Review Article

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On a characterization of exponential, Pearson and Pareto distributions via covariance and pseudo-covariance

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Abstract: Properties of linear regression of order statistics and their functions are usually utilized for the characterization of distributions. In this paper, based on such statistics, the concept of Pearson covariance and the pseudo-covariance measure of dependence is used to characterize the exponential, Pearson and Pareto distributions.

Keywords: characterization, order statistics, linear regression, covariance, pseudo-covariance, uniform distribution, exponential distribution, Pearson type I distribution, Pareto distribution of second kind

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† *Dedicated to the memory of Professor Dominik Szynal.*

1 Introduction

Let $X_1, \ldots, X_n$ be a random sample on X with absolutely continuous distribution function F and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. Define

$$Z_j = \frac{1}{n-j} \sum_{i=1}^{n-j} (X_{j+i:n} - X_{j:n}), \quad j = 1, \ldots, n-1.$$

Let F_0, F_1, F_2 be the c.d.f. of the two-parameter exponential, the Pearson type I and the Pareto distribution of second kind given by

$$F_0(x) = \begin{cases} 1 - e^{-(x-\mu)/\theta}, & x > \mu; \quad \theta > 0, \\ 0, & \text{elsewhere;} \end{cases}$$

$$F_1(x) = \begin{cases} 1 - \left(\frac{\nu - x}{\nu - \mu}\right)^{\theta}, & -\infty < \mu < x < \nu < \infty; \quad \theta > 0, \\ 0, & \text{elsewhere;} \end{cases}$$

$$F_2(x) = \begin{cases} 1 - \left(\frac{\mu + \gamma}{x + \gamma}\right)^{\theta}, & x > \mu; \quad \theta > 0; \quad \gamma > -\infty, \\ 0, & \text{elsewhere.} \end{cases}$$

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There is rich literature on characterizations of the aforementioned distributions via linear regression of order statistics or their functions. We mention here the studies of Dallas [1], Wang and Shrivastava [2], Nagaraja and Nevzorov [3], Dębińska and Wesołowski [4] and Bieniek and Szynal [5].

The main aim of this paper is to characterize the aforementioned distributions via two measures of dependence, namely, the Pearson covariance and the pseudo-covariance measure recently proposed by Pawlas and Szynal [6]. Let us recall these notions.

The Pearson covariance is defined by

$$\mathrm{Cov}(X, Y) = EXY - EXEY,$$

$X \in L^1$, $Y \in L^1$, $XY \in L^1$ (where L^1 is the space of integrable variables), and the pseudo-covariance

$$\mathrm{Cov}^{(PD)}(X, Y) = \left(\int_0^1 |y(p)|^2 \mathrm{d}p \right)^{\frac{1}{2}} \left(\int_0^1 (E(X|Y = y(p)) - EX)^2 \mathrm{d}p \right)^{\frac{1}{2}}, \tag{1}$$

where $X \in L^1$, $Y \in L^0$ (where L^0 is the space of all random variables having a continuous distribution) and for $p \in (0, 1)$, $y(p)$ is the quantile function of the random variable Y.

Taking into account that

$$|\mathrm{Cov}(X, Y)| \le \mathrm{Cov}^{(P,D)}(X, Y),$$

we see that pseudo-covariance provides information on dependent uncorrelated random variables.

Example 1.1. (Pawlas and Szynal [6])
Let the cumulative density function $f_{X,Y}(x, y)$ of a random vector (X, Y) have the form:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}[1 + xy(x^2 - y^2)], & |x| < 1 \wedge |y| < 1, \\ 0, & |x| \ge 1 \vee |y| \ge 1. \end{cases}$$

The marginal distributions of X and Y are uniform on $(-1, 1)$, $f_{X,Y}(x, y) \ne f_X(x) \cdot f_Y(y)$, so X and Y are dependent. The random variables X and Y are uncorrelated ($\mathrm{Cov}(X, Y) = 0$), while $\mathrm{Cov}^{(PD)}(X, Y) = \frac{2\sqrt{21}}{315} \ne 0$.

2 Characterization via covariances

First we recall the results of Wang and Shrivastava [2].

