\frac{1}{2}}^*$, then $\{x\}$ is a supra γ-open set.

Conversely, let F be any μ_γ-g.closed set in the *STS* (X, μ). We have to show that F is supra γ-closed (i.e., $cl_\gamma^\mu(F) = F$ by Lemma 3.18 (3b)). It is sufficient to show that $cl_\gamma^\mu(F) \subseteq F$. Let $x \in cl_\gamma^\mu(F)$. By hypothesis $\{x\}$ is supra γ-closed or supra γ-open $\forall x \in X$. So there are two cases:

Case 1: If $\{x\}$ is supra γ-closed. Let $x \notin F$, then $x \in cl_\gamma^\mu(F) \backslash F$ contains a non-empty supra γ-closed set $\{x\}$. Since F is μ_γ-g.closed and according to Theorem 4.3, we obtain a contradiction. Hence, it must be that $x \in F$. This follows that $cl_\gamma^\mu(F) \subseteq F$ and so $cl_\gamma^\mu(F) = F$. Hence, by Lemma 3.18 (3b) F is supra γ-closed in (X, μ). Therefore, (X, μ) is μ_γ-$T_{\frac{1}{2}}^*$ space.

Case 2: If $\{x\}$ is supra γ-open. Then by Theorem 3.17, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $cl_\gamma^\mu(F) \subseteq F$. Thus, by Lemma 3.18 (3b), F is supra γ-closed. Thus, (X, μ) is μ_γ-$T_{\frac{1}{2}}^*$ space. □

Theorem 5.3. *Suppose that γ is a supra open operator on μ. An STS (X, μ) is a μ_γ-T_0^* iff $cl_\gamma^\mu(\{x\}) \neq cl_\gamma^\mu(\{y\})$, $\forall x, y \in X$ with $x \neq y$.*

Proof. (Necessity) Let $x, y \in X$ with $x \neq y$, where (X, μ) be a μ_γ-T_0^* space. Thus, $\exists U \in \mu_\gamma$ with $x \in U$ and $y \notin \gamma(U)$. Since γ is a supra open operator on μ, then $\exists W \in \mu_\gamma$ with $x \in W$ and $W \subseteq \gamma(U)$. So, $y \in X \backslash \gamma(U) \subseteq X \backslash W$. Since $X \backslash W$ is a supra γ-closed set in (X, μ).Therefore, we obtain that $cl_\gamma^\mu(\{y\}) \subseteq X \backslash W$ and hence $cl_\gamma^\mu(\{x\}) \neq cl_\gamma^\mu(\{y\})$.

(Sufficiency) Suppose that $cl_\gamma^\mu(\{x\}) \neq cl_\gamma^\mu(\{y\}) \forall x, y \in X$ with $x \neq y$. Now, we assume that $\exists z \in X$ such that $z \in cl_\gamma^\mu(\{x\})$, but $z \notin cl_\gamma^\mu(\{y\})$. If $x \in cl_\gamma^\mu(\{y\})$, then $\{x\} \subseteq cl_\gamma^\mu(\{y\})$, which implies that $cl_\gamma^\mu(\{x\}) \subseteq cl_\gamma^\mu(\{y\})$ (by Lemma 3.18 (4)). Therefore, $z \in cl_\gamma^\mu(\{y\})$. This contradiction shows that $x \notin cl_\gamma^\mu(\{y\})$. Thus, $\exists U \in \mu$ such that $x \in U$ and $\gamma(U) \cap \{y\} = \phi$ Hence, we obtain $x \in U$ and $y \notin \gamma(U)$. It gives that the *STS* (X, μ) is μ_γ-T_0^*. □

Theorem 5.4. *An STS (X, μ) is μ_γ-T_0 iff μ_γ-$cl^\mu(\{x\}) \neq \mu_\gamma$-$cl^\mu(\{y\})$, $\forall$ $x, y \in X$ with $x \neq y$.*

Proof. (Necessity) Let X be a μ_γ-T_0 space and $x, y \in X$ with $x \neq y$. Then $\exists U \in \mu_\gamma$ (say $x \in U$, but $y \notin U$). So $X \backslash U$ is a supra γ-closed set, which does not contain x, but contains y. Since μ_γ-$cl^\mu(\{y\})$ is the smallest supra γ-closed set containing y, μ_γ-$cl^\mu(\{y\}) \subseteq X \backslash U$, and so $x \notin \mu_\gamma$-$cl^\mu(\{y\})$. Therefore, μ_γ-$cl^\mu(\{x\}) \neq \mu_\gamma$-$cl^\mu(\{y\})$.

(Sufficiency) Let μ_γ-$cl^\mu(\{x\}) \neq \mu_\gamma$-$cl^\mu(\{y\})\ \forall x, y \in X$ with $x \neq y$. Now, let $z \in X$ such that $z \in \mu_\gamma$-$cl^\mu(\{x\})$, but $z \notin \mu_\gamma$-$cl^\mu(\{y\})$. Now, we claim that $x \in \mu_\gamma$-$cl^\mu(\{y\})$. For, if $x \in \mu_\gamma$-$cl^\mu(\{y\})$, then $\{x\} \subseteq \mu_\gamma$-$cl^\mu(\{y\})$, which implies that μ_γ-$cl^\mu(\{x\}) \subseteq \mu_\gamma$-$cl^\mu(\{y\})$. This is contradiction to the fact that $z \notin \mu_\gamma$-$cl^\mu(\{y\})$. Hence, x belongs to the supra γ-open set $X \backslash \mu_\gamma$-$cl^\mu(\{y\})$ to which y does not belong. It gives that X is μ_γ-T_0 space. □

From Theorem 5.3, Theorem 5.4 and the fact that $cl_\gamma^\mu(A) = \mu_\gamma$-$cl^\mu(A)\forall A \subseteq X$ holds under Theorem 3.23 (1) that γ is a supra open operator on μ, so we have the following corollary.

