Research Article

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Structure of *n*-quasi left *m*-invertible and related classes of operators

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Abstract: Given Hilbert space operators $T, S \in B(\mathcal{H})$, let Δ and $\delta \in B(B(\mathcal{H}))$ denote the elementary operators $\Delta_{T,S}(X) = (L_T R_S - I)(X) = TXS - X$ and $\delta_{T,S}(X) = (L_T - R_S)(X) = TX - XS$. Let $d = \Delta$ or δ. Assuming T commutes with S^*, and choosing X to be the positive operator $S^{*n}S^n$ for some positive integer n, this paper exploits properties of elementary operators to study the structure of n-quasi $[m, d]$-operators $d^m_{T,S}(X) = 0$ to bring together, and improve upon, extant results for a number of classes of operators, such as n-quasi left m-invertible operators, n-quasi m-isometric operators, n-quasi m-self-adjoint operators and n-quasi (m, C) symmetric operators (for some conjugation C of $\mathcal{H}$). It is proved that S^n is the perturbation by a nilpotent of the direct sum of an operator $S_1^n = (S|_{\overline{S^n(\mathcal{H})}})^n$ satisfying $d^m_{T_1,S_1}(I_1) = 0$, $T_1 = T|_{\overline{S^n(\mathcal{H})}}$, with the 0 operator; if S is also left invertible, then S^n is similar to an operator B such that $d^m_{B^*,B}(I) = 0$. For power bounded S and T such that $ST^* - T^*S = 0$ and $\Delta_{T,S}(S^{*n}S^n) = 0$, S is polaroid (i.e., isolated points of the spectrum are poles). The product property, and the perturbation by a commuting nilpotent property, of operators T, S satisfying $d^m_{T,S}(I) = 0$, given certain commutativity properties, transfers to operators satisfying $S^{*n}d^m_{T,S}(I)S^n = 0$.

Keywords: Hilbert space, elementary operators, n-quasi m-left invertible operator, poles, product of operators, perturbation by nilpotents

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1 Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach space $\mathcal{X}$ (resp., Hilbert space $\mathcal{H}$) into itself. Given operators $T, S \in B(\mathcal{X})$, let L_T and $R_S \in B(B(\mathcal{X}))$ denote, respectively, the operators

$$L_T(X) = TX, R_S(X) = XS$$

of left multiplication by T and right multiplication by S. The elementary operators $\Delta_{T,S}$ and $\delta_{T,S} \in B(B(\mathcal{X}))$ are then defined by

$$\Delta_{T,S}(X) = (L_T R_S - I)(X) = TXS - X$$

and

$$\delta_{T,S}(X) = (L_T - R_S)(X) = TX - XS.$$

Let $d_{T,S} \in B(B(\mathcal{X}))$ denote either of the operators $\Delta_{T,S}$ and $\delta_{T,S}$. Let I denote the identity of $B(\mathcal{X})$ and let $m \geq 1$ be some integer. Then

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$$\Delta^m_{T,S}(I) = \Delta_{T,S}(\Delta^{m-1}_{T,S})(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j} \tag{1}$$

and

$$\delta^m_{T,S}(I) = \delta_{T,S}(\delta^{m-1}_{T,S}(I)) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{j}. \tag{2}$$

We say in the following that an operator $S \in B(\mathcal{X})$ is an $m-(d,T)$ operator if $d^m_{T,S}(I) = 0$. Examples of $m-(d,T)$ operators $S \in B(\mathcal{X})$ occur quite naturally. Thus, if an operator $S \in B(\mathcal{X})$ is *m-left invertible* by $T \in B(\mathcal{X})$, then

$$\Delta^m_{T,S}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j} = 0$$

[1–3]; if $S \in B(\mathcal{X})$ is *m-isometric*, then

$$\Delta^m_{S^*,S}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}S^{*(m-j)}S^{m-j} = 0$$

[4–6]; if $S \in B(\mathcal{H})$ is *m-self-adjoint*, then

$$\delta^m_{S^*,S}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}S^{*(m-j)}S^{j} = 0$$

[7]; and if $S \in B(\mathcal{H})$ is *(m, C)-isometric* for some conjugation C of $\mathcal{H}$, then

$$\delta^m_{S^*,CSC}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}S^{*(m-j)}CS^{j}C = 0$$