Proposition 2.1. *Suppose the distribution F of X is continuous with finite first moment. Then for some j, $1 \le j \le n - 1$ it holds that*

(*a*) $E(Z_j|X_{j:n} = x) = \theta$ *if and only if X follows exponential distribution with θ and some μ;*

(*b*) $E(Z_j|X_{j:n} = x) = -\frac{1}{1+\theta}x + 1 + \frac{\mu}{1+\theta}$ *if and only if X follows Pearson type I distribution with θ, μ and $\nu = \theta + \mu + 1$;*

(*c*) $E(Z_j|X_{j:n} = x) = \frac{1}{\theta-1}x + 1 - \frac{\mu}{\theta-1}$ *if and only if X follows Pareto distribution with θ, μ and $\nu = \mu + 1 - \theta$.*

Moreover, we should mention the result of Swanepoel [7]:

$$X \sim F_0 \text{ if and only if } \mathrm{Cov}(Z_1, X_{1:n}) = 0.$$

Before giving the main result we point out that for an exponential distribution F_0 with $\mu = 0$ and $\theta = 1$ we have

$$\mathrm{Cov}^{(P,D)}(Z_j^0, X_{j:n}^0) = \left(\int_0^1 (x^o(p))^2 \mathrm{d}p\right)^{\frac{1}{2}} \left(\int_0^1 (E(Z_j|X_{j:n} = x(p)) - EZ_j)^2 \mathrm{d}p\right)^{\frac{1}{2}} = 0, \tag{2}$$

where $X^0 = X - EX$ and $x^o(p) = x(p) - EX_{j:n}$, as $EZ_j = 1$ and $E(Z_j|X_{j:n} = x) = \theta = 1$. Taking into account the relation

$$|\mathrm{Cov}(Z_j, X_{j:n})| = |\mathrm{Cov}(Z_j^0, X_{j:n}^0)| \le \mathrm{Cov}^{(PD)}(Z_j^0, X_{j:n}^0),$$

we conclude that $\mathrm{Cov}(Z_j, X_{j:n}) = 0$.

Now consider the order statistics $Y_{1:n} \le Y_{2:n} \le \ldots \le Y_{n:n}$ related to the sequence of i.i.d. random variables $Y_1, Y_2, \ldots, Y_n$ with distribution function F_0 with μ and $\theta > 0$. Then for any $n = 1, 2, \ldots$ we have

$$(Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}) = (\theta X_{1:n} + \mu, \theta X_{2:n} + \mu, \ldots, \theta X_{n:n} + \mu), \tag{3}$$

where $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ is the order statistics related to a sample from the standard exponential distribution. For $j = 1, \ldots, n-1$, we can write

$$EZ_j = \frac{1}{n-j} \sum_{i=1}^{n-j} (EY_{j+i:n} - EY_{j:n}) = \frac{\theta}{n-j} \sum_{i=1}^{n-j} (EX_{j+i:n} - EX_{j:n}) = \theta.$$

From Proposition 2.1, we know that $E(Z_j|X_{j:n} = x) = \theta$, which gives (2). Hence, $\mathrm{Cov}(Z_j, X_{j:n}) = 0$.

One can also show that for the Pearson distribution F_1 and the Pareto distribution F_2 the equality

$$|\mathrm{Cov}(Z_j, X_{j:n})| = \mathrm{Cov}(Z_j^0, X_{j:n}^0) = \mathrm{Cov}^{(P,D)}(Z_j^0, X_{j:n}^0), \tag{4}$$

is satisfied. For the Pearson distribution F_1, we obtain

$$\begin{aligned}\mathrm{Cov}(Z_j, X_{j:n}) &= E(X_{j:n}E(Z_j|X_{j:n})) - EX_{j:n}EZ_j \\ &= E\left(X_{j:n}\left(-\frac{1}{1+\theta}X_{j:n} + 1 + \frac{\mu}{1+\theta}\right)\right) \\ &\quad - EX_{j:n}\left(-\frac{1}{1+\theta}EX_{j:n} + 1 + \frac{\mu}{1+\theta}\right) = -\frac{1}{1+\theta}\mathrm{Var}(X_{j:n}).\end{aligned}$$

From the other side we have

$$\left(\int_0^1 (E(Z_j^0|X_{j:n}^0 = x_{j:n}^0(p)) - EZ_j^0)^2 \mathrm{d}p\right)^{\frac{1}{2}} = \frac{1}{1+\theta}\sqrt{\mathrm{Var}(X_{j:n})},$$

which gives (4).

