Research Article

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Almost graded multiplication and almost graded comultiplication modules

Abstract: Let G be a group with identity e, R be a G-graded commutative ring with a nonzero unity 1 and M be a G-graded R-module. In this article, we introduce and study the concept of almost graded multiplication modules as a generalization of graded multiplication modules; a graded R-module M is said to be almost graded multiplication if whenever $a \in h(R)$ satisfies $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(M)$, then $(0 :_M a) = \{0\}$. Also, we introduce and study the concept of almost graded comultiplication modules as a generalization of graded comultiplication modules; a graded R-module M is said to be almost graded comultiplication if whenever $a \in h(R)$ satisfies $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(M)$, then $aM = M$. We investigate several properties of these classes of graded modules.

Keywords: graded multiplication module, graded comultiplication module

MSC 2020: 16W50, 13A02

1 Introduction

Throughout this article, G will be a group with identity e and R a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \bigoplus_{g \in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Also, we set $h(R) = \bigcup_{g \in G} R_g$. Moreover, it has been proved in [1] that R_e is a subring of R and $1 \in R_e$. Let I be an ideal of a graded ring R. Then I is said to be graded ideal if $I = \bigoplus_{g \in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g \in G} x_g$, where $x_g \in I$ for all $g \in G$. An ideal of a graded ring need not be graded; see the following example:

Example 1.1. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $I = \langle 1 + i \rangle$ is an ideal of R with $1 + i \in I$. If I is G-graded, then $1 \in I$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbb{Z}$. Thus, $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not G-graded.

Let R be a G-graded ring and I be a graded ideal of R. Then R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$.

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, we set $h(M) = \bigcup_{g \in G} M_g$.

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Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g \in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g \in G} x_g$, where $x_g \in N$ for all $g \in G$. An R-submodule of a graded R-module need not be graded. Let M be a G-graded R-module and N be a graded R-submodule of M. Then M/N is a graded R-module by $(M/N)_g = (M_g + N)/N$ for all $g \in G$.

Lemma 1.2. [2, Lemma 2.1] *Let R be a G-graded ring and M be a G-graded R-module.*
(1) *If N and K are graded R-submodules of M, then $N + K$ and $N \cap K$ are graded R-submodules of M.*
(2) *If N is a graded R-submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded R-submodules of M. Moreover, $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*

A graded R-module M is said to be graded multiplication if for every graded R-submodule N of M, $N = IM$ for some graded deal I of R. In this case, it is known that $I = (N :_R M)$. Graded multiplication modules were first introduced and studied by Escoriza and Torrecillas in [3], and further results were obtained by several authors, see for example [4]. In [5], Atani introduced the concept of graded prime submodules; a proper graded R-submodule N of a graded R-module M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $r \in (N : M)$ or $m \in N$. A graded R-module M is said to be graded prime if $\{0\}$ is a graded prime R-submodule of M. The set of all graded prime submodules of M is denoted by GSpec(M). In [6], a graded R-module M is said to be graded weak multiplication if GSpec(M) = ϕ or for every graded prime R-submodule N of M, $N = IM$ for some graded deal I of R.

Graded semiprime submodules have been introduced by Lee and Varmazyar in [7]. A proper graded R-submodule N of M is said to be graded semiprime if whenever I is a graded ideal of R and K is a graded R-submodule of M such that $I^n K \subseteq N$ for some positive integer n, then $IK \subseteq N$. A graded R-module M is said to be graded semiprime if $\{0\}$ is a graded semiprime R-submodule of M. Graded semiprime submodules are also studied in [8]. The set of all graded semiprime R-submodules of M is denoted by GSSpec(M). Motivated from the concepts of graded multiplication modules in [3] and graded weak multiplication modules in [6], a new class of graded R-modules has been introduced in [9], called graded semiprime multiplication modules. A graded R-module M is said to be graded semiprime multiplication if GSSpec(M) = $\varnothing$ or for every graded semiprime R-submodule N of M, $N = IM$ for some graded ideal I of R.