[8]. Operators $S \in m-(d,T)$, in particular the classes consisting of m-isometric and (m,C)-isometric operators [9], have been studied in a number of papers in the recent past (see cited references for further references). A generalization of the class consisting of m-isometric (resp., (m,C)-isometric) operators which has drawn some attention in the recent past is that of the n-quasi m-isometric (resp., n-quasi (m,C)-isometric) operators, where an operator $S \in B(\mathcal{H})$ is said to be n-quasi m-isometric (resp., n-quasi (m,C)-isometric) for some integer $n \geq 1$ if $S^{*n}\Delta^m_{S^*,S}(I)S^n = \Delta^m_{S^*,S}(S^{*n}S^n) = 0$ (respectively, $S^{*n}\Delta_{S^*,CSC}(I)S^n = 0$) [10,11]. In keeping with current terminology [10–12], we say in the following that an operator $S \in B(\mathcal{H})$ *is n-quasi* $[m,d]$*-intertwined by* $T \in B(\mathcal{H})$ (*equivalently, T is an n-quasi* $[m,d]$*-intertwining of* S) for some integer $n \geq 1$ if

$$S^{*n}d^m_{T,S}(I)S^n = 0.$$

It is immediate from the definition that if $S \in B(\mathcal{H})$ is n-quasi $[m,d]$-intertwined by T, $[S,T^*] = ST^* - T^*S = 0$ (thus $S^{*n}d^m_{T,S}(I)S^n = d^m_{T,S}(S^{*n}S^n) = 0$), $T_1^* = T^*|_{\overline{S^n(\mathcal{H})}}$ and $S_1 = S|_{\overline{S^n(\mathcal{H})}}$, then $d^m_{T_1,S_1}(I_1) = 0$. Choosing $T = S^*$, we prove in the following that if $S^{*n}d^m_{S^*,S}(I)S^n = 0$ and if $d = \Delta$ (resp., $d = \delta$ and S is injective), then there exist a positive operator Q and an operator A such that $\Delta^m_{A^*,A}(Q) = 0$ and S^n is similar to A (resp., $\delta^m_{A^*,A}(Q) = 0$ and $\delta_{A,S^n}(P) = 0$, P a quasi-affinity). Furthermore, if S is left invertible, then there exists an operator $B \in B(\mathcal{H})$ such that S^n is similar to B and $d^m_{B^*,B}(I) = 0$.

Left m-invertible Banach space (as also m-isometric, m-self-adjoint Hilbert space [7]) operators are known to satisfy the properties that: if $S_i, T_i \in B(\mathcal{X})$, $i = 1,2$, are such that S_i is left m_i-invertible by T_i and $[S_1,S_2] = 0 = [T_1,T_2]$, then S_1S_2 is left (m_1+m_2-1)-invertible by T_1T_2; if $S_1 \in B(\mathcal{X})$ is left m_1-invertible by $T_1 \in B(\mathcal{X})$ and $N_1 \in B(\mathcal{X})$ is an n_1-nilpotent operator which commutes with S_1, then $S_1 + N_1$ is left (m_1+n_1-1)-invertible by T_1 [3]. These results, which hold equally well for $[m,d]$-intertwinings, have extensions to n-quasi $[m,d]$-intertwining (Hilbert space) operators S, T. Let us say that $S_1 \in B(\mathcal{H})$ *is* $n(S)$*-quasi* $[m,d]$*-intertwined by* $T_1 \in B(\mathcal{H})$ *for some operator* $S \in B(\mathcal{H})$ *if*