Using similar arguments for the Pareto distribution of second kind we have

$$\mathrm{Cov}(Z_j, X_{j:n}) = E\left(X_{j:n}\left(\frac{1}{\theta-1}X_{j:n} + 1 + \frac{\mu}{\theta-1}\right)\right) - EX_{j:n}\left(\frac{1}{\theta-1}EX_{j:n} + 1 + \frac{\mu}{\theta-1}\right) = \frac{1}{\theta-1}\mathrm{Var}(X_{j:n})$$

and

$$\left(\int_0^1 (E(Z_j^0|X_{j:n}^0 = x_{j:n}^0(p)) - EZ_j^0)^2 \mathrm{d}p\right)^{\frac{1}{2}} = \frac{1}{\theta-1}\sqrt{\mathrm{Var}(X_{j:n})},$$

which gives (4).

Our characterization result is as follows.

Theorem 2.1. *Let $X_1, \ldots, X_n$ be $n \geq 2$ i.i.d. random variables X with an absolutely continuous distribution F and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics.*

The equality

$$|\mathrm{Cov}(Z_j, X_{j:n})| = \mathrm{Cov}(Z_j^0, X_{j:n}^0) = \mathrm{Cov}^{(P,D)}(Z_j^0, X_{j:n}^0), \tag{5}$$

holds true if and only if $F \in \{F_0, F_1, F_2\}$.

Proof. We stated above that property (5) is satisfied by the distribution functions F_0, F_1, F_2. Moreover, we know that

$$|\mathrm{Cov}(Z_j, X_{j:n})| = |\mathrm{Cov}(Z_j^0, X_{j:n}^0)| \leq \mathrm{Cov}^{(PD)}(Z_j^0, X_{j:n}^0) = \left(\int_0^1 (x^o(p))^2 \mathrm{d}p \right)^{\frac{1}{2}} \left(\int_0^1 (E(Z_j | X_{j:n} = x(p)) - EZ_j)^2 \mathrm{d}p \right)^{\frac{1}{2}}.$$

But the above Schwarz inequality reduces to equality if and only if for a real number c

$$c(x(p) - EX_{j:n}) = E(Z_j | X_{j:n} = x(p)) - EZ_j. \tag{6}$$

Furthermore, we know (Wang and Shrivastava [2]) that

$$E(Z_j | X_{j:n} = x(p)) = \frac{\int_{x(p)}^{\infty} x \mathrm{d}F(x)}{\bar{F}(x(p))} - x(p).$$

But

$$\int_{x(p)}^{\infty} x \mathrm{d}F(x) = -\int_{x(p)}^{\infty} x \mathrm{d}(1 - F(x)) = -x(1 - F(x))|_{x(p)}^{\infty} + \int_{x(p)}^{\infty} \bar{F}(x) \mathrm{d}x = x(p)\bar{F}(x(p)) + \int_{x(p)}^{\infty} \bar{F}(x) \mathrm{d}x.$$

Therefore, (6) takes the form

$$c(x(p) - EX_{j:n}) = \frac{\int_{x(p)}^{\infty} x \mathrm{d}F(x)}{\bar{F}(x(p))} - x(p) - EZ_j,$$

where $\bar{F}(x) = 1 - F(x)$. Taking into account that

$$c(x(p) - EX_{j:n}) = \frac{x(p)\bar{F}(x(p)) + \int_{x(p)}^{\infty} \bar{F}(x) \mathrm{d}x}{\bar{F}(x(p))} - x(p) - EZ_j,$$

we get

$$c(x(p) - EX_{j:n}) = \frac{\int_{x(p)}^{\infty} \bar{F}(x) \mathrm{d}x}{\bar{F}(x(p))} - EZ_j. \tag{7}$$

By differentiating both sides of (7), we obtain

$$cx'(p) = \frac{-x'(p)\bar{F}^2(x(p)) + f(x(p))x'(p) \int_{x(p)}^{\infty} \bar{F}(x) \mathrm{d}x}{\bar{F}^2(x(p))}.$$

The above equality can be rewritten in the following form:

$$-\frac{(c+1)\bar{F}(x(p))x'(p)}{\int_{x(p)}^{\infty} \bar{F}(x) \mathrm{d}x} = -\frac{f(x(p))x'(p)}{\bar{F}(x(p))}. \tag{8}$$