Corollary 5.5. *Suppose that γ is a supra open operator on μ. An STS (X, μ) is μ_γ-T_0^* iff (X, μ) is μ_γ-T_0.*

Theorem 5.6. *For an STS (X, μ) with an operator γ on μ. Then the following conditions are true:*
1. *(X, μ) is μ_γ-T_1^*.*
2. *The set $\{x\}$ is supra γ-closed $\forall x \in X$.*
3. *(X, μ) is μ_γ-T_1.*

Proof. (1) ⇒ (2) Let x be a point of a μ_γ-T_1^* space. Then $\forall y \in X\backslash\{x\}$, $\exists V_y \in \mu$ such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X\backslash\{x\}$. This implies that

$$X\backslash\{x\} = \bigcup\{\gamma(V_y) : y \in X\backslash\{x\}\}.$$

It is shown that $X\backslash\{x\} \in \mu_\gamma$. Hence, $\{x\}$ is supra γ-closed in (X, μ).

(2) ⇒ (3) Suppose every singleton set in X is supra γ-closed. Let $x, y \in X$ with $x \neq y$. Then by (2), the sets $X\backslash\{x\} \in \mu_\gamma$ and $X\backslash\{y\} \in \mu_\gamma$ with $y \in X\backslash\{x\}$ but $x \notin X\backslash\{x\}$ and $x \in X\backslash\{y\}$ but $y \notin X\backslash\{y\}$. Thus, (X, μ) is μ_γ-T_1.

(3) ⇒ (1) It is shown that if $x \in U$, where $U \in \mu_\gamma$, then $\exists V \in \mu$ with $x \in V \subseteq \gamma(V) \subseteq U$. Hence, by using (3), we have that (X, μ) is μ_γ-T_1^*. □

Theorem 5.7. *Let (X, μ) be an STS and γ be an operator on μ. Then the following statements are equivalent:*
1. *X is μ_γ-T_2.*
2. *If $x \in X$, then $\exists U \in \mu_\gamma$ with $x \in U$ and $y \notin \mu_\gamma$-$cl^\mu(U)\ \forall y \in X$ with $x \neq y$.*
3. *$\bigcap\{\mu_\gamma$-$cl^\mu(U) : U \in \mu_\gamma\} = \{x\}\forall x \in X$.*

Proof. (1) ⇒ (2) Let X be any μ_γ-T_2 space and $\forall x, y \in X$ with $x \neq y$, then $\exists U, V \in \mu_\gamma$ with $x \in U$, $y \in V$ and $U \cap V = \phi$. This implies that $U \subseteq X\backslash H$ and hence μ_γ-$cl^\mu(\{U\}) \subseteq X\backslash V$ since $X\backslash V$ is supra γ-closed in X and $y \notin X\backslash V$. Thus, $y \notin \mu_\gamma$-$cl^\mu(U)$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Let $x, y \in X$ with $x \neq y$. By (3), $\exists U \in \mu_\gamma$ with $x \in U$ and $y \notin \mu_\gamma$-$cl^\mu(U)$. Then $y \in X\backslash\mu_\gamma$-$cl^\mu(U)$ and $X\backslash\mu_\gamma$-$cl^\mu(U) \in \mu_\gamma$. Thus,

$$U \cap X\backslash\mu_\gamma\text{-}cl^\mu(U) = \phi.$$

Hence, X is μ_γ-T_2. □

Theorem 5.8. *For any STS (X, μ) and any operator γ on μ, the following properties hold.*
1. *Each μ_γ-T_j space is μ_γ-T_{j-1}, where $j \in \{2, 1\}$.*
2. *Each μ_γ-T_j space is μ_γ-T_j^*, where $j \in \{2, 0\}$.*
3. *Each μ_γ-T_2^* space is μ_γ-T_1^*.*
4. *Each μ_γ-T_1^* space is μ_γ-$T_{\frac{1}{2}}^*$.*
5. *Each μ_γ-$T_{\frac{1}{2}}^*$ space is μ_γ-T_0^*.*
6. *Each μ_γ-$T_{\frac{1}{2}}^*$ space is μ_γ-T_0.*
7. *Each μ_γ-T_j^* space is S-T_j, where $j \in \{2, 1, 0\}$.*

Proof. The proofs are obvious from their definitions and hence they are omitted. □

Observe that the converse of each part of Theorem 5.8 is not true as shown by the following examples.

Example 5.9. Suppose $X = \{a, b, c\}$ and $\mu = P(X)$. Define an operator γ on μ as follows: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} A & \text{if } c \in A; \\ X & \text{if } c \notin A. \end{cases}$$

Thus, the space (X, μ) is μ_γ-T_0, but (X, μ) is not μ_γ-T_1.

Example 5.10. Let $X = \{a, b, c\}$ and $\mu = P(X)$. Define an operator γ on μ as follows: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\}; \\ X & \text{otherwise.} \end{cases}$$

(i) Thus, STS (X, μ) is μ_γ-T_1 space, but (X, μ) is not μ_γ-T_2.
(ii) Thus, STS (X, μ) is μ_γ-T_1^* space, but (X, μ) is not μ_γ-T_2^*.

Example 5.11. The STS (X, μ) in Example 3.12 is both μ_γ-T_0 and μ_γ-T_0^*, but (X, μ) is not μ_γ-$T_{\frac{1}{2}}^*$.

Example 5.12. Let $X = \{a, b, c\}$ and $\mu = P(X)$. Let $\gamma : \mu \to P(X)$ be an operator on μ defined as follows: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\}; \\ X & \text{otherwise.} \end{cases}$$

Obviously, $\mu_\gamma = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Thus, the STS (X, μ) is μ_γ-$T_{\frac{1}{2}}^*$, but (X, μ) is not μ_γ-T_1^*.