$$S^{*n}d^m_{T_1,S_1}(I)S^n = 0.$$

We prove that if $S_i, T_i \in B(\mathcal{H})$ $(i = 1, 2)$ are some operators such that S_1 is $n(S)$-quasi $[m_1, d]$-intertwined by T_1, S_2 is $[m_2, d]$-intertwined by T_2, $[S_1, S_2] = 0 = [T_1, T_2]$ and $[S, S_i] = 0 = [S, T_i^*]$ $(i = 1, 2)$, then S_1S_2 is $n(S)$-quasi $[(m_1 + m_2 - 1), d]$-intertwined by T_1T_2. For an n-quasi m_1-isometric $S \in B(\mathcal{H})$ and an m_2-isometric $T \in B(\mathcal{H})$ such that S, T commute, this implies that ST is an n-quasi $(m_1 + m_2 - 1)$-isometry. Again, if S is $n(S)$-quasi $[m, d]$-intertwined by T, $N_i \in B(\mathcal{H})$ are nilpotent operators $(i = 1, 2)$, $[S, N_1] = 0 = [S, T^*]$, $[N_2, T] = 0 = [S, N_2^*]$ and S is injective in the case in which $d = \delta$, then $(S^* + N_1^*)^{n+n_1-1}d^{m+n_1+n_2-2}_{T+N_2,S+N_1}(I)(S + N_1)^{n+n_1-1} = 0$. Translated to left invertible n-quasi m-isometric operators $S \in B(\mathcal{H})$ such that S commutes with an n_1-nilpotent operator $N \in B(\mathcal{H})$, this implies that there exists an m-isometric operator $B \in B(\mathcal{H})$ such that $(S + N)^{n+n_1-1}$ is similar to B.

Recall that a Banach space operator $A \in B(\mathcal{X})$ is *polaroid* if the isolated points of the spectrum of A, points $\in \operatorname{iso}\sigma(A)$, are poles of (the resolvent of) A. It is known, [6, Theorem 2.4], that contractive (more generally, power bounded) m-isometric Banach space operators S (i.e., contractions, respectively, power bounded, $S \in B(\mathcal{X})$ such that $\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S^{m-j}x\|^2 = 0$ for all $x \in \mathcal{X}$) are isometric, hence polaroid. This result extends to power bounded $S, T \in B(\mathcal{X})$ such that $\Delta^m_{T,S}(I) = 0$. We prove in the following that the nth power (hence the operator itself) of an n-quasi m-isometric operator in $B(\mathcal{H})$ is polaroid whenever it is a contraction (more generally, power bounded). Indeed, we prove more: Power bounded operators $S, T \in B(\mathcal{H})$ such that $[S, T^*] = 0$ and $\Delta^m_{T,S}(S^{*n}S^n) = 0$ are polaroid.

The rest of this paper is organized as follows. We introduce our notation/terminology, along with some complementary results, in Section 2. Here we have a first look at the structure of n-quasi $[m, d]$-operators. Section 3 is devoted to proving the polaroid property for n-quasi left m-invertible operators, Section 4 considers the product of an n-quasi $[m_1, d]$-operator with an $[m_2, d]$-operator and Section 5 deals with perturbation by nilpotents. As we point out at various points in the paper, our results represent a considerable improvement upon various extant results.

2 Complementary results

Given a Banach space operator $A \in B(\mathcal{X})$, we denote the isolated points of the spectrum $\sigma(A)$ (resp., the approximate point spectrum $\sigma_a(A)$, the surjectivity spectrum $\sigma_{su}(A)$) of A by $\operatorname{iso}\sigma(A)$ (resp., $\operatorname{iso}\sigma_a(A)$, $\operatorname{iso}\sigma_{su}(A)$). Let $A - \lambda$ denote $A - \lambda I$. The operator A is said to have SVEP, *the single-valued extension property*, at a point λ of the complex plane $\mathbb{C}$ if, for every neighborhood O_λ of λ, the only analytic function $f : O_\lambda \to \mathcal{X}$ satisfying $(A - \mu)f(\mu) = 0$ for all $\mu \in O_\lambda$ is the function $f \equiv 0$; we say that A *has SVEP* if it has SVEP at every $\lambda \in \mathbb{C}$. The ascent $\operatorname{asc}(A)$ (resp., descent $\operatorname{dsc}(A)$) of A is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n\mathcal{X} = A^{n+1}\mathcal{X}$); if no such integer exists, then $\operatorname{asc}(A) = \infty$ (resp., $\operatorname{dsc}(A) = \infty$). It is well known, [13–16], that $\operatorname{asc}(A) < \infty$ implies A has SVEP at 0 and $\operatorname{dsc}(A) < \infty$ implies A^*, the dual operator, has SVEP at 0, and that finite ascent and descent imply their equality. A point $\lambda \in \operatorname{iso}\sigma(A)$ is a pole of (the resolvent of) A if $\operatorname{asc}(A - \lambda) = \operatorname{dsc}(A - \lambda) < \infty$.