From the above it follows that $c + 1 > 0$. Now integrating both sides of (8), we have

$$(c+1)\ln \int_{x(p)}^{\infty} \bar{F}(x)\mathrm{d}x = \ln\ \bar{F}(x(p)) + \ln A,\quad A > 0.$$

Hence, we obtain

$$\int_{x(p)}^{\infty} \bar{F}(x)\mathrm{d}x = (A\bar{F}(x(p)))^{\frac{1}{c+1}}. \tag{9}$$

Letting $x(p) = y$ in (9), we note that

$$\int_{y}^{\infty} \bar{F}(x)\mathrm{d}x = (A\bar{F}(y)))^{\frac{1}{c+1}}. \tag{10}$$

Thus, we have

$$\bar{F}(y) = \frac{A^{\frac{1}{c+1}}}{c+1} f(y)(\bar{F}(y))^{\frac{-c}{c+1}}.$$

Now consider the following cases:

Case 1. $-\frac{c}{c+1} = 1\left(c = -\frac{1}{2}\right)$. Then

$$f(x) = \text{constant},$$

i.e., we obtain a uniform distribution on an interval (a, b), where $b - a = \frac{A^2}{2}$.

Case 2. $c \neq -\frac{1}{2}$. Then

$$(1 - F(x))^{\frac{2c+1}{c+1}} = \frac{A^{\frac{1}{c+1}}}{c+1}\frac{\mathrm{d}F(x)}{\mathrm{d}x},$$

which gives

$$\int \mathrm{d}x = \frac{A^{\frac{1}{c+1}}}{c+1}\int (1 - F(x))^{-\frac{2c+1}{c+1}}\mathrm{d}F(x). \tag{11}$$

Assume that $\frac{2c+1}{c+1} = 1\ (c = 0)$. Then

$$\int \mathrm{d}x = A\int (1 - F(x))^{-1}\mathrm{d}F(x),$$

which gives

$$-x = A\ \ln(1 - F(x)) + B.$$

Let $S = (x_{\min}, x_{\max})$ be the support of the distribution F. For $x = x_{\min}$, we have $F(x_{\min}) = 0$ and $-x_{\min} = B$. Hence,

$$\ln(1 - F(x)) = -\frac{x - x_{\min}}{A},$$

which gives

$$F(x) = 1 - e^{-\frac{x - x_{\min}}{A}},\quad \text{for } x > x_{\min}, A > 0 \text{ (exponential case)}.$$

Case 3. $c \neq 0$. From (11) we get

$$x = \frac{A^{\frac{1}{c+1}}}{c}(1 - F(x))^{-\frac{c}{c+1}} + B.$$

For $-\frac{c}{c+1} > 0$ $(c \neq -\frac{1}{2}, c \in (-1, 0))$, we have

$$F(x) = 1 - \left(\frac{x - B}{\frac{A^{\frac{1}{c+1}}}{c}} \right)^{-\frac{c+1}{c}}.$$

But

$$x_{\min} = \frac{A^{\frac{1}{c+1}}}{c} + B, \quad x_{\max} = B$$

and

$$F(x) = 1 - \left(\frac{x - x_{\max}}{x_{\min} - x_{\max}} \right)^{-\frac{c+1}{c}} = 1 - \left(\frac{x_{\max} - x}{x_{\max} - x_{\min}} \right)^{-\frac{c+1}{c}}.$$

Thus, we can write

$$F(x) = \begin{cases} 1 - \left(\frac{v - x}{v - \mu} \right)^{\theta}, & -\infty < \mu < x < v < \infty, \quad \theta > 0; \\ 0 & \text{elsewhere (Pearson case).} \end{cases}$$

Case 4. $-\frac{c}{c+1} < 0$ $(c > 0)$. Then we have

$$F(x) = 1 - \left(\frac{\frac{A^{\frac{1}{c+1}}}{c}}{x - B} \right)^{\frac{c+1}{c}}$$

and consequently

$$F(x) = \begin{cases} 1 - \left(\frac{\mu + \gamma}{x + \gamma} \right)^{\theta}, & x > \mu, \ \theta > 0, \ \gamma > -\infty; \\ 0 & \text{elsewhere (Pareto case).} \end{cases}$$

□

3 Final remarks

The characterization condition of Wang and Shrivastava [2] does not include the case of uniform distribution on $(0, 1)$, which we get for $c = -\frac{1}{2}$. The characterization of power distribution via covariances is presented also by Pawlas and Szynal [8].

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