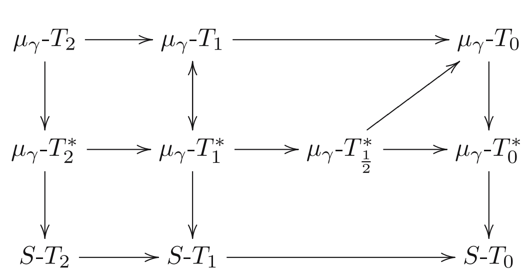
Example 5.13. Consider $X = \{a, b, c\}$ and $\mu = P(X)$. Define an operator γ on μ as follows: $\forall A \in \mu$

$$\gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{b\}; \\ \{b, c\} & \text{if } A = \{c\} \text{ or } \{b, c\}; \\ X & \text{otherwise.} \end{cases}$$

Thus, the STS (X, μ) is μ_γ-T_0^*, but (X, μ) is not μ_γ-T_0.

Example 5.14. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Define an operator $\gamma : \mu \to P(X)$ by $\gamma(A) = X$, $\forall A \in \mu$. Thus, the STS (X, μ) is S-T_j, but (X, μ) is not μ_γ-T_j^* for $j \in \{2, 1, 0\}$

Remark 5.15. By Theorem 5.6 and Theorem 5.8, we obtain the following diagram of implications.



6 $S_{(\gamma,\beta)}$-Continuous functions

Throughout this section and Section 7, let (X, μ) and (Y, ν) be STS and let $\gamma : \mu \to P(X)$ and $\beta : \nu \to P(Y)$ be operators on ST μ and ST ν, respectively.

Definition 6.1. A function $f : (X, \mu) \to (Y, \nu)$ is said to be $S_{(\gamma,\beta)}$-continuous if $\forall x \in X$ and $\forall V \in \nu$ in Y with $f(x) \in V$, $\exists U \in \mu$ in X with $x \in U$ and $f(\gamma(U)) \subseteq \beta(V)$.

Theorem 6.2. *Let $f : (X, \mu) \to (Y, \nu)$ be an $S_{(\gamma,\beta)}$-continuous function, then,*
1. $f(cl^{\mu}_{\gamma}(A)) \subseteq cl^{\nu}_{\beta}(f(A))$, $\forall A \subseteq (X, \mu)$.
2. $f^{-1}(F)$ is supra γ-closed set in (X, μ), $\forall$ supra β-closed set F of (Y, ν).

Proof.
(1) Let $y \in f(cl^{\mu}_{\gamma}(A))$ and $V \in \nu$ in Y with $y \in V$. Then by hypothesis, $\exists x \in X$ and $\exists U \in \mu$ in X with $x \in U$ and $f(x) = y$ and $f(\gamma(U)) \subseteq \beta(V)$. Since $x \in cl^{\mu}_{\gamma}(A)$, then $\gamma(U) \cap A \neq \phi$. Hence, $\phi \neq f(\gamma(U) \cap A) \subseteq f(\gamma(U)) \cap f(A) \subseteq \beta(V) \cap f(A)$. This implies that $y \in cl^{\nu}_{\beta}(f(A))$. Therefore, $f(cl^{\mu}_{\gamma}(A)) \subseteq cl^{\nu}_{\beta}(f(A))$.
(2) Let F be any supra β-closed set of (Y, ν). So, $f^{-1}(F) \subseteq (X, \mu)$. Then by using (1), we have

$$f(cl^{\mu}_{\gamma}(f^{-1}(F))) \subseteq cl^{\nu}_{\beta}(F) = F.$$

Hence, $cl^{\mu}_{\gamma}(f^{-1}(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is supra γ-closed in (X, μ). □

Theorem 6.3. *Items (1) and (2) in the aforementioned theorem are equivalent to each other if either the space (Y, ν) is supra β-regular or the operator β is supra open.*

Proof. The implications: "$S_{(\gamma,\beta)}$-continuity of f" $\Rightarrow$ (1) $\Rightarrow$ (2) follow from the proof of Theorem 6.2. Then, when the space (Y, ν) is supra β-regular, we prove the implication: (2) $\Rightarrow$ $S_{(\gamma,\beta)}$-continuity of f. Let $x \in X$ and let $V \in \nu$ in Y with $f(x) \in V$. Since (Y, ν) is a supra β-regular space, then by Theorem 3.14, $V \in \nu_{\beta}$ in Y. By using (2) of Theorem 6.2, $f^{-1}(V) \in \mu_{\gamma}$ in X with $x \in f^{-1}(V)$. So $\exists U \in \mu$ in X with $x \in U$ and $\gamma(U) \subseteq f^{-1}(V)$. This implies that $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, f is $S_{(\gamma,\beta)}$-continuous.

Now, when β is a supra open operator, we show the implication: (2) $\Rightarrow$ $S_{(\gamma,\beta)}$-continuity of f. Let $x \in X$ and let $V \in \nu$ in Y with $f(x) \in V$. Since β is a supra open operator, then $\exists W \in \nu_{\beta}$ in Y with $f(x) \in W$ and $W \subseteq \beta(V)$. By using (2) of Theorem 6.2, $f^{-1}(W) \in \mu_{\gamma}$ in X with $x \in f^{-1}(W)$. So $\exists U \in \mu$ in X with $x \in U$ and $\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))$. This implies that $f(\gamma(U)) \subseteq \beta(V)$. Hence, f is $S_{(\gamma,\beta)}$-continuous. □

Definition 6.4. A function $f : (X, \mu) \to (Y, \nu)$ is said to be
1. ν_{β}-closed if the image of each supra γ-closed set of X is supra β-closed in Y.
2. $S_{(id,\beta)}$-closed if the image of each supra closed set of X is supra β-closed in Y.

