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Research Article

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An iterative algorithm for the system of split mixed equilibrium problem

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Abstract: In this article, a new problem that is called system of split mixed equilibrium problems is introduced. This problem is more general than many other equilibrium problems such as problems of system of equilibrium, system of split equilibrium, split mixed equilibrium, and system of split variational inequality. A new iterative algorithm is proposed, and it is shown that it satisfies the weak convergence conditions for nonexpansive mappings in real Hilbert spaces. Also, an application to system of split variational inequality problems and a numeric example are given to show the efficiency of the results. Finally, we compare its rate of convergence other algorithms and show that the proposed method converges faster.

Keywords: split problem, equilibrium problem, nonexpansive mappings, fixed point

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1 Introduction

Let H be a real Hilbert space and C be a nonempty, closed, and convex subset of H. For $i = 1, 2,\ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a family of bifunctions such that $F_i(x, x) = 0$ for $i = 1, 2,\ldots, N$. We define the following problems:

1. The equilibrium problem is to find $x^* \in C$ such that $F_1(x^*, x) \geq 0$ for all $x \in C$.
2. The system of equilibrium problems is to find $x^* \in C$ such that $F_i(x^*, x) \geq 0$ for all $x \in C$.

Although the theory of equilibrium problems was first introduced by Fan [1] in 1972, the most significant contributions to this problem were made by Blum and Oettli [2] and Noor and Oettli [3] in 1994. The equilibrium problem has a great impact on the development of several branches of pure and applied sciences, and it provides a natural and unified framework for solving several problems arising in physics, engineering, economics, game theory, image reconstruction, transportation, network, and elasticity. It can also be reformulated in the form of different mathematical problems such as an optimization problem, a convex feasibility problem (see [4]), a variational inequality problem (see [5]), a minimization problem (see [6]), a minimax inequality problem, a fixed point problem, a complementarity problem, a saddle point problem, or a Nash equilibrium problem in noncooperative games (see [2]). Therefore, it is natural to extend such a problem to more general problems in several ways. The system of equilibrium problems and mixed equilibrium problems was introduced and studied by some authors in, for instance, [15–19].

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Recently, Moudafi [7] introduced a split equilibrium problem which is a generalization of several optimization problems such as split feasibility problem, split inclusion problem, split variational inequality problem, and split common fixed point problem, see, e.g., [8–14]. By combining the ideas of split equilibrium problem with the system of equilibrium problems, in 2016, the system of split equilibrium problems and mixed equilibrium problems was introduced by Ugwunnadi and Ali [20] and Onjai-uea and Phuengrattana [21], respectively, see also [22–24]. These problems are defined as follows:

1. The split equilibrium problem is to find $x^* \in C$ such that $F_1(x^*, x) \geq 0$, for all $x \in C$, and such that $y^* = Ax^* \in Q$ solves $G_1(y^*, y) \geq 0$ for all $y \in Q$;
2. The system of split equilibrium problems is to find $x^* \in C$ such that $F_i(x^*, x) \geq 0$, for all $x \in C$, and such that $y^* = Ax^* \in Q$ solves $G_i(y^*, y) \geq 0$ for all $y \in Q$, where $\{F_i\}$ and $\{G_i\}$ are families of bifunctions;
3. The mixed equilibrium problem is to find $x^* \in C$ such that $F_1(x^*, x) + \varphi(x) - \varphi(x^*) \geq 0$ for all $x \in C$;
4. The split mixed equilibrium problem is to find $x^* \in C$ such that $F_1(x^*, x) + \varphi(x) - \varphi(x^*) \geq 0$ for all $x \in C$ and such that $y^* = Ax^* \in Q$ solves $F_2(y^*, y) + \phi(y) - \phi(y^*) \geq 0$ for all $y \in Q$,

where C and Q are nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ are families of bifunctions satisfying $F_i(x, x) = 0$ for all $x \in C$ and $G_i(y, y) = 0$ for all $Y \in Q$, for $I = 1, 2, \ldots, N$, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functions such that $C \cap \mathrm{dom}\varphi \neq \emptyset$ and $Q \cap \mathrm{dom}\phi \neq \emptyset$.

In recent years, many authors have made several efforts to develop implementable iterative methods for solving all these problems. In 2016, Suantai et al. [10] considered the split equilibrium problem and proposed the following iterative algorithm to find a common solution of fixed point problem for a nonspreading multivalued mapping and the split equilibrium problem:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Su_n, \forall n \in \mathbb{N}. \end{cases}$$

They proved a weak convergence theorem for the iterative sequence. In the same year, Ugwunnadi and Ali [20] established the following algorithm to solve the system of split equilibrium problems and showed that the sequence generated by their algorithm converges strongly to the common solution of considered problem and fixed point problem for a finite family of continuous pseudocontractive mappings.

$$\begin{cases} y_n = P_C(x_n + \lambda B(\mathcal{J}_n^M - I)Ax_n), \\ z_n = \beta_n y_n + (1 - \beta_n)T_{[n]r_n} y_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n \mu G)z_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\mathcal{J}_n^M = T_{s_{M,n}}^M T_{s_{M-1,n}}^M \cdots T_{s_{2,n}}^2 T_{s_{1,n}}^1$, $\mathcal{J}_n^0 = I$, and $T_r^g x = \left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$.

One year later, Onjai-uea and Phuengrattana [21] proposed another iterative algorithm to find a solution for the split mixed equilibrium problem for λ-hybrid multivalued mappings. They proved that the sequence generated by the following iterative algorithm converges weakly to a common solution of fixed point problem and split mixed equilibrium problem.

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)w_n, w_n \in Su_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n)z_n, z_n \in Sy_n, n \in \mathbb{N}. \end{cases} \tag{1.2}$$

Motivated and inspired by these problems and iterative methods, we introduce a new problem called system of split mixed equilibrium problems, which generalizes all these problems stated above and propose a new iterative algorithm to find a common solution of fixed point problem and system of split mixed equilibrium problems. We prove that sequence generated by our algorithm converges weakly to the solution. Also, we give some corollaries and numeric results to show that our results generalize and extend many results in the literature.

2 Preliminaries

Throughout this article, we use $\mathbb{N}$ and $\mathbb{R}$ to represent the set of natural and real numbers, respectively, "$\to$" for strong convergence of a sequence and "$\rightharpoonup$" for the weak convergence. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ a bounded linear operator, $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ nonlinear bifunctions, and S be a mapping from C onto H. A point $x \in C$ is called a fixed point of S if $Sx = x$ and the set of fixed points of S is denoted by $F(S)$. A mapping $S : C \to H$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|,$$

firmly nonexpansive if

$$\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle$$

and ν-inverse strongly monotone (ν-ism) if

$$\langle Sx - Sy, x - y\rangle \geq \nu\|Sx - Sy\|^2$$

for all $x, y \in C$. It is easy to see from the Schwarz inequality that every firmly nonexpansive mapping is also a nonexpansive mapping.

Lemma 1. [25] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Then $F(S)$ is closed and convex.*

Lemma 2. [26] *Let C be a nonempty closed convex subset of a real Hilbert space H, and S be a nonexpansive self-mapping on C. If $F(S) \neq \emptyset$, then $I - S$ is demiclosed at 0; i.e., if $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, then $(I - S)x = 0$, i.e., $x \in F(S)$. Here, I is the identity mapping of H.*

Lemma 3. *Let H be a Hilbert space and $\{x_n\}$ a sequence in H. Let $u, v \in H$ be such that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 4. [27] *Let H be a real Hilbert space. Then, we have*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle,$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. Also, if $\{x_n\}$ is a sequence in H weakly converging to $z \in H$, then

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

We need the following assumptions to solve a mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$ and a mapping φ:

(A1) $F(x, x) = 0, \forall x \in C$,

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$,

(A3) $\lim_{\lambda\to 0} F(\lambda z + (1 - \lambda)x, y) \leq F(x, y)$ for all $x, y, z \in C$,

(A4) $\forall x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous,

(A5) for each $x \in C$, $\lambda \in (0, 1]$, and $r > 0$, there exist a bounded subset $D \subseteq C$ and $a \in C$ such that for any $z \in C\backslash D$,

$$F(z, a) + \varphi(a) - \varphi(z) + \frac{1}{r}\langle a - z, z - x\rangle < 0.$$

(A6) C is a bounded set.