For a given operator $A \in B(\mathcal{X})$, let $\Pi_a(A) = \{\lambda \in \operatorname{iso}\sigma_a(A)$: *there exists an integer* $d \geq 1$ *such that* $\operatorname{asc}(A - \lambda) \leq d$ *and* $(A - \lambda)^{d+1}$ *is closed*$\}$ = *set of left poles of* A, and let $\Pi(A) = \{\lambda \in \operatorname{iso}\sigma(A) : \operatorname{asc}(A - \lambda) = \operatorname{dsc}(A - \lambda) < \infty\}$ = *set of poles of* A. Then $\Pi(A) \subseteq \Pi_a(A)$ and a necessary and sufficient condition for $\lambda \in \Pi_a(A)$ to imply $\lambda \in \Pi(A)$ is that A^* has SVEP at λ [13]. *We say that A is polaroid* (*resp. left polaroid*) if $\{\lambda \in \sigma(A) : \lambda \in \operatorname{iso}\sigma(A)\} = \Pi(A)$ (resp., $\{\lambda \in \sigma(A) : \lambda \in \operatorname{iso}\sigma_a(A)\} = \Pi_a(A)$). To every $\lambda \in \operatorname{iso}\sigma(A)$, there corresponds a decomposition

$$\mathcal{X} = H_0(A - \lambda) \oplus K(A - \lambda),$$

where $H_0(A - \lambda)$, the quasinilpotent part of $A - \lambda$, and $K(A - \lambda)$, the analytic core of $A - \lambda$, are the sets

$$H_0(A - \lambda) = \left\{x \in \mathcal{X} : \lim_{n\to\infty} \|(A - \lambda)^n x\|^{\frac{1}{n}} = 0\right\}$$

and

$$K(A-\lambda) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, (A-\lambda)x_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, \ldots\}$$

[13]. $H_0(A-\lambda)$ and $K(A-\lambda)$ are generally non-closed hyperinvariant subspaces of $A-\lambda$ such that $(A-\lambda)^{-p}(0) \subseteq H_0(A-\lambda)$ for all positive integers p and $(A-\lambda)K(A-\lambda) = K(A-\lambda)$. A necessary and sufficient condition for a $\lambda \in \text{iso}\sigma(A)$ to be a pole of A is that $H_0(A-\lambda) = (A-\lambda)^{-n}(0)$ for some integer $n > 0$. (The number n is then said to be the order of the pole at λ; if $n = 1$, then the pole is said to be a simple pole.)

Similarities preserve spectrum (hence, isolated points of the spectrum), the ascent and the descent. Hence: *Similarities preserve the polaroid property*. Recall that an $A \in B(\mathcal{X})$ is an isometry if $\|Ax\| = \|x\|$ for all $x \in \mathcal{X}$. Isometries are normaloid operators, i.e., if an $A \in B(\mathcal{X})$ is isometric, then $\|A\|$ equals the spectral radius $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$. The inverse of an isometry, whenever it exists as a bounded operator, is again an isometry. Since the restriction of an isometry to an invariant subspace is again an isometry, isometries are *totally hereditarily normaloid operators* (see [17]). Conclusion: *Invertible isometries are polaroid* ([17]; see also [15, Theorem 1.5.13]).

Given operators $S, T \in B(\mathcal{X})$, it is seen that

$$\Delta^{m+k}_{T,S}(I) = (L_T R_S - I)^k (\Delta^m_{T,S}(I)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} T^{k-j} \Delta^m_{T,S}(I) S^{k-j}$$

and

$$\delta^{m+k}_{T,S}(I) = (L_T - R_S)^k (\delta^m_{T,S}(I)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} T^{k-j} \delta^m_{T,S}(I) S^{j}$$

for all integers $m, k \ge 1$. Hence:

Lemma 2.1. *If $d^m_{T,S}(I) = 0$, then $d^t_{S,T}(I) = 0$ for all integers $t \ge m$.*

For an operator $S \in B(\mathcal{H})$, let $\overline{S^n(\mathcal{H})}$ denote the closure of the range of S^n, and let $S^{*-n}(0)$ denote the kernel of S^{*n}. If an operator $T \in B(\mathcal{H})$ is such that $[S, T^*] = ST^* - T^*S = 0$, then $\mathcal{H}$ has a direct sum decomposition $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$, and S, T^* have upper triangular representations

$$S = \begin{pmatrix} S_1 & S_0 \\ 0 & S_2 \end{pmatrix}, \quad T^* = \begin{pmatrix} T_1^* & T_0^* \\ 0 & T_2^* \end{pmatrix}, \tag{3}$$