Theorem 6.5. *Suppose that a function $f : (X, \mu) \to (Y, \nu)$ is both $S_{(\gamma,\beta)}$-continuous and $S_{(id,\beta)}$-closed, then:*
1. *The image $f(A)$ is ν_{β}-g closed in (Y, ν), $\forall \mu_{\gamma}$-g closed set A of (X, μ).*
2. *The inverse set $f^{-1}(B)$ is μ_{γ}-g closed in (X, μ), $\forall \nu_{\beta}$-g closed set B of (Y, ν).*

Proof.
(1) Let $U \in \nu_{\beta}$ with $f(A) \subseteq U$. Since f is $S_{(\gamma,\beta)}$-continuous function, then by using Theorem 6.2 (2), $f^{-1}(U) \in \mu_{\gamma}$ in X. Since A is μ_{γ}-g closed and $A \subseteq f^{-1}(U)$, then we have $cl^{\mu}_{\gamma}(A) \subseteq f^{-1}(U)$, and hence $f(cl^{\mu}_{\gamma}(A)) \subseteq U$. Thus, by Lemma 3.18 (1), $cl^{\mu}_{\gamma}(A)$ is a supra closed set and since f is an $S_{(id,\beta)}$-closed, then $f(cl^{\mu}_{\gamma}(A))$ is supra β-closed set in Y. Therefore, $cl^{\nu}_{\beta}(f(A)) \subseteq cl^{\nu}_{\beta}(f(cl^{\mu}_{\gamma}(A))) = f(cl^{\mu}_{\gamma}(A)) \subseteq U$. This implies that $f(A)$ is ν_{β}-g.closed in (Y, ν).

(2) Let $V \in \mu_\gamma$ in X with $f^{-1}(B) \subseteq V$. Let $C = cl^{\mu}_{\gamma}(f^{-1}(B)) \cap (X\backslash V)$, then by Lemma 3.18 (1), C is a supra closed set in (X, μ). Since f is the $S_{(id,\beta)}$-closed function, then $f(C)$ is supra β-closed in (Y, ν). Since f is the $S_{(\gamma,\beta)}$-continuous function, then by using Theorem 6.2 (1), we have $f(C) = f(cl^{\mu}_{\gamma}(f^{-1}(B)))\cap f(X\backslash V) \subseteq cl^{\nu}_{\beta}(B) \cap f(X\backslash V) \subseteq cl^{\nu}_{\beta}(B) \cap (Y\backslash B) = cl^{\nu}_{\beta}(B)\backslash B$. This implies from Theorem 4.3 that $f(C) = \phi$, and hence $C = \phi$. So, $cl^{\mu}_{\gamma}(f^{-1}(B)) \subseteq V$. Thus, $f^{-1}(B)$ is μ_γ-g closed in (X, μ). □

Theorem 6.6. *Let $f : (X, \mu) \to (Y, \nu)$ be a surjective, $S_{(\gamma,\beta)}$-continuous and $S_{(id,\beta)}$-closed function. If (X, μ) is μ_γ-$T^*_{\frac{1}{2}}$, then (Y, ν) is ν_β-$T^*_{\frac{1}{2}}$.*

Proof. Let V be a ν_β-g closed set of (Y, ν). Since f is $S_{(\gamma,\beta)}$-continuous and $S_{(id,\beta)}$-closed function. Then by Theorem 6.5 (2), $f^{-1}(V)$ is μ_γ-g.closed in (X, μ). Since (X, μ) is μ_γ-$T^*_{\frac{1}{2}}$, then we have $f^{-1}(V)$ is supra γ-closed set in X. Again, since f is the $S_{(id,\beta)}$-closed function, then $f(f^{-1}(V))$ is supra β-closed in Y. Therefore, V is supra β-closed in Y since f is surjective. Hence, (Y, ν) is ν_β-$T^*_{\frac{1}{2}}$ space. □

Theorem 6.7. *Let $f : (X, \mu) \to (Y, \nu)$ be an injective, $S_{(\gamma,\beta)}$-continuous and $S_{(id,\beta)}$-closed function. If (Y, ν) is ν_β-$T^*_{\frac{1}{2}}$ space, then (X, μ) is μ_γ-$T^*_{\frac{1}{2}}$ space.*

Proof. Let U be any μ_γ-g closed set of (X, μ). Since f is $S_{(\gamma,\beta)}$-continuous and $S_{(id,\beta)}$-closed function. Then by Theorem 6.5 (1), $f(U)$ is ν_β-g closed in (Y, ν). Since (Y, ν) is ν_β-$T^*_{\frac{1}{2}}$, then $f(U)$ is supra β-closed in Y. Again, since f is $S_{(\gamma,\beta)}$-continuous, so by Theorem 6.2 (2), $f^{-1}(f(U))$ is supra γ-closed in (X, μ). Thus, U is supra γ-closed in (X, μ) because f is injective. Thus, the space (X, μ) is μ_γ-$T^*_{\frac{1}{2}}$. □

Theorem 6.8. *If a function $f : (X, \mu) \to (Y, \nu)$ is injective $S_{(\gamma,\beta)}$-continuous and the space (Y, ν) is ν_β-T^*_2, then the STS (X, μ) is μ_γ-T^*_2.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is an injective function and (Y, ν) is a ν_β-T^*_2 space. Then $\exists U_1 \in \nu$ and $U_2 \in \nu$ in Y with $f(x_1) \in U_1$, $f(x_2) \in U_2$ and $\beta(U_1) \cap \beta(U_2) = \phi$. Since f is $S_{(\gamma,\beta)}$-continuous, $\exists V_1 \in \mu$ and $V_2 \in \mu$ in X with $x_1 \in V_1$, $x_2 \in V_2$, $f(\gamma(V_1)) \subseteq \beta(U_1)$ and $f(\gamma(V_2)) \subseteq \beta(U_2)$. Hence, $\beta(U_1) \cap \beta(U_2) = \phi$. Thus, (X, μ) is μ_γ-T^*_2. □

Theorem 6.9. *If a function $f : (X, \mu) \to (Y, \nu)$ is injective $S_{(\gamma,\beta)}$-continuous and the space (Y, ν) is ν_β-T^*_j, then the space (X, μ) is μ_γ-T^*_j for $j \in \{0, 1\}$.*

Proof. The proof is similar to Theorem 6.8. □

Definition 6.10. A function $f : (X, \mu) \to (Y, \nu)$ is said to be $S_{(\gamma,\beta)}$-homeomorphism if f is bijective, $S_{(\gamma,\beta)}$-continuous and f^{-1} is $\mu_{(\beta,\gamma)}$-continuous.