Lemma 5. [28] *Let C be a nonempty closed convex subset of a Hilbert space H_1 and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex mapping such that $C \cap \operatorname{dom}\varphi = \emptyset$. Suppose that bifunction $F : C \times C \to \mathbb{R}$ and a mapping φ satisfy Conditions (A1)–(A6). For $r > 0$ and $x \in H_1$, let $T_r^F : H_1 \to C$ be a mapping defined by*

$$T_r^F(x) = \left\{z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}. \tag{2.1}$$

Assume that either (A5) or (A6) holds. Then:
(*i*) *for each $x \in H_1$, $T_r^F x \neq \emptyset$,*
(*ii*) *T_r^F is single valued,*
(*iii*) *T_r^F is firmly nonexpansive,*
(*iv*) *$F(T_r^F) = MEP(F, \varphi)$ and it is closed and convex.*

Let $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex mapping such that $Q \cap \operatorname{dom}\phi = \emptyset$. Suppose that bifunction $G : Q \times Q \to \mathbb{R}$ and a mapping ϕ satisfy Conditions (A1)–(A6). For $s > 0$ and $u \in H_2$. Let $T_s^G : H_2 \to Q$ be a mapping defined by

$$T_s^G(u) = \left\{v \in Q : G(v, w) + \phi(w) - \phi(v) + \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w \in Q\right\}. \tag{2.2}$$

Then clearly T_s^G satisfies (i)–(iv) of Lemma 5, and $F(T_s^G) = \mathrm{MEP}(G, \phi)$.

3 Main results

First, we introduce the system of split mixed equilibrium problems in the following form:

Definition 1. Let C_i and Q_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$, $1 \leq i \leq N$, nonlinear bifunctions and let $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$ and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions such that $C_i \cap \operatorname{dom}\varphi_i \neq \emptyset$ and $Q_i \cap \operatorname{dom}\phi_i \neq \emptyset$. The system of split mixed equilibrium problems is to find $x^* \in C = \bigcap_{i=1}^N C_i$ such that

$$F_i(x^*, x) + \varphi_i(x) - \varphi_i(x^*) \geq 0, \forall x \in C_i, \tag{3.1}$$

and such that $y^* = Ax^* \in Q = \bigcap_{i=1}^N Q_i$ solves

$$G_i(y^*, y) + \phi_i(y) - \phi_i(y^*) \geq 0, \forall y \in Q_i. \tag{3.2}$$

The solution set of system of split mixed equilibrium problems (3.1) and (3.2) is denoted by

$$\mathrm{SSMEP}(F_i, \varphi_i, G_i, \phi_i) = \{x^* \in C : x^* \in \textstyle\bigcap_{i=1}^N \mathrm{MEP}(F_i, \varphi_i) \text{ and } Ax^* \in \bigcap_{i=1}^N \mathrm{MEP}(G_i, \phi_i)\},$$

where $\mathrm{MEP}(F_i, \varphi_i)$ is the set of solutions of mixed equilibrium problem, i.e.,

$$\mathrm{MEP}(F_i, \varphi_i) := \{x^* \in C_i : F_i(x^*, x) + \varphi_i(x) - \varphi_i(x^*) \geq 0, \forall x \in C_i\}.$$

Remark 1. In Definition 1, if
1. $N = 1$, then the system of split mixed equilibrium problems is reduced to the split mixed equilibrium problem studied in, e.g., [21].
2. $\varphi = \phi = 0$, then the system of split mixed equilibrium problems is reduced to the system of split equilibrium problems studied in, e.g., [20].
3. $N = 1$ and $\varphi = \phi = 0$, then the system of split mixed equilibrium problems is reduced to the split equilibrium problem studied in, e.g., [7,9–11].

4. $H_1 = H_2$, $A = I$, $F_i = G_i$, and $\varphi_i = \phi_i$, then the system of split mixed equilibrium problems is reduced to the system of mixed equilibrium problems.
5. $H_1 = H_2$, $A = I$, $F_i = G_i$, and $\varphi_i = \phi_i = 0$, then the system of split mixed equilibrium problems is reduced to the system of equilibrium problems studied in, e.g., [16,17].

Theorem 1. *Let C_i and Q_i, $1 \le i \le N$, be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator and $S : C \to C$ a nonexpansive mapping, where $C = \bigcap_{i=1}^{N} C_i$. Let $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ be nonlinear bifunctions satisfying Assumptions (A1)–(A6), $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$ and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ proper lower semicontinuous and convex functions such that $C_i \cap dom\varphi_i \neq \emptyset$ and $Q_i \cap dom\phi_i \neq \emptyset$ and let G_i be upper semicontinuous in the first argument. Assume that $\Gamma = F(S) \cap SSMEP(F_i, \varphi_i, G_i, \phi_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n S u_n + (1-\alpha_n) S y_n, \\ y_n = \beta_n x_n + (1-\beta_n) S z_n, \\ z_n = \delta_n u_n + (1-\delta_n) S u_n, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{G_1})A) u_{n,1}, \\ u_{n,1} = T_{r_n}^{F_2}(I - \gamma A^*(I - T_{r_n}^{G_2})A) u_{n,2}, \\ \vdots \\ u_{n,N-2} = T_{r_n}^{F_{N-1}}(I - \gamma A^*(I - T_{r_n}^{G_{N-1}})A) u_{n,N-1}, \\ u_{n,N-1} = T_{r_n}^{F_N}(I - \gamma A^*(I - T_{r_n}^{G_N})A) x_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.3}$$

where $a_n, \beta_n, \delta_n \in (0,1)$, $r_n \in (0,\infty)$, and $\gamma \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(*i*) $0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < 1$,
(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(*iii*) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,
(*iv*) $0 < \liminf_{n\to\infty} r_n$.

Then the sequence $\{x_n\}$ generated by (3.3) converges weakly to $p \in \Gamma$.

Proof. We divide our proof into six steps.
Step 1. In the first step, we show that $A^*(I - T_{r_n}^{Gi})A$ is a $\frac{1}{L}$-ism for all $i = 1, 2, \ldots, N$. Since $T_{r_n}^{Gi}$ is firmly nonexpansive and $I - T_{r_n}^{Gi}$ is 1-ism, by using that A^* is adjoint of A, we have

$$\begin{aligned} \|A^*(I - T_{r_n}^{Gi})Ax - A^*(I - T_{r_n}^{Gi})Ay\|^2 &= \langle A^*(I - T_{r_n}^{Gi})A(x-y), A^*(I - T_{r_n}^{Gi})A(x-y)\rangle \\ &= \langle (I - T_{r_n}^{Gi})A(x-y), AA^*(I - T_{r_n}^{Gi})A(x-y)\rangle \\ &\le L\langle (I - T_{r_n}^{Gi})A(x-y), (I - T_{r_n}^{Gi})A(x-y)\rangle \\ &= L\|(I - T_{r_n}^{Gi})A(x-y)\|^2 \\ &\le L\langle A(x-y), (I - T_{r_n}^{Gi})A(x-y)\rangle \\ &= L\langle x-y, A^*(I - T_{r_n}^{Gi})Ax - A^*(I - T_{r_n}^{Gi})Ay\rangle \end{aligned}$$

for all $x, y \in H_1$. So, $A^*(I - T_{r_n}^{Gi})A$ is a $\frac{1}{L}$-ism for all $i = 1, 2, \ldots, N$. On the other hand, since $0 < \gamma < \frac{1}{L}$, we get $I - \gamma A^*(I - T_{r_n}^{G_i})A$ is a nonexpansive mapping.