In [10], Atani introduced the concept of graded weakly prime submodules over graded commutative rings; where a graded proper R-submodule N of a graded R-module M is said to be graded weakly prime R-submodule of M if whenever $r \in h(R)$ and $m \in h(M)$ such that $0 \neq rm \in N$, then either $m \in N$ or $r \in (N :_R M)$. One can easily see that every graded prime submodule is graded weakly prime. However, the converse is not true in general; for example, $\{0\}$ is graded weakly prime submodule by definition but $\{0\}$ need not be graded prime submodule. In [11], several results on graded weakly prime submodules have been proved and investigated to introduce the concept of graded quasi multiplication modules; a graded R-module M is said to be graded quasi multiplication if for every graded weakly prime R-submodule N of M, $N = IM$ for some graded ideal I of R.

A proper graded R-submodule N of a graded R-module M is said to be graded 2-absorbing if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. The set of all graded 2-absorbing R-submodules of M is denoted by GABSpec(M). This concept has been first introduced and studied in [12], and then generalized into graded n-absorbing submodules in [13]. In [11], a parallel study given in [9] has been followed to investigate the new class of graded absorbing multiplication modules, by first providing many interesting results on graded 2-absorbing submodules. A graded R-module M is said to be graded absorbing multiplication if GABSpec(M) = ϕ or for every graded 2-absorbing R-submodule N of M, $N = IM$ for some graded deal I of R.

So, most of all generalizations for graded multiplication modules were fixing on changing the graded R-submodule N from a general graded submodule to graded prime, graded semiprime, graded weakly prime or graded 2-absorbing submodule. In [14], an R-module M is said to be a quasi multiplication module if whenever $\text{Ann}_R(rM) = \text{Ann}_R(M)$ for each $r \in R$, then $(0 :_M r) = \{0\}$. In this article, we follow [14] to explore another technique to generalize the concept of graded multiplication modules. First, we need to introduce the following:

Proposition 1.3. *Let M be a G-graded R-module and N a graded R-submodule of M. If I is a graded ideal of R, then $(N :_M I) = \{m \in M : Im \subseteq N\}$ is a graded R-submodule of M.*

Proof. Clearly, $(N :_M I)$ is an R-submodule of M. Let $m \in (N :_M I)$. Then $Im \subseteq N$. Now, $m = \sum_{g \in G} m_g$, where $m_g \in M_g$ for all $g \in G$. Let $x \in I$. Then $x_g \in I$ for all $g \in G$ since I is graded. Assume that $h \in G$. Then $x_h m_g \in M_{hg} \subseteq h(M)$ for all $g \in G$ such that $\sum_{g \in G} x_h m_g = x_h \left(\sum_{g \in G} m_g \right) = x_h m \in N$. Since N is graded, $x_h m_g \in N$ for all $g \in G$ which implies that $\sum_{h \in G} x_h m_g \in N$ for all $g \in G$, and then $x m_g \in N$ for all $g \in G$. So, $Im_g \subseteq N$ for all $g \in G$, and hence $m_g \in (N :_M I)$ for all $g \in G$. Therefore, $(N :_M I)$ is a graded R-submodule of M. □

Similarly, one can prove the following:

Proposition 1.4. *Let M be a G-graded R-module and $a \in h(R)$. Then $(0 :_M a) = \{m \in M : am = 0\}$ is a graded R-submodule of M.*

In this article, we introduce and study the concept of almost graded multiplication modules; a graded R-module M is said to be almost graded multiplication if whenever $a \in h(R)$ satisfies $\text{Ann}_R(aM) = \text{Ann}_R(M)$, then $(0 :_M a) = \{0\}$.