Theorem 6.11. *Suppose that a function $f : (X, \mu) \to (Y, \nu)$ is an $S_{(\gamma,\beta)}$-homeomorphism. If (X, μ) is μ_γ-$T^*_{\frac{1}{2}}$, then (Y, ν) is ν_β-$T^*_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y, ν). Then $\exists x \in X$ with $y = f(x)$. Then by hypothesis and Theorem 5.2, we get $\{x\}$ is supra γ-closed or supra γ-open in (X, μ). By using Theorem 6.2, we obtain $\{y\}$ is supra β-closed or supra β-open. Thus, by Theorem 5.2, the space (Y, ν) is ν_β-$T^*_{\frac{1}{2}}$. □

7 $S_{(\gamma,\beta)}$-closed graphs and strongly $S_{(\gamma,\beta)}$-closed graphs

In this section, we further investigate general operator approaches of closed graphs of functions. Let $(X \times Y, \tau \times \sigma)$ be the product space of the STS (X, μ) and (Y, ν), and let $\rho : \mu \times \nu \to P(X \times Y)$ be an operator on $\mu \times \nu$.

Definition 7.1. The graph $G(f)$ of a function $f:(X,\mu)\to(Y,\nu)$ is called $S_{(\gamma,\beta)}$-closed if $\forall(x,y)\in(X\times Y)\backslash G(f)$, $\exists U\in\mu$ in X and $V\in\nu$ in Y with $x\in U$, $y\in V$ and $(\gamma(U)\times\beta(V))\cap G(f)=\phi$.

Lemma 7.2. *A function $f:(X,\mu)\to(Y,\nu)$ has $S_{(\gamma,\beta)}$-closed graph iff $\forall(x,y)\in(X\times Y)\backslash G(f)$, $\exists U\in\mu$ in X and $V\in\nu$ in Y with $x\in U$, $y\in V$ and $f(\gamma(U))\cap\beta(V)=\phi$.*

Proof. The proof is directly from the above definition. □

Theorem 7.3. *If $f:(X,\mu)\to(Y,\nu)$ is an $S_{(\gamma,\beta)}$-continuous function and (Y,ν) is a ν_β-T_2^* space, then f has an $S_{(\gamma,\beta)}$-closed graph.*

Proof. Let $(x,y)\in(X\times Y)\backslash G(f)$. Then $y\neq f(x)$, and since (Y,ν) is ν_β-T_2^*, $\exists U,V\in\nu$ in Y with $f(x)\in U$, $y\in V$ and $\beta(U)\cap\beta(V)=\phi$. Since f is $S_{(\gamma,\beta)}$-continuous, then $\exists W\in\mu$ in X with $x\in W$ and $f(\gamma(W))\subseteq\beta(U)$. Thus, $f(\gamma(W))\cap\beta(V)=\phi$. Therefore, by using Lemma 7.2, f has an $S_{(\gamma,\beta)}$-closed graph. □

Theorem 7.4. *If $f:(X,\mu)\to(Y,\nu)$ is an $S_{(\gamma,\beta)}$-continuous injective function with an $S_{(\gamma,\beta)}$-closed graph, then (X,μ) is a μ_γ-T_2^* space.*

Proof. Let $x_1,x_2\in X$ with $x_1\neq x_2$. Then $f(x_1)\neq f(x_2)$. This implies that $(x_1,f(x_2))\in(X\times Y)\backslash G(f)$. Since f has an $S_{(\gamma,\beta)}$-closed graph, then by using Lemma 7.2, $\exists U\in\mu$ in X and $V\in\nu$ in Y with $x_1\in U$, $f(x_2)\in V$ and $f(\gamma(U))\cap\beta(V)=\phi$. Since f is $S_{(\gamma,\beta)}$-continuous, then $\exists W\in\mu$ in X with $x_2\in W$ and $f(\gamma(W))\subseteq\beta(V)$. Thus, $f(\gamma(U))\cap f(\gamma(W))=\phi$. Therefore, $\gamma(U)\cap\gamma(W)=\phi$. Hence, (X,μ) is μ_γ-T_2^*. □

Definition 7.5. The graph $G(f)$ of a function $f:(X,\mu)\to(Y,\nu)$ is called strongly $S_{(\gamma,\beta)}$-closed if $\forall(x,y)\in(X\times Y)\backslash G(f)$, $\exists U\in\mu_\gamma$ in X and $V\in\nu_\beta$ in Y with $x\in U$, $y\in V$ and $(U\times V)\cap G(f)=\phi$.

Lemma 7.6. *A function $f:(X,\mu)\to(Y,\nu)$ has strongly $S_{(\gamma,\beta)}$-closed graph iff $\forall(x,y)\in(X\times Y)\backslash G(f)$, $\exists U\in\mu_\gamma$ in X and $V\in\nu_\beta$ in Y with $x\in U$, $y\in V$ and $f(U)\cap V=\phi$.*

Proof. Obvious. □

Definition 7.7. A function $f:(X,\mu)\to(Y,\nu)$ is said to be $S_{(\gamma,\beta)}$-irresolute if $\forall x\in X$ and $\forall V\in\nu_\beta$ with $f(x)\in V$, $\exists U\in\mu_\gamma$ with $x\in U$ and $f(U)\subseteq V$.