Step 2. In the second step, we show that sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded. Let $q \in \Gamma$. It means that q is a fixed point of the mappings S, $T_{r_n}^{Fi}$ and $I - \gamma A^*(I - T_{r_n}^{G_i})A$. Since $T_{r_n}^{Fi}$ and $I - \gamma A^*(I - T_{r_n}^{G_i})A$ are nonexpansive mappings, we have

$$\|u_{n,N-1} - q\| = \|T_{r_n}^{F_N}(I - \gamma A^*(I - T_{r_n}^{G_N})A)x_n - T_{r_n}^{F_N}(I - \gamma A^*(I - T_{r_n}^{G_N})A)q\| \le \|x_n - q\| \tag{3.4}$$

and

$$\begin{aligned} \|u_n - q\| &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{G_1})A)u_{n,1} - T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{G_1})A)q\| \le \|u_{n,1} - q\| \\ &\le \|T_{r_n}^{F_2}(I - \gamma A^*(I - T_{r_n}^{G_2})A)u_{n,2} - T_{r_n}^{F_2}(I - \gamma A^*(I - T_{r_n}^{G_2})A)q\| \\ &\le \|u_{n,2} - q\| \le \cdots \le \|u_{n,N-1} - q\| \le \|x_n - q\|. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\|z_n - q\| = \|\delta_n u_n + (1 - \delta_n)Su_n - q\| \le \delta_n\|u_n - q\| + (1 - \delta_n)\|Su_n - q\| \le \|u_n - q\| \le \|x_n - q\|. \tag{3.6}$$

From (3.6), we have

$$\begin{aligned} \|y_n - q\| &= \|\beta_n x_n + (1 - \beta_n)Sz_n - q\| \\ &\le \beta_n\|x_n - q\| + (1 - \beta_n)\|Sz_n - q\| \\ &\le \beta_n\|x_n - q\| + (1 - \beta_n)\|z_n - q\| \le \|x_n - q\|. \end{aligned} \tag{3.7}$$

So, we have from (3.5) and (3.7) that

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n Su_n + (1 - \alpha_n)Sy_n - q\| \le \alpha_n\|Su_n - q\| + (1 - \alpha_n)\|Sy_n - q\| \\ &\le \alpha_n\|u_n - q\| + (1 - \alpha_n)\|y_n - q\| \le \|x_n - q\|. \end{aligned} \tag{3.8}$$

Hence, it follows from (3.8) that the sequence $\{\|x_n - q\|\}$ is nonincreasing and bounded below. Therefore, we get $\{\|x_n - q\|\}$ and so $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are convergent (so they are bounded) sequences.

Step 3. In this step, we show that $\|u_n - x_n\| \to 0$. For this, we need to show $\|u_n - u_{n,1}\| \to 0$, $\|u_{n,i} - u_{n,i+1}\| \to 0$ $(1 \le i \le N - 2)$ and $\|u_{n,N-1} - x_n\| \to 0$. Let $J_i = T_{r_n}^{F_i}(I - \gamma A^*(I - T_{r_n}^{G_i})A)$. So we can write $u_n = J_1J_2\dots J_N x_n$. Also, we know that the mapping J_i is nonexpansive mapping and $q \in \Gamma$ is a fixed point of J_i. Thus, we have

$$\begin{aligned} \|u_n - q\|^2 &= \|J_1u_{n,1} - q\|^2 \\ &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{G_1})A)u_{n,1} - q\|^2 \\ &\le \|u_{n,1} - q - \gamma A^*(I - T_{r_n}^{G_1})Au_{n,1}\|^2 \\ &\le \|u_{n,1} - q\|^2 + \gamma^2\|A^*(I - T_{r_n}^{G_1})Au_{n,1}\|^2 - 2\gamma\langle u_{n,1} - q, A^*(I - T_{r_n}^{G_1})Au_{n,1}\rangle \\ &= \|u_{n,1} - q\|^2 + \gamma^2\langle A^*(I - T_{r_n}^{G_1})Au_{n,1}, A^*(I - T_{r_n}^{G_1})Au_{n,1}\rangle + 2\gamma\langle A(q - u_{n,1}), (I - T_{r_n}^{G_1})Au_{n,1}\rangle \\ &= \|u_{n,1} - q\|^2 + \gamma^2\langle (I - T_{r_n}^{G_1})Au_{n,1}, AA^*(I - T_{r_n}^{G_1})Au_{n,1}\rangle + 2\gamma\langle A(q - u_{n,1}) \\ &\quad + (I - T_{r_n}^{G_1})Au_{n,1}, (I - T_{r_n}^{G_1})Au_{n,1}\rangle - 2\gamma\langle (I - T_{r_n}^{G_1})Au_{n,1}, (I - T_{r_n}^{G_1})Au_{n,1}\rangle \\ &\le \|u_{n,1} - q\|^2 + L\gamma^2\langle (I - T_{r_n}^{G_1})Au_{n,1}, (I - T_{r_n}^{G_1})Au_{n,1}\rangle + 2\gamma\langle Aq - T_{r_n}^{G_1}Au_{n,1}, (I - T_{r_n}^{G_1})Au_{n,1}\rangle \\ &\quad - 2\gamma\langle (I - T_{r_n}^{G_1})Au_{n,1}, (I - T_{r_n}^{G_1})Au_{n,1}\rangle \\ &\le \|u_{n,1} - q\|^2 + L\gamma^2\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2 + 2\gamma\frac{1}{2}\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2 - 2\gamma\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2 \\ &= \|u_{n,1} - q\|^2 + \gamma(L\gamma - 1)\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n Su_n + (1 - \alpha_n)Sy_n - q\|^2 \\ &\le \alpha_n\|Su_n - q\|^2 + (1 - \alpha_n)\|Sy_n - q\|^2 \\ &\le \alpha_n\|u_n - q\|^2 + (1 - \alpha_n)\|y_n - q\|^2 \\ &\le \alpha_n(\|u_{n,1} - q\|^2 + \gamma(L\gamma - 1)\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2) + (1 - \alpha_n)\|x_n - q\|^2 \\ &\le \|x_n - q\|^2 + \alpha_n\gamma(L\gamma - 1)\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2, \end{aligned}$$

we obtain

$$-\alpha_n\gamma(L\gamma - 1)\|(I - T_{r_n}^{G_1})Au_{n,1}\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

If we take limit from both sides, we have $\|(I - T_{r_n}^{G_1})Au_{n,1}\| \to 0$. Since $u_n = J_1 u_{n,1} = J_1 J_2 u_{n,2} = \cdots = J_1 J_2 \ldots J_{N-1} u_{n,N-1} = J_1 J_2 \ldots J_{N-1} J_N x_n$, in a similar way, we see that

$$\lim_{n\to\infty} \|(I - T_{r_n}^{G_i})Au_{n,i}\| = 0 \tag{3.9}$$

for $i = 1, 2, \ldots, N-1$ and

$$\lim_{n\to\infty} \|(I - T_{r_n}^{G_N})Ax_n\| = 0. \tag{3.10}$$

Also, since $T_{r_n}^{F_N}$ is firmly nonexpansive, we get

$$\begin{aligned}
\|u_{n,N-1} - q\|^2 &= \|T_{r_n}^{F_N}(I - \gamma A^*(I - T_{r_n}^{G_N})A)x_n - T_{r_n}^{F_N} q\|^2 \\
&\leq \langle T_{r_n}^{F_N}(I - \gamma A^*(I - T_{r_n}^{G_N})A)x_n - T_{r_n}^{F_N} q, (I - \gamma A^*(I - T_{r_n}^{G_N})A)x_n - q\rangle \\
&= \langle u_{n,N-1} - q, (I - \gamma A^*(I - T_{r_n}^{G_N})A)x_n - q\rangle \\
&= \frac{1}{2}(\|u_{n,N-1} - q\|^2 + \|(I - \gamma A^*(I - T_{r_n}^{G_N})A)x_n - q\|^2 - \|u_{n,N-1} - x_n - \gamma A^*(I - T_{r_n}^{G_N})Ax_n\|^2) \\
&\leq \frac{1}{2}(\|u_{n,N-1} - q\|^2 + \|x_n - q\|^2 - (\|u_{n,N-1} - x_n\|^2 + \gamma^2\|A^*(I - T_{r_n}^{G_N})Ax_n\|^2 \\
&\quad - 2\gamma\langle u_{n,N-1} - x_n, A^*(I - T_{r_n}^{G_N})Ax_n\rangle)).
\end{aligned}$$