Graded comultiplication modules have been introduced by Toroghy and Farshadifar in [15]; a graded R-module M is said to be graded comultiplication if for every graded R-submodule N of M, $N = (0 :_M I)$ for some graded ideal I of R, or equivalently, $N = (0 :_M \text{Ann}_R(N))$. A generalization for graded comultiplication modules has been introduced and studied in [16]; a graded R-module M is said to be graded weak comultiplication if for every graded prime R-submodule N of M, $N = (0 :_M I)$ for some graded ideal I of R. In [14], an R-module M is said to be a quasi comultiplication module if whenever $\text{Ann}_R(rM) = \text{Ann}_R(M)$ for each $r \in R$, then $rM = M$. In this article, we follow [14] to introduce and study the concept of almost graded comultiplication modules; a graded R-module M is said to be almost graded comultiplication if whenever $a \in h(R)$ satisfies $\text{Ann}_R(aM) = \text{Ann}_R(M)$, then $aM = M$.

2 Almost graded multiplication modules

In this section, we introduce and study the concept of almost graded multiplication modules as a generalization of graded multiplication modules.

Definition 2.1. A graded R-module M is said to be almost graded multiplication if whenever $a \in h(R)$ satisfies $\text{Ann}_R(aM) = \text{Ann}_R(M)$, then $(0 :_M a) = \{0\}$.

Proposition 2.2. *Every graded multiplication module is almost graded multiplication.*

Proof. Let M be a graded multiplication R-module. Suppose that $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Then $(0 :_M a)$ is a graded R-submodule of M, and then there exists a graded ideal I of R such that $(0 :_M a) = IM$. It follows that $I \subseteq \text{Ann}_R(aM)$. So, $I \subseteq \text{Ann}_R(M)$ and hence $(0 :_M a) = \{0\}$. Thus, M is an almost graded multiplication R-module. □

The next example shows that the converse of Proposition 2.2 is not true in general.

Example 2.3. Consider $R = \mathbb{Z}$, $M = (\mathbb{Z}_2 \oplus \mathbb{Z}_2)[i]$ and $G = \mathbb{Z}_2$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $M_1 = i(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. Clearly, $\text{Ann}_R(M) = 2\mathbb{Z}$. Let $a \in h(R)$. If a is even, then $\text{Ann}_R(aM) = \text{Ann}_R(\{0\}) = R = \mathbb{Z} \neq \text{Ann}_R(M)$. If a is odd, then $\text{Ann}_R(aM) = 2\mathbb{Z} = \text{Ann}_R(M)$, and $(0 :_M a) = \{0\}$. Hence, M is almost graded multiplication. On the other hand, $N = \mathbb{Z}_2 \oplus \{0\}$ is a graded R-submodule of M such that $(N :_R M)M = \{0\} \neq N$, so M is not graded multiplication.

Proposition 2.4. *Let M be a graded R-module. If $(0 :_M a) = ((0 :_M a) :_R M)M$ for each $a \in h(R)$, then M is an almost graded multiplication R-module.*

Proof. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Then $(0 :_M a) = ((0 :_M a) :_R M)M = \text{Ann}_R(aM)M = \text{Ann}_R(M)M = \{0\}$. Therefore, M is an almost graded multiplication R-module. □

Proposition 2.5. *Let M be a graded R-module. If $(Rx :_R M) = \text{Ann}_R(M)$ implies $Rx = \{0\}$ for every $x \in h(M)$, then M is an almost graded multiplication R-module.*

Proof. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Suppose that $x \in (0 :_M a)$. Then $x_g \in (0 :_M a)$ for all $g \in G$ since $(0 :_M a)$ is graded. Let $g \in G$. Then $(Rx_g :_R M) \subseteq ((0 :_M a) :_R M) = \text{Ann}_R(aM) = \text{Ann}_R(M)$. Thus, $(Rx_g :_R M) = \text{Ann}_R(M)$ as the reverse inclusion is obvious. Now by assumption, $Rx_g = \{0\}$, which implies that $x_g = 0$ for all $g \in G$ as $1 \in R$. So, $x = \sum_{g \in G} x_g = 0$. Hence, $(0 :_M a) = \{0\}$, so M is an almost graded multiplication R-module. □

Let M be a G-graded R-module. Then the set of all homogeneous zero divisors of M is given by

$$Z_G(M) = \{m \in h(M) : rm = 0 \text{ for some nonzero } r \in h(R)\}.$$

The dual notion of $Z_G(M)$ is given by

$$W_G(M) = \{a \in h(R) : aM \neq M\}.$$

Let M and S be two G-graded R-modules. An R-homomorphism $f : M \to S$ is said to be graded R-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$. In [17], a graded R-module M is said to be graded Hopfian (resp. graded co-Hopfian) if every surjective (resp. injective) graded R-endomorphism of M is a graded R-isomorphism.