Theorem 7.8. *If $f:(X,\mu)\to(Y,\nu)$ is an $S_{(\gamma,\beta)}$-irresolute function and (Y,ν) is a ν_β-T_2 space, then f has a strongly $S_{(\gamma,\beta)}$-closed graph.*

Proof. Let $(x,y)\in(X\times Y)\backslash G(f)$. Then $y\neq f(x)$, and since (Y,ν) is ν_β-T_2, $\exists\, U,V\in\nu_\beta$ in Y with $f(x)\in U$, $y\in V$ and $\beta(U)\cap V=\phi$. Since f is $S_{(\gamma,\beta)}$-irresolute, then $\exists W\in\mu_\gamma$ in X with $x\in W$ and $f(W)\subseteq U$. Thus, $f(W)\cap V=\phi$. Therefore, by using Lemma 7.6, f has a strongly $S_{(\gamma,\beta)}$-closed graph. □

Theorem 7.9. *If $f:(X,\mu)\to(Y,\nu)$ is an $S_{(\gamma,\beta)}$-irresolute injective function with a strongly $S_{(\gamma,\beta)}$-closed graph, then (X,μ) is a μ_γ-T_2 space.*

Proof. Let $x_1,x_2\in X$ with $x_1\neq x_2$. Then $f(x_1)\neq f(x_2)$. This implies that $(x_1,f(x_2))\in(X\times Y)\backslash G(f)$. Since f has a strongly $S_{(\gamma,\beta)}$-closed graph, then by using Lemma 7.6, $\exists U\in\mu_\gamma$ in X and $V\in\nu_\beta$ in Y with $x_1\in U$, $f(x_2)\in V$ and $f(U)\cap V=\phi$. Since f is $S_{(\gamma,\beta)}$-irresolute, then $\exists W\in\mu_\gamma$ in X with $x_2\in W$ and $f(W)\subseteq V$. Thus, $f(U)\cap f(W)=\phi$. Therefore, $U\cap W=\phi$. Hence, (X,μ) is μ_γ-T_2. □

Lemma 7.10. *Suppose $\gamma : \mu \to P(X)$ and $\beta : \nu \to P(Y)$ are operators on μ and ν respectively. Then we have*
1. *If $f : (X, \mu) \to (Y, \nu)$ has a strongly $S_{(\gamma,\beta)}$-closed graph, then it has an $S_{(\gamma,\beta)}$-closed graph.*
2. *If $f : (X, \mu) \to (Y, \nu)$ has an $S_{(\gamma,\beta)}$-closed graph, then it has an S-closed graph.*

Proof.
(1) Let $x \in X$, $y \in Y$ with $f(x) \neq y$. Since f has a strongly $S_{(\gamma,\beta)}$-closed graph, then by Lemma 7.6, $\exists U \in \mu_\gamma$ in X and $V \in \nu_\beta$ in Y with $x \in U$, $f(y) \in V$ and $f(U) \cap V = \phi$. Thus, $\exists E \in \mu$ in X with $x \in E \subseteq \gamma(E) \subseteq U$, and $F \in \nu$ in Y with $y \in F \subseteq \beta(F) \subseteq V$. Therefore, $f(\gamma(E)) \cap \beta(F) = \phi$. Hence, by Lemma 7.2, f has an $S_{(\gamma,\beta)}$-closed graph.
(2) Let $x \in X$, $y \in Y$ with $f(x) \neq y$. Since f has an $S_{(\gamma,\beta)}$-closed graph, then by Lemma 7.2, $\exists U \in \mu$ in X and $V \in \nu$ in Y with $x \in U$, $f(y) \in V$ and $f(\gamma(U)) \cap \beta(V) = \phi$. Thus, $f(U) \cap V = \phi$. Hence, f has an S-closed graph. □

Remark 7.11. From Lemma 7.10, we obtain the following diagram.

$$\text{strongly } S_{(\gamma,\beta)}\text{-closed graph} \to S_{(\gamma,\beta)}\text{-closed graph} \to S\text{-closed graph}$$

Theorem 7.12. *Suppose that γ and β are operators on μ and ν respectively. If STS (X, μ) and STS (Y, ν) are supra γ-regular and supra β-regular spaces, then the following are equivalent for any function $f : (X, \mu) \to (Y, \nu)$:*
1. *f has strongly $S_{(\gamma,\beta)}$-closed graph.*
2. *f has $S_{(\gamma,\beta)}$-closed graph.*
3. *f has S-closed graph.*

Proof. Follows directly from Theorem 3.14 and the above diagram. □

Definition 7.13. An operator $\rho : \mu \times \nu \to P(X \times Y)$ is said to be supra associated with γ and β if $\rho(U \times V) = \gamma(U) \times \beta(V)$ holds $\forall U \in \mu$ and $\forall\ V \in \nu$.

Definition 7.14. The operator $\rho : \mu \times \nu \to P(X \times Y)$ is said to be supra regular with respect to γ and β if $\forall (x, y) \in X \times Y$ and $\forall W \in (\mu \times \nu)$ in $X \times Y$ with $(x, y) \in W$, $\exists U \in \mu$ in X and $V \in \nu$ in Y with $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \rho(W)$.