So, we have

$$\begin{aligned}
\|u_{n,N-1} - q\|^2 &\leq \|x_n - q\|^2 - \|u_{n,N-1} - x_n\|^2 + 2\gamma\langle u_{n,N-1} - x_n, A^*(I - T_{r_n}^{G_n})Ax_n\rangle \\
&\leq \|x_n - q\|^2 - \|u_{n,N-1} - x_n\|^2 + 2\gamma\|u_{n,N-1} - x_n\|\|A^*(I - T_{r_n}^{G_n})Ax_n\|.
\end{aligned}$$

Last inequality with inequalities (3.5) and (3.7) implies that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|\alpha_n S u_n + (1 - \alpha_n)S y_n - q\|^2 \\
&\leq \alpha_n\|u_n - q\|^2 + (1 - \alpha_n)\|y_n - q\|^2 \\
&\leq \alpha_n\|u_{n,N-1} - q\|^2 + (1 - \alpha_n)\|x_n - q\|^2 \\
&\leq \alpha_n(\|x_n - q\|^2 - \|u_{n,N-1} - x_n\|^2 + 2\gamma\|u_{n,N-1} - x_n\|\|A^*(I - T_{r_n}^{G_N})Ax_n\|) + (1 - \alpha_n)\|x_n - q\|^2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\alpha_n\|u_{n,N-1} - x_n\|^2 &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\gamma\alpha_n\|u_{n,N-1} - x_n\|\|A^*(I - T_{r_n}^{G_N})Ax_n\| \\
&\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\gamma\alpha_n M\|A^*(I - T_{r_n}^{G_N})Ax_n\|,
\end{aligned} \tag{3.11}$$

where $M = \sup_{n\in\mathbb{N}}\{\|u_{n,N-1} - x_n\|\}$. Therefore, it follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} \|u_{n,N-1} - x_n\| = 0. \tag{3.12}$$

Similarly, we have

$$\begin{aligned}
\|u_{n,N-2} - q\|^2 &= \|T_{r_n}^{F_{N-1}}(I - \gamma A^*(I - T_{r_n}^{G_{N-1}})A)u_{n,N-1} - T_{r_n}^{F_{N-1}} q\|^2 \\
&\leq \langle T_{r_n}^{F_{N-1}}(I - \gamma A^*(I - T_{r_n}^{G_{N-1}})A)u_{n,N-1} - T_{r_n}^{F_{N-1}} q, (I - \gamma A^*(I - T_{r_n}^{G_{N-1}})A)u_{n,N-1} - q\rangle \\
&= \langle u_{n,N-2} - q, (I - \gamma A^*(I - T_{r_n}^{G_{N-1}})A)u_{n,N-1} - q\rangle \\
&= \frac{1}{2}(\|u_{n,N-2} - q\|^2 + \|(I - \gamma A^*(I - T_{r_n}^{G_{N-1}})A)u_{n,N-1} - q\|^2 - \|u_{n,N-2} - u_{n,N-1} \\
&\quad - \gamma A^*(I - T_{r_n}^{G_{N-1}})Au_{n,N-1}\|^2) \\
&\leq \frac{1}{2}(\|u_{n,N-2} - q\|^2 + \|u_{n,N-1} - q\|^2 - (\|u_{n,N-2} - u_{n,N-1}\|^2 + \gamma^2\|A^*(I - T_{r_n}^{G_{N-1}})Au_{n,N-1}\|^2 \\
&\quad - 2\gamma\langle u_{n,N-2} - u_{n,N-1}, A^*(I - T_{r_n}^{G_{N-1}})Au_{n,N-1}\rangle)).
\end{aligned}$$

So, we have

$$\begin{aligned}\|u_{n,N-2}-q\|^2 &\le \|u_{n,N-1}-q\|^2 - \|u_{n,N-2}-u_{n,N-1}\|^2 + 2\gamma\langle u_{n,N-2}-u_{n,N-1}, A^*(I-T_{r_n}^{G_{N-1}})Au_{n,N-1}\rangle \\ &\le \|x_n-q\|^2 - \|u_{n,N-2}-u_{n,N-1}\|^2 + 2\gamma\|u_{n,N-2}-u_{n,N-1}\|\|A^*(I-T_{r_n}^{G_{N-1}})Au_{n,N-1}\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \|\alpha_n Su_n + (1-\alpha_n)Sy_n - q\|^2 \\ &\le \alpha_n\|u_n-q\|^2 + (1-\alpha_n)\|y_n-q\|^2 \\ &\le \alpha_n\|u_{n,N-2}-q\|^2 + (1-\alpha_n)\|x_n-q\|^2 \\ &\le \alpha_n(\|x_n-q\|^2 - \|u_{n,N-2}-u_{n,N-1}\|^2 \\ &\quad + 2\gamma\|u_{n,N-2}-u_{n,N-1}\|\|A^*(I-T_{r_n}^{G_{N-1}})Au_{n,N-1}\|) + (1-\alpha_n)\|x_n-q\|^2 \\ &\le \|x_n-q\|^2 - \alpha_n\|u_{n,N-2}-u_{n,N-1}\|^2 + 2\gamma\alpha_n\|u_{n,N-2}-u_{n,N-1}\|\|A^*(I-T_{r_n}^{G_{N-1}})Au_{n,N-1}\|.\end{aligned}$$

Hence, we obtain

$$\alpha_n\|u_{n,N-2}-u_{n,N-1}\|^2 \le \|x_n-q\|^2 - \|x_{n+1}-q\|^2 2\gamma\alpha_n M_1\|A^*(I-T_{r_n}^{G_{N-1}})Au_{n,N-1}\|,$$

where $M_1 = \sup_{n\in\mathbb{N}}\{\|u_{n,N-1}-u_{n,N-1}\|\}$. Therefore, it follows from (3.9) that

$$\lim_{n\to\infty}\|u_{n,N-2}-u_{n,N-1}\| = 0. \tag{3.13}$$

With a similar way, we have

$$\lim_{n\to\infty}\|u_{n,i}-u_{n,i+1}\| = 0 \text{ and } \lim_{n\to\infty}\|u_n-u_{n,1}\| = 0. \tag{3.14}$$