where

$$S_2^n = 0 \quad \text{and} \quad [S_1, T_1^*] = 0.$$

The hypothesis $S^{*n} d^m_{T,S}(I) S^n = 0$ implies that if $d = \Delta$, then

$$\begin{aligned} S^{*n}\Delta^m_{T,S}(I)S^n = 0 &\Leftrightarrow S^{*n}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j}\right\}S^n = 0 \\ &\Leftrightarrow \begin{pmatrix} S_1^{*n} & 0 \\ X^* & 0 \end{pmatrix}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}\begin{pmatrix} T_1^{m-j}S_1^{m-j} & X_{1j} \\ X_{2j} & X_{3j} \end{pmatrix}\right\}\begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix} = 0 \end{aligned}$$

and if $d = \delta$, then

$$\begin{aligned} S^{*n}\delta^m_{T,S}(I)S^n = 0 &\Leftrightarrow S^{*n}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{j}\right\}S^n = 0 \\ &\Leftrightarrow \begin{pmatrix} S_1^{*n} & 0 \\ X^* & 0 \end{pmatrix}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}\begin{pmatrix} T_1^{m-j}S_1^{j} & X_{1j} \\ X_{2j} & X_{3j} \end{pmatrix}\right\}\begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix} = 0 \end{aligned}$$

for some operators X and X_{ij} ($i = 1, 2, 3$). Hence

$$S_1^{*n}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{m-j}S_1^{m-j}\right\}S_1^n = 0 \Leftrightarrow \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{m-j}S_1^{m-j} = 0$$

and

$$S_1^{*n}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{m-j}S_1^{j}\right\}S_1^n = 0 \Leftrightarrow \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{m-j}S_1^{j} = 0,$$

i.e., $d^m_{T_1,S_1}(I) = 0$. Consequently, [2, Remark 2.7] and Lemma 2.1, $d^m_{T_1^p,S_1^p}(I) = 0$ for every integer $p \geq 1$. Hence,

$$S^{*n}d^m_{T^p,S^p}(I)S^n = 0, \quad \text{for all integers } p \geq 1.$$

The observations that

$$\Delta^{m+1}_{A,B}(I) = A\Delta^m_{A,B}(I)B - \Delta^m_{A,B}(I), \quad \delta^{m+1}_{A,B}(I) = A\delta^m_{A,B}(I) - \delta^m_{A,B}(I)B$$

lead to the implication

$$d^m_{T_1,S_1}(I_1) = 0 \Leftrightarrow d^t_{T_1,S_1}(I_1) = 0 \quad \text{for all integers } t \geq m,$$

and hence

$$S^{*n}d^m_{T,S}(I)S^n = 0 \Rightarrow S^{*n}d^t_{T,S}(I)S^n = 0 \quad \text{for all integers } t \geq m.$$

If we let X denote the operator

$$X = \sum_{j=0}^{n-1} S_1^{n-1-j}S_0S_2^j,$$

then

$$S^n = \begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix}.$$

Now if $S^{*n}\Delta^m_{T,S}(I)S^n = 0$, then $\Delta^m_{T_1,S_1}(I_1) = 0$ implies S_1 is (m-left invertible, hence) left invertible. Consequently, if S_1 has a dense range (or, equivalently, S_1^* has SVEP at 0), then the operator S^n is similar to $A = S_1^n \oplus 0$ (with the similarity implemented by the invertible operator $E = \begin{pmatrix} S_1^n & S_1^nX \\ 0 & 1 \end{pmatrix}$). Observe that the operator A is not left m-invertible (i.e., there does not exist an operator $B \in B(\mathcal{H})$ such that $\Delta^m_{B,A}(I) = 0$). Letting $T = S^*$ (so that $S^{*n}\Delta^m_{S^*,S}(I)S^n = 0$ – such operators have been called *n-quasi m-isometric* [11]), it then follows that S_1^n is m-isometric and, if S_1 has a dense range, S^n is similar to A. Operators $S \in B(\mathcal{H})$ for which $\delta^m_{S^*,S}(I) = 0$ are called *m-self-adjoint operators* [7]. If $S^{*n}\delta^m_{S^*,S}(I)S^n = 0$ (i.e., if S is *n-quasi m-self-adjoint*), then (S_1, hence) S_1^p is m-self-adjoint for all integers $p \geq 1$ [7]. More is true, as we prove in the following.