Proposition 2.6. *Let M be a G-graded R-module. If M is graded co-Hopfian, then $W_G(M) = Z_G(R/\text{Ann}_R(M))$ if and only if M is almost graded multiplication.*

Proof. Suppose that $W_G(M) = Z_G(R/\text{Ann}_R(M))$. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. If $aM \neq M$, then $a \in W_G(M)$, and then by assumption, there exists $b \in h(R) - \text{Ann}_R(M)$ such that $ab \in \text{Ann}_R(M)$. Hence, $b \in \text{Ann}_R(aM) = \text{Ann}_R(M)$, which is a contradiction. So, $aM = M$, and then $(0 :_M a) = \{0\}$ as M is graded co-Hopfian. Therefore, M is almost graded multiplication. Conversely, certainly, $Z_G(R/\text{Ann}_R(M)) \subseteq W_G(M)$. Let $a \in W_G(M)$. Then $a \in h(R)$ such that $aM \neq M$. Now, as M is graded co-Hopfian, $(0 :_M a) \neq \{0\}$. So, by assumption, we can choose $b \in \text{Ann}_R(aM) - \text{Ann}_R(M)$. It follows that $a \in Z_G(R/\text{Ann}R(M))$, as needed. □

Proposition 2.7. *Let M be a G-graded R-module. If $b \in h(R) - W_G(M)$ such that bM is an almost graded multiplication R-module, then M is an almost graded multiplication R-module.*

Proof. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Then $ab \in h(R)$ such that $\text{Ann}_R(bM) = \text{Ann}_R(abM)$, and then by assumption, we have that $(0 :_{bM} a) = \{0\}$. Let $x \in (0 :_M a)$. Then $ax = 0$. Since $b \notin W_G(M)$, $bM = M$, and then $x = by$ for some $y \in M$. Hence, $aby = 0$, which implies that $by \in (0 :_{bM} a) = \{0\}$. Therefore, $x = by = 0$, and hence $(0 :_M a) = \{0\}$. Thus, M is an almost graded multiplication R-module. □

Proposition 2.8. *Every graded prime R-module is an almost graded multiplication R-module.*

Proof. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Suppose that $x \in (0 :_M a)$. Then $ax = 0$. Since $\{0\}$ is a graded prime R-submodule of M, either $x = 0$ or $a \in (0 :_R M)$. If $a \in (0 :_R M)$, then $aM = \{0\}$, and then $\text{Ann}_R(M) = \text{Ann}_R(aM) = \text{Ann}_R(\{0\}) = R$, which means that $RM = \{0\}$. Again since $\{0\}$ is a graded prime R-submodule of M, either $M = \{0\}$ or $R \subseteq (0 :_R M)$. If $R \subseteq (0 :_R M)$, then $R = (0 :_R M)$ which is impossible since $(0 :_R M)$ is a graded prime ideal of R by [5, Lemma 2.1]. So, $x = 0$, as needed. □

Remark 2.9. The converse of Proposition 2.8 is not true in general, because if it is true, then by Proposition 2.2, we have that every graded multiplication module is graded prime which is not true, as {0} is a graded multiplication module which is not a graded prime module.

The next proposition shows that the converse of Proposition 2.8 will be true if R is integral domain and M is faithful.