Theorem 7.15. *Let $\rho : \mu \times \mu \to P(X \times X)$ be a supra associated operator with γ and γ. If $f : (X, \mu) \to (Y, \nu)$ is an $S_{(\gamma,\beta)}$-continuous function and (Y, ν) is a ν_β-T_2^* space, then the set $A = \{(x, y) \in X \times X : f(x) = f(y)\}$ is supra ρ-closed of $(X \times X, \mu \times \mu)$.*

Proof. We have to show that $cl_\rho^{\mu\times\mu}(A) \subseteq A$. Let $(x, y) \in (X \times X)\backslash A$. Since (Y, ν) is ν_β-T_2^*. Then $\exists U, V \in \nu$ in Y with $f(x) \in U$, $f(y) \in V$ and $\beta(U) \cap \beta(V) = \phi$. Furthermore, for U and V, $\exists G, H \in \mu$ in X with $x \in G$, $y \in H$ and $f(\gamma(G)) \subseteq \beta(U)$ and $f(\gamma(H)) \subseteq \beta(V)$ because f is $S_{(\gamma,\beta)}$-continuous. Thus, we obtain $(x, y) \in \gamma(G) \times \gamma(H) = \rho(G \times H) \cap A = \phi$ since $G \times H \in \mu \times \mu$. This gives that $(x, y) \notin cl_\rho^{\mu\times\mu}(A)$. Hence, the proof is complete. □

Corollary 7.16. *Suppose $\rho : \mu \times \mu \to P(X \times X)$ is supra associated operator with γ and γ, and it is supra regular with γ and γ. An STS (X, μ) is μ_γ-T_2^* iff the diagonal set $\Delta = \{(x, x) : x \in X\}$ is supra ρ-closed of $(X \times X, \mu \times \mu)$.*

Theorem 7.17. *Let $\rho : \mu \times \nu \to P(X \times Y)$ be a supra associated operator with γ and β. If $f : (X, \mu) \to (Y, \nu)$ is $S_{(\gamma,\beta)}$-continuous and (Y, ν) is ν_β-T_2^*, then the graph of f, $G(f) = \{(x, f(x)) \in X \times Y\}$ is a supra ρ-closed set of $(X \times Y, \mu \times \nu)$.*

Proof. Similar to Theorem 7.15. □

Definition 7.18. Let (X, μ) be an STS and γ be an operator on μ. A subset S of X is said to be μ_γ-compact if $\forall$ supra open cover $\{U_i, i \in \mathbb{N}\}$ of S, $\exists$ a finite subfamily $\{U_1, U_2,\ldots, U_n\}$ with $S \subseteq \gamma(U_1) \cup \gamma(U_2) \cup \ldots \cup \gamma(U_n)$.

Theorem 7.19. *Suppose that γ is supra regular and $\rho : \mu \times \nu \to P(X \times Y)$ is supra regular with respect to γ and β. Let $f : (X, \mu) \to (Y, \nu)$ be a function whose graph $G(f)$ is supra ρ-closed in $(X \times Y, \mu \times \nu)$. If a subset S is $S_{(\gamma,\beta)}$-compact in (Y, ν), then $f^{-1}(S)$ is supra γ-closed in (X, μ).*

Proof. Suppose that $f^{-1}(S)$ is not supra γ-closed, then $\exists$ a point x with $x \in cl^\mu_\gamma(f^{-1}(S))$ and $x \notin f^{-1}(S)$. Since $(x, s) \notin G(f)$ and $\forall s \in S$ and $cl^{\mu\times\mu}_\rho(G(f)) \subseteq G(f)$, $\exists W \in (\mu \times \nu)$ in $X \times Y$ with $(x, s) \in W$ and $\beta(W) \cap G(f) = \phi$. By supra regularity of ρ, $\forall s \in S$ we can take $U(s) \in \nu$ and $V(s) \in \nu$ in Y with $x \in U(s)$, $s \in V(s)$ and $\gamma(U(s)) \times \beta(V(s)) \subseteq \rho(W)$. Then we have $f(\gamma(U(s))) \cap \beta(V(s)) = \phi$. Since $\{V(s) : s \in S\}$ is supra open cover of S, then by μ_γ-compactness $\exists$ a finite number $s_1, s_2,\ldots, s_n \in S$ with $S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup \ldots \cup \beta(V(s_n))$. By the supra regularity of γ, $\exists$ a supra open set $U \in \mu$ in X with $x \in U$ and $\gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap \ldots \cap \gamma(U(s_n))$. Therefore, we have $\gamma(U) \cap f^{-1}(S) \subseteq U(s_i) \cap f^{-1}(\beta(V(s_i))) = \phi$. This shows that $x \notin cl^\mu_\gamma(f^{-1}(S))$. This is a contradiction. Thus, $f^{-1}(S)$ is supra γ-closed in X. □

8 Conclusion

In this paper, the notion of an operator γ on supra open sets has been studied and the notion of supra γ-open sets of an STS (X, μ) has been defined. The notions of $\mu\gamma$-g closed sets and operator on subspace ST have been presented and investigated. New $\mu\gamma$-separation axioms have been introduced and explored. Some characterizations of $S(\gamma,\beta)$-continuous functions have been studied and some properties of $S(\gamma,\beta)$-closed graph and strongly $S(\gamma,\beta)$-closed graph have been given. Several examples have been exhibited to validate the discussed results.

In the upcoming works, we plan to study these concepts on the contents of supra soft topological spaces [35] and binary STS [36].

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References

[1] S. Kasahara, *Operation compact spaces*, Math. Japonica **24** (1979), no. 1, 97–105.
[2] D. S. Jankovic, *On functions with α-closed graphs*, Glas. Mat. **18** (1983), no. 38, 141–148.
[3] H. Ogata, *Operation on topological spaces and associated topology*, Math. Japonica **36** (1991), no. 1, 175–184.
[4] E. Rosas and J. Vielma, *Operator-compact and operator-connected spaces*, Sci. Math. **1** (1998), no. 2, 203–208.
[5] N. Kalaivani and G. S. S. Krishnan, *Operation approaches on α-γ-open sets in topological spaces*, Int. J. Math. Anal. **7** (2013), no. 10, 491–498.