Given a positive operator ($0 \leq$) $Q \in B(\mathcal{H})$, we say that the operator $S \in B(\mathcal{H})$ is $[m, Q]$-isometric (resp., $[m, Q]$-self-adjoint) if $\Delta^m_{S^*,S}(Q) = 0$ (resp., $\delta^m_{S^*,S}(Q) = 0$); we say that $S \in [m, d(Q)]$ if $d^m_{S^*,S}(Q) = 0$, $d = \Delta$ or δ. We assume in the following that $S_1^n = (S|_{\overline{S^n(\mathcal{H})}})^n$ has the polar decomposition $S_1^n = U_1P_1$. It is then clear that U_1 is an isometry and $P_1 \geq 0$ is invertible in the case in which S is n-quasi m-isometric, and U_1 is isometric and $P_1 \geq 0$ is injective in the case in which S is n-quasi m-self-adjoint and injective. Define the operator $P \in B(\overline{S^n(\mathcal{H})} \oplus S^{*-n}(0))$ by $P = P_1 \oplus I_2$.

Proposition 2.2. *Let $S \in B(\mathcal{H})$ be such that $S^{*n}d^m_{S^*,S}(I)S^n = 0$ for some integers $m, n \geq 1$.*

(*i*) *If $d = \Delta$, then there exist operators $Q, A \in B(\mathcal{H})$ such that $Q \geq 0$, $\Delta^m_{A^*,A}(Q) = 0$ and S^n is similar to A.*

(*ii*) *If $d = \delta$ and the operator S is injective, then there exist operators $Q, A \in B(\mathcal{H})$ such that $Q \geq 0$, $\delta^m_{A^*,A}(Q) = 0$ and $\delta_{A,S^n}(P) = 0$.*

(*iii*) *If S is left invertible, then there exists an operator $B \in B(\mathcal{H})$ such that $d^m_{B^*,B}(I) = 0$ and S^n is similar to B.*

Proof. The hypothesis $S^{*n}d^m_{S^*,S}(I)S^n = 0$ implies $d^m_{S_1^{*p},S_1^p}(I) = 0$, and hence

$$S^{*n}d^m_{S^{*p},S^p}(I)S^n = 0 \quad \text{for all integers } p \geq 1.$$

Let, as above,

$$S^n = \begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix}.$$

Define the operators A_1, A and Q by

$$A_1 = P_1U_1, \quad A = \begin{pmatrix} A_1 & P_1X \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} I_1 & U_1^*X \\ X^*U_1 & X^*X \end{pmatrix}.$$

Let I_2 denote (as above) the identity of $B(S^{-*n}(0))$.

(i). If $d = \Delta$, then (upon letting $p = n$ in the above) we have:

$$\begin{aligned}
S^{*n}\Delta^m_{S^{*n},S^n}(I)S^n = 0 &\Leftrightarrow S^{*n}\left\{\sum_{j=0}^m (-1)^j \binom{m}{j} S^{*n(m-j)}S^{n(m-j)}\right\}S^n = 0 \\
&\Leftrightarrow (P_1 \oplus I_2)\begin{pmatrix} U_1^* & 0 \\ X^* & 0 \end{pmatrix}\left\{\sum_{j=0}^m (-1)^j \binom{m}{j}\begin{pmatrix} S_1^{*n} & 0 \\ X^* & 0 \end{pmatrix}^{m-j}\begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix}^{m-j}\right\}\begin{pmatrix} U_1 & X \\ 0 & 0 \end{pmatrix}(P_1 \oplus I_2) = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j}\begin{pmatrix} U_1^* & 0 \\ X^* & 0 \end{pmatrix}\begin{pmatrix} P_1U_1^* & 0 \\ X^* & 0 \end{pmatrix}^{m-j}\begin{pmatrix} U_1P_1 & X \\ 0 & 0 \end{pmatrix}^{m-j}\begin{pmatrix} U_1 & X \\ 0 & 0 \end{pmatrix} = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j}\begin{pmatrix} A_1^* & 0 \\ X^*P_1 & 0 \end{pmatrix}^{m-j}\begin{pmatrix} U_1^* & 0 \\ X^* & 0 \end{pmatrix}\begin{pmatrix} U_1 & X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 & P_1X \\ 0 & 0 \end{pmatrix}^{m-j} = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j}\begin{pmatrix} A_1^* & 0 \\ X^*P_1 & 0 \end{pmatrix}^{m-j}\begin{pmatrix} I_1 & U_1^*X \\ X^*U_1 & X^*X \end{pmatrix}\begin{pmatrix} A_1 & P_1X \\ 0 & 0 \end{pmatrix}^{m-j} = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j}\begin{pmatrix} A_1^* & 0 \\ X^*P_1 & 0 \end{pmatrix}^{m-j} Q \begin{pmatrix} A_1 & P_1X \\ 0 & 0 \end{pmatrix}^{m-j} = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)}QA^{m-j} = 0 \\
&\Leftrightarrow \Delta^m_{A^*,A}(Q) = 0.
\end{aligned}$$