Since,

$$\|u_n-x_n\| \le \|u_n-u_{n,1}\| + \|u_{n,1}-u_{n,2}\| + \cdots + \|u_{n,N-2}-u_{n,N-1}\| + \|u_{n,N-1}-x_n\|$$

using (3.12), (3.13), and (3.14), we obtain

$$\lim_{n\to\infty}\|u_n-x_n\| = 0. \tag{3.15}$$

Step 4. Now, we show that $\lim_{n\to\infty}\|u_n-Su_n\| = 0$. Since,

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \|\alpha_n Su_n + (1-\alpha_n)Sy_n - q\|^2 \\ &\le \alpha_n\|u_n-q\|^2 + (1-\alpha_n)\|y_n-q\|^2 \\ &\le \alpha_n\|x_n-q\|^2 + (1-\alpha_n)(\beta_n\|x_n-q\|^2 + (1-\beta_n)\|Sz_n-q\|^2) \\ &\le \|x_n-q\|^2 + (1-\alpha_n)(1-\beta_n)\|z_n-q\|^2 \\ &\le \|x_n-q\|^2 + (1-\alpha_n)(1-\beta_n)(\|\delta_n u_n + (1-\delta_n)Su_n - q\|^2) \\ &\le \|x_n-q\|^2 + (1-\alpha_n)(1-\beta_n)(\delta_n\|u_n-q\|^2 \\ &\quad + (1-\delta_n)\|Su_n-q\|^2 - \delta_n(1-\delta_n)\|Su_n-u_n\|^2) \\ &\le \|x_n-q\|^2 - \delta_n(1-\alpha_n)(1-\beta_n)(1-\delta_n)\|Su_n-u_n\|^2,\end{aligned}$$

we get,

$$\delta_n(1-\alpha_n)(1-\beta_n)(1-\delta_n)\|Su_n-u_n\|^2 \le \|x_n-q\|^2 - \|x_{n+1}-q\|^2.$$

Therefore, we obtain

$$\lim_{n\to\infty}\|Su_n-u_n\| = 0. \tag{3.16}$$

Step 5. In this step, we show that $\omega_w(x_n) \subset \Gamma$, where $\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x\}\}$. It is clear that $\omega_w(x_n) \neq \varnothing$ because of boundedness of $\{x_n\}$. Let us assume that p is an arbitrary element of $\omega_w(x_n)$. It means that there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup p$. Using (3.15), we know that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup p$. By (3.16) and Lemma 2, we have $p \in F(S)$.

Now, we show that $p \in \bigcap_{i=1}^{N} \mathrm{MEP}(F_i, \varphi_i)$. Since $u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{G_1})A)u_{n,1}$, we get

$$F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - u_{n,1} + \gamma A^*(I - T_{r_n}^{G_1})Au_{n,1}\rangle \geq 0, \quad \forall y \in C_1.$$

So, we can write

$$F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - u_{n,1}\rangle + \frac{1}{r_n}\langle y - u_n, \gamma A^*(I - T_{r_n}^{G_1})Au_{n,1}\rangle \geq 0, \quad \forall y \in C_1.$$

Since F_1 is a monotone mapping, we have

$$\varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - u_{n,1}\rangle + \frac{1}{r_n}\langle y - u_n, \gamma A^*(I - T_{r_n}^{G_1})Au_{n,1}\rangle \geq F_1(y, u_n),$$

and hence

$$\varphi_1(y) - \varphi_1(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - u_{n_i,1}\rangle + \frac{1}{r_{n_i}}\langle y - u_{n_i}, \gamma A^*(I - T_{r_{n_i}}^{G_1})Au_{n_i,1}\rangle \geq F_1(y, u_{n_i}),$$

for all $y \in C_1$. It follows from weakly convergence of u_{n_i} to p, Condition (iv), (3.9), (3.14) and the proper lower semicontinuity of φ_1 that

$$F_1(y, p) + \varphi_1(p) - \varphi_1(y) \leq 0, \quad \forall y \in C_1.$$

Let $y_\lambda = \lambda y + (1 - \lambda)p$, for all $\lambda \in (0, 1]$ and $y \in C_1$. It is clear that $y_\lambda \in C_1$. So, last inequality holds for $y = y_\lambda$, that is,

$$F_1(y_\lambda, p) + \varphi_1(p) - \varphi_1(y_\lambda) \leq 0.$$

From Assumptions (A1)–(A6) and last inequality, we have

$$\begin{aligned} 0 &= F_1(y_\lambda, y_\lambda) + \varphi_1(y_\lambda) - \varphi_1(y_\lambda) \\ &\leq \lambda F_1(y_\lambda, y) + (1 - \lambda)F_1(y_\lambda, p) + \lambda\varphi_1(y) + (1 - \lambda)\varphi_1(p) - \lambda\varphi_1(y_\lambda) - (1 - \lambda)\varphi_1(y_\lambda) \\ &= \lambda(F_1(y_\lambda, y) + \varphi_1(y) - \varphi_1(y_\lambda)) + (1 - \lambda)(F_1(y_\lambda, p) + \varphi_1(p) - \varphi_1(y_\lambda)) \\ &\leq \lambda(F_1(y_\lambda, y) + \varphi_1(y) - \varphi_1(y_\lambda)). \end{aligned}$$

Therefore, we have

$$F_1(y_\lambda, y) + \varphi_1(y) - \varphi_1(y_\lambda) \geq 0, \quad \forall y \in C_1.$$

By taking limit as $\lambda \to 0$, we get

$$F_1(p, y) + \varphi_1(y) - \varphi_1(p) \geq 0, \quad \forall y \in C_1,$$

that is, $p \in \mathrm{MEP}(F_1, \varphi_1)$. Similarly, since $u_{n,i} = J_{i+1}u_{n,i+1}$ for $1 \leq i \leq N - 2$, $u_{n,N-1} = J_N x_n$ it follows from (3.9), (3.10), (3.12), and (3.14) that $p \in \mathrm{MEP}(F_i, \varphi_i)$ for $1 \leq i \leq N$. So, we obtain that $p \in \bigcap_{i=1}^{N} \mathrm{MEP}(F_i, \varphi_i)$ for $y \in C = \bigcap_{i=1}^{N} C_i$. On the other hand, since A is a bounded linear operator, we get $Ax_{n_i} \rightharpoonup Ap$. Then, from (3.9), (3.10), and (3.15), we have $T_{r_{n_i}}^{G_k}Ax_{n_i} \rightharpoonup Ap$, for $k = 1, 2, \ldots, N$. So, from definition of $T_{r_{n_i}}^{G_k}Ax_{n_i}$, we get

$$G_k(T_{r_{n_i}}^{G_k}Ax_{n_i}, y) + \phi_k(y) - \phi_k(T_{r_{n_i}}^{G_k}Ax_{n_i}) + \frac{1}{r_{n_i}}\langle y - T_{r_{n_i}}^{G_k}Ax_{n_i}, T_{r_{n_i}}^{G_k}Ax_{n_i} - Ax_{n_i}\rangle \geq 0,$$

for all $y \in Q_k$. It follows from weakly convergence of $T_{r_{n_i}}^{G_k}Ax_{n_i}$ to Ap and upper semicontinuity in the first argument of G_k that

$$G_k(Ap, y) + \phi_k(y) - \phi_k(Ap) \geq 0, \quad \forall y \in Q_k.$$

This implies that $Ap \in \mathrm{MEP}(G_i, \phi_i)$ and so $Ap \in \bigcap_{i=1}^{N} \mathrm{MEP}(G_i, \phi_i)$ for $y \in Q = \bigcap_{i=1}^{N} Q_i$. Hence, $p \in \mathrm{SSMEP}(F_i, \varphi_i, G_i, \phi_i)$ and so $p \in \Gamma$.

Step 6. Finally, we show that $x_n \rightharpoonup p \in \Gamma$. If we assume that there exist subsequences $\{x_{n_i}\}$ and $\{x_{n_k}\}$ which converge weakly to p and q, respectively, then we obtain from Step 2 that $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty}\|x_n - q\|$ exist. So, from Lemma 3, we have $p = q$. It means that $\omega_w(x_n)$ is a singleton set. This completes the proof. □

Remark 2. In Theorem 1, if
1. $N = 1$, then we derive the split mixed equilibrium problems which were introduced by Onjai-uea and Phuengrattana [21]. So, our problem generalizes their problem. Also, if we choose $\delta_n = 1$ for all $n \in \mathbb{N}$, then we derive their iterative algorithm for nonexpansive mappings.
2. $\varphi_i = \phi_i = 0$, then the sequence $\{x_n\}$ generated by (3.3) converges weakly to a solution of system of split equilibrium problems.