Proposition 2.10. *Let M be an almost graded multiplication R-module. If R is an integral domain and M is faithful, then M is a graded prime R-module.*

Proof. Let $a \in h(R)$ such that $(0 :_M a) \neq M$. Clearly, $\text{Ann}_R(M) \subseteq \text{Ann}_R(aM)$. Now, let $b \in \text{Ann}_R(aM)$. Then $ba \in \text{Ann}_R(M) = \{0\}$. Since R is an integral domain and $(0 :_M a) \neq M$, we have that $b \in \text{Ann}_R(M)$. So, $\text{Ann}_R(aM) \subseteq \text{Ann}_R(M)$. Thus, $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Hence, $(0 :_M a) = \{0\}$ since M is almost graded multiplication. Therefore, M is a graded prime R-module. □

Graded second modules have been introduced by Ansari-Toroghy and Farshadifar in [18]; a graded R-module M is said to be graded second if $M \neq \{0\}$ and for each $a \in h(R)$, the graded R-homomorphism $f : M \to M$ defined by $f(x) = ax$ is either surjective or zero. Graded second submodules have been wonderfully studied by Çeken and Alkan in [19]. The next proposition shows that the converse of Proposition 2.8 will be true if M is a graded second R-module.

Proposition 2.11. *Let M be an almost graded multiplication R-module. If M is graded second, then M is a graded prime R-module.*

Proof. Let $a \in h(R)$. Then by assumption, $aM = \{0\}$ or $\text{Ann}_R(M) = \text{Ann}_R(M/(0 :_M a)) = \text{Ann}_R(aM)$. Hence, $aM = \{0\}$ or $(0 :_M a) = \{0\}$, as needed. □

3 Almost graded comultiplication modules

In this section, we introduce and study the concept of almost graded comultiplication modules as a generalization of graded comultiplication modules.

Definition 3.1. A graded R-module M is said to be almost graded comultiplication if whenever $a \in h(R)$ satisfies $\text{Ann}_R(aM) = \text{Ann}_R(M)$, then $aM = M$.

Proposition 3.2. *Every graded comultiplication module is almost graded comultiplication.*

Proof. Let M be a graded comultiplication R-module. Assume that $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Now, aM is a graded R-submodule of M, so as M is graded comultiplication, we have that $aM = (0 :_M I)$ for some graded ideal of R, and then $IaM = \{0\}$, which means that $I \subseteq \text{Ann}_R(aM) = \text{Ann}_R(M)$. Let $x \in M$. Then $Ix = \{0\}$, and then $x \in (0 :_M I) = aM$. So, $M \subseteq aM$, but clearly, $aM \subseteq M$, so $aM = M$. Hence, M is an almost graded comultiplication R-module. □

The next example shows that the converse of Proposition 3.2 is not true in general.

Example 3.3. Consider the details of Example 2.3. Similarly, one can prove that M is almost graded comultiplication. On the other hand, $N = \mathbb{Z}_2 \oplus \{0\}$ is a graded R-submodule of M such that $(0 :_M \text{Ann}_R(N)) = M \neq N$, so M is not graded comultiplication.

The next two examples prove that the concepts of almost graded multiplication modules and almost graded comultiplication modules are totally different. The next example shows that not every almost graded multiplication module is almost graded comultiplication.

Example 3.4. Consider $R = \mathbb{Z}$, $M = (\mathbb{Z} \oplus \mathbb{Z})[i]$ and $G = \mathbb{Z}_2$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z} \oplus \mathbb{Z}$ and $M_1 = i(\mathbb{Z} \oplus \mathbb{Z})$. Now, for each $a \in h(R)$, $\text{Ann}_R(aM) = \text{Ann}_R(M) = \{0\}$ and $(0 :_M a) = \{0\}$, so we have M is an almost graded multiplication R-module. On the other hand, $a = 2 \in h(R)$ such that $aM \neq M$, which implies that M is not an almost graded comultiplication R-module.

The next example shows that not every almost graded comultiplication module is almost graded multiplication.