[6] N. Ahmad and B. A. Asaad, *More properties of an operation on semi-generalized open sets*, Ital. J. Pure Appl. Math. **39** (2018), 608–627.

[7] T. V. An, D. X. Cuong and H. Maki, *On operation-preopen sets in topological spaces*, Sci. Math. Jpn. **68** (2008), no. 1, 11–30, (e-2008), 241–260.

[8] B. A. Asaad, *Some applications of generalized open sets via operations*, New Trends Math. Sci. **5** (2017), no. 1, 145–157.

[9] B. A. Asaad and N. Ahmad, *Operation on semi generalized open sets with its separation axioms*, AIP Conf. Proc. **1905** (2017), 020001, DOI: 10.1063/1.5012141.

[10] B. A. Asaad and Z. A. Ameen, *Some properties of an operation on gα-open sets*, New Trends Math. Sci. **7** (2019), no. 2, 150–158.

[11] C. Carpintero, N. Rajesh and E. Rosas, *Operation approaches on b-open sets and applications*, Bol. Soc. Paran. Mat. **30** (2012), no. 1, 21–33.

[12] N. Kalaivani, A. I. El-Maghrabi, D. Saravanakumar, and G. Sai Sundara Krishnan, *Operation-compact spaces, regular spaces and normal spaces with α-γ-open sets in topological spaces*, J. Interdiscip. Math. **20** (2017), no. 2, 427–441.

[13] G. S. S. Krishnan, M. Ganster and K. Balachandran, *Operation approaches on semi-open sets and applications*, Kochi. J. Math. **2** (2007), 21–33.

[14] S. Tahiliani, *Operation approach to β-open sets and applications*, Math. Commun. **16** (2011), 577–591.

[15] P. L. Powar, B. A. Asaad, K. Rajak, and R. Kushwaha, *Operation on fine topology*, Eur. J. Pure Appl. **12** (2019), no. 3, 960–977.

[16] A. S. Mashhour, A. A. Allam, F. S. Mahmoud, and F. H. Khedr, *On supra topological spaces*, Indian J. Pure Appl. Math. **14** (1983), no. 4, 502–510.

[17] A. M. Kozae, M. Shokry and M. Zidan, *Supra topologies for digital plane*, AASCIT Commun. **3** (2016), no. 1, 1–10.

[18] T. M. Al-shami, *Some results related to supra topological spaces*, J. Adv. Stud. Topol. **7** (2016), no. 4, 283–294.

[19] T. M. Al-shami, *Utilizing supra α-open sets to generate new types of supra compact and supra Lindelöf spaces*, Facta Univ. Ser. Math. Inform. **32** (2017), no. 1, 151–162.

[20] T. M. Al-shami, *Supra semi-compactness via supra topological spaces*, J. Taibah Univ. Sci. **12** (2018), no. 3, 338–343.

[21] T. M. Al-shami, B. A. Asaad and M. A. El-Gayar, *Various types of supra pre-compact and supra pre-Lindelöf spaces*, Missouri J. Math. Sci. **32** (2020), no. 1, 1–20.

[22] J. M. Mustafa, *Supra b-compact and supra b-Lindelöf spaces*, J. Math. Appl. **36** (2013), 79–83.

[23] T. M. Al-shami, *Paracompactness on supra topological spaces*, J. Linear. Topolog. Algebra. **9** (2020), no. 2, 1–7.

[24] T. M. Al-shami, E. A. Abo-Tabl and B. A. Asaad, *Investigation of limit points and separation axioms using supra β-open sets*, Missouri J. Math. Sci. **32** (2020), no. 2, 171–187.

[25] T. M. Al-shami, E. A. Abo-Tabl, B. A. Asaad, and M. A. Arahet, *Limit points and separation axioms with respect to supra semi-open sets*, Eur. J. Pure Appl. **13** (2020), no. 3, 427–443.

[26] T. M. Al-shami, B. A. Asaad and M. K. EL-Bably, *Weak types of limit points and separation axioms on supra topological spaces*, Adv. Math. Sci. J. **9** (2020), no. 10, 8017–8036.

[27] M. E. El-Shafei, A. H. Zakari and T. M. Al-shami, *Some applications of supra preopen sets*, J. Math. **2020** (2020), 9634206.

[28] S. Jafari and S. Tahiliani, *Supra β-open sets and supra β-continuity on topological spaces*, Annales Univ. Sci. Budapest **56** (2013), 1–9.

[29] T. M. Al-shami, *On supra semi open sets and some applications on topological spaces*, J. Adv. Stud. Topol. **8** (2017), no. 2, 144–153.

[30] R. Devi, S. Sampathkumar and M. Caldas, *On α-open sets and Sα-continuous maps*, General Math. **16** (2008), 77–84.

[31] O. R. Sayed, *Supra pre-open sets and supra pre-continuous on topological spaces*, Ser. Math. Inform. **20** (2010), 79–88.

[32] O. R. Sayed and T. Noiri, *On supra b-open sets and supra b-continuity on topological spaces*, Eur. J. Pure Appl. **3** (2010), 295–302.

[33] J. M. Mustafa, *Totally supra b-continuous and slightly supra b-continuous functions*, Stud. Univ. Babeş-Bolyai Math. **57** (2012), no. 1, 135–144.

[34] J. M. Mustafa and H. A. Qoqazeh, *Supra D-sets and associated separation axioms*, Int. J. Pure Appl. Math. **80** (2012), 657–663.

[35] T. M. Al-shami and M. E. El-Shafei, *Two new types of separation axioms on supra soft separation spaces*, Demonstr. Math. **52** (2019), no. 1, 147–165.

[36] M. L. Thivagar and J. Kavitha, *On binary structure of supra topological spaces*, Bol. Soc. Paran. Mat. **35** (2017), no. 3, 25–37.