Now, we give the following theorem for the system of split variational inequality problems.

Theorem 2. *Let $C_i, Q_i, C, Q, H_1, H_2, A$, and S be chosen as in* Theorem 1. *Let the bifunctions $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ be defined by $F_i(x, y) = \langle A_i(x^*), y - x^*\rangle$ and $G_i(u, v) = \langle B_i(u^*), v - u^*\rangle$, respectively, where $A_i : C_i \to H_1$ and $B_i : Q_i \to H_2$ are monotone mappings. Then, the sequence $\{x_n\}$ generated by* (3.3) *converges weakly to a solution of system of split variational inequality problems which is to find a point $x^* \in C$ such that*

$$\langle A_i(x^*), y - x^*\rangle \geq 0, \forall y \in C_i$$

and such that $u^* = Ax^* \in Q$ solves

$$\langle B_i(u^*), v - u^*\rangle \geq 0, \forall v \in Q_i.$$

4 Numerical examples

Now, we give a numerical example to support our proof.

Example 1. Let $H_1 = H_2 = \mathbb{R}$, $C_i = [-i, 0]$, $Q_i = [-10 - i, 0]$, $\varphi_i(x) = \phi_i(x) = 0$, $F_i : C_i \times C_i \to H_1$, $F_i(x, y) = ix(y - x)$, $G_i : Q_i \times Q_i \to H_2$, $G_i(x, y) = (10 + i)x(y - x)$, $1 \leq i \leq N$, $S : C \to H_1$,$Sx = \frac{x}{2}$, $A : H_1 \to H_2$, $Ax = \frac{x}{2}$, where $C = \bigcap_{i=1}^N C_i = [-1, 0]$ and $Q = \bigcap_{i=1}^N Q_i = [-11, 0]$. It is clear that F_i and G_i satisfy Assumptions (A1)–(A6), the set of fixed point of S, $F(S)$ is $\{0\}$, the adjoint operator A^* of A is defined by $A^*x = \frac{x}{2}$ from H_2 to H_1 and the spectral radius of A^*A is $L = \frac{1}{2}$. First, we find a common solution $x^* \in C$ for the following system of mixed equilibrium problems:

$$F_i(x^*, x) + \varphi_i(x) - \varphi_i(x^*) \geq 0, \quad \forall x \in C_i,$$

for $1 \leq i \leq N$. Since we choose the mapping φ as 0, the point x^* has to be a solution for the inequality $ix^*(x - x^*) \geq 0$ for all $x \in [-i, 0]$. This problem has a unique solution $x^* = 0$. It is obvious that the point $y^* = Ax^* = 0$ is a solution for the following system of mixed equilibrium problems:

$$G_i(y^*, y) + \phi_i(y) - \phi_i(y^*) \geq 0, \quad \forall y \in Q_i,$$

for $1 \leq i \leq N$, that is, $y^* = 0$ solves the inequality $(10 + i)y^*(y - y^*) \geq 0$ for all $y \in [-10 - i, 0]$. So, we obtain that $x^* = 0$ is a common solution for the system of split mixed equilibrium problems and fixed point problem, i.e., $0 \in \Gamma = F(S) \cap \mathrm{SSMEP}(F_i, \varphi_i, G_i, \phi_i)$.

Next, we compute $T_{r_n}^{F_i}(I - \gamma A^*(I - T_{r_n}^{Gi})A)x$. From Assumptions (A1)–(A6), it is known that the mappings $T_{r_n}^{F_i}$ and $T_{r_n}^{Gi}$ are single value mappings. Let $T_{r_n}^{Gi}Ax = z$. Then, we have

$$\begin{aligned} T_{r_n}^{Gi}Ax = z \;\; &\Leftrightarrow \;\; G_i(z, y) + \phi_i(y) - \phi_i(z) + \frac{1}{r_n}(y - z)(z - Ax) \geq 0, \quad \forall y \in Q_i \\ &\Leftrightarrow \;\; (10 + i)z(y - z) + \frac{1}{r_n}(y - z)\left(z - \frac{x}{2}\right) \geq 0, \quad \forall y \in Q_i \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow\ (y-z)\left((10+i)z+\frac{1}{r_n}\left(z-\frac{x}{2}\right)\right)\geq 0,\quad \forall y\in Q_i\\
&\Leftrightarrow\ z\left(10+i+\frac{1}{r_n}\right)-\frac{x}{2r_n}=0\\
&\Leftrightarrow\ z=\frac{x}{2+2(10+i)r_n}.
\end{aligned}$$

So, we get

$$(I-\gamma A^*(I-T_{r_n}^{Gi})A)x\ =x-A^*\left(\frac{x}{2}-T_{r_n}^{Gi}Ax\right)=x-\frac{x}{4}+\frac{x}{4+4(10+i)r_n}\ =\frac{3x}{4}+\frac{x}{4+4(10+i)r_n}.$$

On the other hand, let $T_{r_n}^{F_i}u=w$. Then, we have

$$\begin{aligned}
T_{r_n}^{Fi}u=w\ &\Leftrightarrow F_i(w,v)+\varphi_i(v)-\varphi_i(w)+\frac{1}{r_n}(v-w)(w-u)\geq 0,\quad \forall v\in C_i\\
&\Leftrightarrow iw(v-w)+\frac{1}{r_n}(v-w)(w-u)\geq 0,\quad \forall v\in C_i\\
&\Leftrightarrow (v-w)\left(iw+\frac{1}{r_n}(w-u)\right)\geq 0,\quad \forall v\in C_i\\
&\Leftrightarrow w\left(i+\frac{1}{r_n}\right)-\frac{u}{r_n}=0\\
&\Leftrightarrow w=\frac{u}{1+ir_n}.
\end{aligned}$$

From the last equality, we obtain that

$$T_{r_n}^{F_i}(I-\gamma A^*(I-T_{r_n}^{Gi})A)x=\frac{1}{1+ir_n}\left(\frac{3x}{4}+\frac{x}{4+4(10+i)r_n}\right).$$

Now, we show that the sequence $\{x_n\}$ generated by our iteration method (3.3) converges weakly to the common solution $x^*=0$. Let $\alpha_n=\frac{n}{n+1}$, $\beta_n=\frac{n}{2n+1}$, $\delta_n=\frac{n}{3n+1}$, $r_n=\frac{n}{4n+1}$, and $\gamma=1$. It is clear that α_n, β_n, δ_n, and r_n satisfy Conditions (i)–(iv) of Theorem 1. Then, Algorithm (3.3) becomes

$$\begin{cases}
x_{n+1}=\dfrac{n}{2n+2}u_n+\dfrac{1}{2n+2}y_n,\\
y_n=\dfrac{n}{2n+1}x_n+\dfrac{n+1}{4n+2}z_n,\\
z_n=\dfrac{n}{3n+1}u_n+\dfrac{2n+1}{6n+2}u_n,\\
u_n=\dfrac{4n+1}{5n+1}\left[\dfrac{3}{4}+\dfrac{1}{4+4.\,(10+1)\frac{n}{4n+1}}\right]u_{n,1},\\
u_{n,1}=\dfrac{4n+1}{6n+1}\left[\dfrac{3}{4}+\dfrac{1}{4+4.\,(10+2)\frac{n}{4n+1}}\right]u_{n,2},\\
\vdots\\
u_{n,N-2}=\dfrac{4n+1}{(3+N)n+1}\left[\dfrac{3}{4}+\dfrac{1}{4+4.\,(10+N-1)\frac{n}{4n+1}}\right]u_{n,N-1},\\
u_{n,N-1}=\dfrac{4n+1}{(4+N)n+1}\left[\dfrac{3}{4}+\dfrac{1}{4+4.\,(10+N)\frac{n}{4n+1}}\right]x_n,\quad \forall n\in\mathbb{N}.
\end{cases}\tag{4.1}$$