Example 3.5. Consider $R = \mathbb{Z}$, $M = (\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty})[i]$ and $G = \mathbb{Z}_2$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and $M_1 = i(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty})$. Now, for each $a \in h(R)$, $\text{Ann}_R(aM) = \text{Ann}_R(M) = \{0\}$ and $aM = M$, so we have M is an almost graded comultiplication R-module. On the other hand, $p \in h(R)$ such that $(0 :_M p) \neq \{0\}$, which implies that M is not an almost graded multiplication R-module.

Proposition 3.6. *Let M be a graded R-module. If $aM = (0 :_M \text{Ann}_R(aM))$ for each $a \in h(R)$, then M is an almost graded comultiplication R-module.*

Proof. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Then we have $aM = (0 :_M \text{Ann}_R(aM)) = (0 :_M \text{Ann}_R(M)) = M$. Hence, M is an almost graded comultiplication R-module. □

The next corollary gives another proof for Proposition 3.2.

Corollary 3.7. *Every graded comultiplication module is almost graded comultiplication.*

Proof. Let M be a graded comultiplication R-module. Then as aM is a graded R-submodule of M for all $a \in h(R)$, we have $aM = (0 :_M \text{Ann}_R(aM))$ for each $a \in h(R)$, and then by Proposition 3.6, M is an almost graded comultiplication R-module. □

In [20], a proper $\mathbb{Z}$-graded R-submodule N of M is said to be graded completely irreducible if whenever $N = \bigcap_{k \in \Delta} N_k$ where $\{N_k\}_{k \in \Delta}$ is a family of $\mathbb{Z}$-graded R-submodules of M, then $N = N_k$ for some $k \in \Delta$. In [21], the concept of graded completely irreducible submodules has been extended into G-graded case, for any group G. It has been proved that every graded R-submodule of M is an intersection of graded completely irreducible R-submodules of M. In many instances, we use the following basic fact without further discussion.

Remark 3.8. Let N and L be two graded R-submodules of M. To prove that $N \subseteq L$, it is enough to prove that if K is a graded completely irreducible R-submodule of M such that $L \subseteq K$, then $N \subseteq K$.

Proposition 3.9. *Let M be a graded R-module. If $\text{Ann}_R(K) = \text{Ann}_R(M)$ implies that $K = M$ for every graded completely irreducible R-submodule K of M, then M is an almost graded comultiplication R-module.*

Proof. Let $a \in h(R)$ such that $\text{Ann}_R(aM) = \text{Ann}_R(M)$. Let K be a graded completely irreducible R-submodule of M such that $aM \subseteq K$. Then $\text{Ann}_R(K) \subseteq \text{Ann}_R(aM) = \text{Ann}_R(M)$, and then $\text{Ann}_R(K) = \text{Ann}_R(M)$. So, by assumption, $K = M$. Therefore, $aM = M$ by Remark 3.8. Thus, M is an almost graded comultiplication R-module. □

Proposition 3.10. *Let M be a graded R-module. If $b \in h(R) - Z_G(M)$ such that bM is an almost graded comultiplication R-module, then M is an almost graded comultiplication R-module.*

Proof. Let $a \in h(R)$ such that $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(aM)$. Then $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(abM)$. So, by assumption, $bM = abM$. Now, let $x \in M$. Then $bx = aby$ for some $y \in M$. So, $b(x - ay) = 0$. Since $b \notin Z_G(M)$, $x = ay$ and so $M \subseteq aM$, as required. $\square$

Proposition 3.11. *Let M be an almost graded comultiplication R-module. If R is an integral domain and M is faithful, then M is a graded second R-module.*

Proof. Let $a \in h(R)$ such that $aM \neq \{0\}$. Since R is an integral domain and M is faithful, we have $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(Ra) = \mathrm{Ann}_R(M) = \{0\}$, which implies that $aM = M$ as M is almost graded comultiplication. $\square$

Proposition 3.12. *Let M be an almost graded comultiplication R-module. If M is graded prime, then M is a graded second R-module.*

Proof. Let $a \in h(R)$ such that $aM \neq \{0\}$. Then $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(aM)$, and then $M = aM$, as needed. $\square$

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