Table 1: Some steps of Algorithm (4.1)

	$x_1 = -0.5$, $N = 5$	$x_1 = -0.3$, $N = 10$	$x_1 = -0.8$, $N = 10$
x_2	-1.56092×10^{-1}	-7.15331×10^{-2}	-1.90755×10^{-1}
x_3	-4.66326×10^{-2}	-1.58648×10^{-2}	-4.23062×10^{-2}
x_4	-1.35621×10^{-2}	-3.35334×10^{-3}	-8.94224×10^{-3}
x_5	-3.87231×10^{-3}	-6.85367×10^{-4}	-1.82764×10^{-3}
x_6	-1.09101×10^{-3}	-1.36677×10^{-4}	-3.64473×10^{-4}
x_7	-3.04315×10^{-4}	-2.67522×10^{-5}	-7.13391×10^{-5}
x_8	-8.42199×10^{-5}	-5.16014×10^{-6}	-1.37604×10^{-5}
x_9	-2.31623×10^{-5}	-9.83664×10^{-7}	-2.6231×10^{-6}
x_{10}	-6.33749×10^{-6}	-1.85707×10^{-7}	-4.9522×10^{-7}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{100}	-1.2431×10^{-58}	-1.99996×10^{-75}	-5.33324×10^{-75}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{1000}	$-5.07494 \times 10^{-584}$	$-2.86696 \times 10^{-764}$	$-7.64523 \times 10^{-764}$

In Table 1, we give some steps of Algorithm (4.1) for some initial values and special N. From the table, it is clear that sequence $\{x_n\}$ generated by Algorithm (4.1) converges weakly to common solution $x^* = 0$.

In Figure 1, we give the graphics of the fitted curves, which are generated according to the values given in Table 1. For the fitting process third-degree polynomials were used.

Example 2. Next, we compare the performance of our Algorithm (3.3) with Algorithm (1.1) of Ugwunnadi and Ali [20]. Let $H_1 = \mathbb{R}^N = H_2$. For $i = 1, 2, \ldots, N$, let $C_i = \{x \in H : \|x\| \leq 1\}$ and $Q_i = [-10, 10] \times [-10, 10] \times \cdots \times [-10, 10]$, where $C = \bigcap_{i=1}^N C_i$ and $Q = \bigcap_{i=1}^N Q_i$. Define the bifunctions $F_i : C_i \times C_i \to \mathbb{R}$ and $G_i : Q_i \times Q_i \to \mathbb{R}$ by $F_i(x, y) = \frac{i}{2}(y^2 - x^2)$ and $G_i(u, v) = -3iu^2 + 2uvi + iv^2$, $\varphi_i : C_i \to \mathbb{R} \cup \{+\infty\}$, and $\phi_i : Q_i \to \mathbb{R} \cup \{+\infty\}$ are defined by $\varphi_i(x) = 0$ for all $x \in C_i$ and $\phi_i(u) = 0$ for all $u \in Q_i$. It is easy to show that

$$T_{r_n}^{F_i} z = \frac{z}{1 + ir_n}, \quad i = 1, 2, \ldots, N,$$

and

$$T_{r_n}^{G_i} w = \frac{w}{1 + 4ir_n}, \quad i = 1, 2, \ldots, N.$$

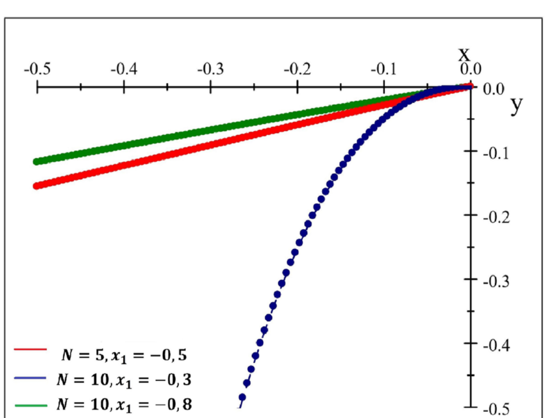

Figure 1: Graphics of the fitted curves generated from Table 1.

Let $S : C \to H_1$ be defined by $Sx = \frac{x}{8}$, which is nonexpansive and $F(S) = \{0\}$. Clearly $\Gamma = \{0\}$. We choose the following parameters $\alpha_n = \frac{n}{2n+3}$, $\beta_n = \frac{5n+7}{12n+13}$, $\delta_n = \frac{2n+1}{6n+8}$, $r_n = \frac{n}{n+4}$, and $\gamma = \frac{1}{2}$. The operator $A : H_1 \to H_2$ is defined by $Ax = 2x$ which is bounded and linear. The adjoint operator of A, i.e., $A^* : H_2 \to H_1$ is defined by $A^*x = 2x$ for all $x \in H_2$. Then Algorithm (3.3) becomes:

$$\begin{cases} x_{n+1} = \dfrac{n}{8(2n+3)}u_n + \dfrac{n+3}{8(2n+3)}y_n, \\ y_n = \dfrac{5n+7}{12n+13}x_n + \dfrac{7n+6}{8(12n+13)}z_n, \\ z_n = \dfrac{2n+1}{6n+3}u_n + \dfrac{4n+2}{8(6n+3)}u_n, \\ u_n = \dfrac{n+4}{2n+4}\left[1 - \left(1 - \dfrac{n+4}{5n+4}\right)\right]2u_{n,1}, \\ u_{n,1} = \dfrac{n+4}{3n+4}\left[1 - \left(1 - \dfrac{n+4}{9n+4}\right)\right]2u_{n,2}, \\ \vdots \\ u_{n,N-2} = \dfrac{n+4}{n(N-1)+4}\left[1 - \left(1 - \dfrac{n+4}{n(4N-7)+4}\right)\right]2u_{n,N-1}, \\ u_{n,N-1} = \dfrac{n+4}{nN+4}\left[1 - \left(1 - \dfrac{n+4}{n(4N-3)+4}\right)\right]2x_n. \end{cases}$$

For Algorithm (1.1), we take $T_i = \frac{x}{8}$ for all $x \in \mathbb{R}^N$, $i = 1, 2, \ldots, M$, $g_k(u, v) = -3ku^2 + 2uvk + kv^2$ for all $u, v \in Q$, and $k = 1, 2, \ldots, N$, $f(x) = \frac{x}{2}$, $Gx = x$ for all $x \in \mathbb{R}^N$. We also choose the following parameters: $\alpha = \frac{1}{n+1}$, $\delta = \frac{n}{3n+5}$, $r_n = \frac{n}{n+4}$, $s_n = \frac{n}{2n+1}$, $\gamma = 0.25$, $\mu = 1$, $M = 1$, and $\beta = 0.54$. The initial value is generated randomly in $(-2, 2)$. We compare the performance of Algorithms (3.3) and (1.1) for different values of N as follows: $N = 20, 50, 100$ and 500. We choose $\|x_{n+1} - x_n\| < 10^{-5}$ as a stopping criterion and plot the graphs of $\|x_{n+1} - x_n\|$ against a number of iterations for each algorithm. The results of the numerical computation are reported in Table 2 and Figures 2–4.

5 Conclusion

In this article, we generalized several equilibrium problems by introducing the system of split mixed equilibrium problems. We established an iterative algorithm and proved that the iterative sequence generated by the algorithm converges weakly to the common solution of considered problems. Since our problem is fairly general, our results are very significant. Also, we substantiated our results by constructing

Table 2: Computation result for Example 2

		Algorithm (3.3)	**Algorithm (1.1)**
$N = 20$	No of iter.	6	15
	CPU time (s)	0.0018	0.0023
$N = 50$	No of iter.	6	16
	CPU time (s)	0.0022	0.0031
$N = 100$	No of iter.	6	16
	CPU time (s)	0.0079	0.0092
$N = 500$	No of iter.	6	16
	CPU time (s)	0.0091	0.0138

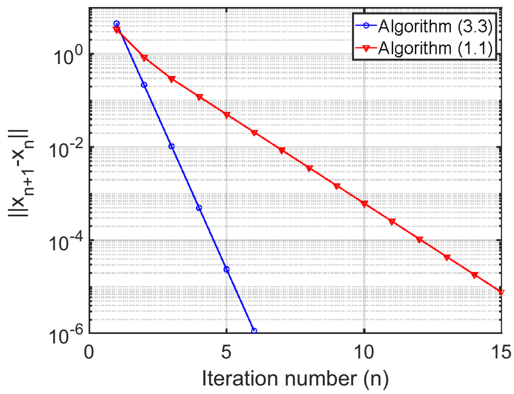

Figure 2: $N = 20$.

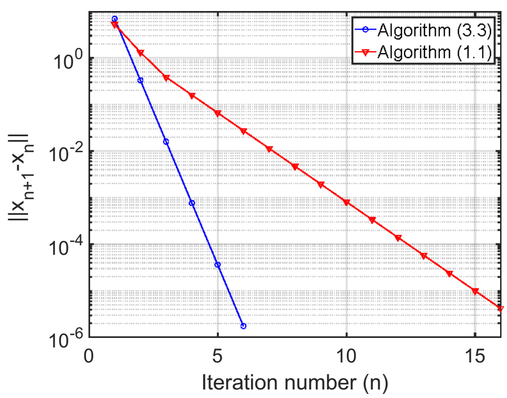

Figure 3: $N = 100$.

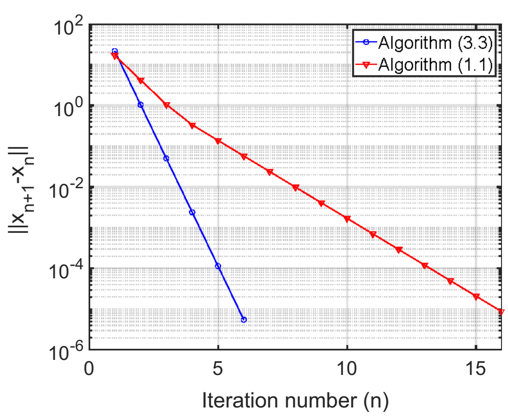

Figure 4: $N = 500$.

a numerical model. In this model, we constructed an iterative sequence by choosing special mappings and sequences, which satisfies the conditions of our theorem and calculated its steps in Mathematica software. As can be seen from the table, iterative sequence converges strongly and hence weakly to the solution. Also, we compare the rate of convergence of our method with the method of Ugwunnadi and Ali [20] and show that our method converges faster than their method.

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References

[1] K. Fan, *A Minimax Inequality and Applications*, in: O. Shisha (ed.), Inequality III, Academic Press, New York, 1972.
[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.
[3] M. A. Noor and W. Oettli, *On general nonlinear complementarity problems and quasi equilibria*, Matematiche (Catania) **49** (1994), 313–331.
[4] S. Suantai, S. Kesornprom and P. Cholamjiak, *A new hybrid CQ algorithm for the split feasibility problem in Hilbert spaces and its applications to compressed sensing*, Mathematics **7** (2019), no. 9, art. 789, DOI: https://doi.org/10.3390/math7090789.
[5] D. V. Hieu and P. Cholamjiak, *Modified extragradient method with Bregman distance for variational inequalities*, Appl. Anal. (2020), DOI: https://doi.org/10.1080/00036811.2020.1757078.
[6] K. Kunrada, N. Pholasa and P. Cholamjiak, *On convergence and complexity of the modified forward-backward method involving new linesearches for convex minimization*, Math. Methods Appl. Sci. **42** (2019), 1352–1362.
[7] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), 275–283.
[8] V. H. Dang, *Parallel extragradient-proximal methods for split equilibrium problems*, Math. Model. Anal. **21** (2016), no. 4, 478–501.
[9] K. R. Kazmi and S. H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem*, J. Egyptian Math. Soc. **21** (2013), no. 1, 44–51.
[10] S. Suantai, P. Cholamjiak, Y. J. Cho, and W. Cholamjiak, *On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces*, Fixed Point Theory Appl. **2016** (2016), art. 35, DOI: https://doi.org/10.1186/s13663-016-0509-4.
[11] L. O. Jolaoso and I. Karahan, *A general alternative regularization method with line search technique for solving split equilibrium and fixed point problems in Hilbert spaces*, Comput. Appl. Math. **39** (2020), art. 150, DOI: https://doi.org/10.1007/s40314-020-01178-8.
[12] S. Suantai, K. Kankam and P. Cholamjiak, *A novel forward-backward algorithm for solving convex minimization problem in Hilbert spaces*, Mathematics **8** (2020), art. 42, DOI: https://doi.org/10.3390/math8010042.
[13] M. A. A. Khan and P. Cholamjiak, *A multi-step approximant for fixed point problem and convex optimization problem in Hadamard spaces*, J. Fixed Point Theory Appl. **22** (2020), art. 62, DOI: https://doi.org/10.1007/s11784-020-00796-3.
[14] Y. Shehu and P. Cholamjiak, *Iterative method with inertial for variational inequalities in Hilbert spaces*, Calcolo **56** (2019), DOI: https://doi.org/10.1007/s10092-018-0300-5.
[15] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), no. 1, 117–136.
[16] N. Buong, *Regularization extragradient method for a system of equilibrium problems*, Comput. Methods Appl. Math. **7** (2007), no. 4, 285–293.
[17] J. K. Kim and N. Buong, *An iteration method for common solution of a system of equilibrium problems in Hilbert spaces*, Fixed Point Theory Appl. (2011), art. 780764, DOI: https://doi.org/10.1155/2011/780764.
[18] I. V. Konnov, S. Schaible and J. C. Yao, *Combined relaxation method for mixed equilibrium problems*, J. Optim. Theory Appl. **126** (2005), no. 2, 309–322.
[19] L. C. Zeng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.
[20] G. C. Ugwunnadi and B. Ali, *Approximation methods for solutions of system of split equilibrium problems*, Adv. Oper. Theory **1** (2016), no. 2, 164–183.
[21] N. Onjai-uea and W. Phuengrattana, *On solving split mixed equilibrium problems and fixed point problems of hybrid-type multivalued mappings in Hilbert spaces*, J. Inequal. Appl. **2017** (2017), art. 137.
[22] I. Karahan, *Strong and weak convergence theorems for split equality generalized mixed equilibrium problem*, Fixed Point Theory Appl. **2016** (2016), art. 101, DOI: https://doi.org/10.1186/s13663-016-0592-6.
[23] L. O. Jolaoso, K. O. Oyewole, C. C. Okeke, and O. T. Mewomo, *A unified algorithm for solving split generalized mixed equilibrium problem and for finding fixed point of nonspreading mapping in Hilbert spaces*, Demonstr. Math. **51** (2018), 211–232.
[24] F. U. Ogbuisi and O. T. Mewomo, *On split generalised mixed equilibrium problems and fixed-point problems with no prior knowledge of operator norm*, J. Fixed Point Theory Appl. **19** (2017), 2109–2128.

[25] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Dekker, New York, 1984.
[26] K. Goebel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, England, 1990.
[27] G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudocontractions in Hilbert space*, J. Math. Anal. Appl. **329** (2007), 336–346.
[28] M. Rahaman, Y. C. Liou, R. Ahmad, and I. Ahmad, *Convergence theorems for split equality generalized mixed equilibrium problems for demi-contractive mappings*, J. Inequal. Appl. **2015** (2015), art. 418.