Set $P_1 \oplus I_2 = P$. Then

$$S^n = \begin{pmatrix} U_1P_1 & X \\ 0 & 0 \end{pmatrix} = P^{-1}\begin{pmatrix} A_1 & P_1X \\ 0 & 0 \end{pmatrix}P = P^{-1}AP,$$

i.e., S^n is similar to A.

(ii). If $d = \delta$, then (following the notation developed above):

$$\begin{aligned}
S^{*n}\delta^m_{S^*,S}(I)S^n = 0 &\Rightarrow S^{*n}\delta^m_{S^{*n},S^n}(I)S^n = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} S^{*n(m-j+1)}S^{n(j+1)} = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} S^{*n(m-j)}(P_1 \oplus I_2)\begin{pmatrix} U_1^* & 0 \\ X^* & 0 \end{pmatrix}\begin{pmatrix} U_1 & X \\ 0 & 0 \end{pmatrix}(P_1 \oplus I_2)S^{nj} = 0 \\
&\Leftrightarrow P\left\{\sum_{j=0}^m (-1)^j \binom{m}{j}\begin{pmatrix} U_1^*P_1 & 0 \\ X^*P_1 & 0 \end{pmatrix}^{m-j}\begin{pmatrix} I_1 & U_1^*X \\ X^*U_1 & X^*X \end{pmatrix}\begin{pmatrix} P_1U_1 & P_1X \\ 0 & 0 \end{pmatrix}^j\right\}P = 0 \\
&\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)}QA^j = 0 \\
&\Leftrightarrow \delta^m_{A^*,A}(Q) = 0.
\end{aligned}$$

Evidently, $P = P_1 \oplus I_2$ is a quasi-affinity such that $\delta_{A,S^n}(P) = 0$.

(iii). Assume now that S is left invertible. Then P and Q (defined as above) are positive invertible, and

$$S^{*n}\Delta^m_{S^*,S}(I)S^n = 0 \Rightarrow S^{*n}\Delta^m_{S^{*n},S^n}(I)S^n = 0 \Rightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)}QA^{m-j} = 0$$
$$\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} \left(Q^{-\frac{1}{2}}A^*Q^{\frac{1}{2}}\right)^{m-j}\left(Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}\right)^{m-j} = 0$$

and

$$S^{*n}\delta^m_{S^*,S}(I)S^n = 0 \Rightarrow S^{*n}\delta^m_{S^{*n}S^n}(I)S^n = 0 \Rightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)}QA^{j} = 0$$
$$\Leftrightarrow \sum_{j=0}^m (-1)^j \binom{m}{j} \left(Q^{-\frac{1}{2}}A^*Q^{\frac{1}{2}}\right)^{m-j}\left(Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}\right)^{j} = 0.$$

Now define $B \in B(\mathcal{H})$ by

$$B = Q^{\frac{1}{2}}AQ^{-\frac{1}{2}};$$

then

$$d^m_{B^*,B}(I) = 0.$$

Since

$$B = Q^{\frac{1}{2}}AQ^{-\frac{1}{2}} = Q^{\frac{1}{2}}PS^nP^{-1}Q^{-\frac{1}{2}} = LS^nL^{-1} \Rightarrow S^n = L^{-1}BL, \quad L = Q^{\frac{1}{2}}P,$$

S^n is similar to B. □

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$ and let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$.

Corollary 2.3. (cf. [18, Corollary 4.3]) *If $d = \Delta$ in the statement of Proposition 2.2 and the operator Q (in the proof of the proposition) is injective, then $\sigma_p(S) \subseteq \partial D$.*

Proof. The hypotheses imply $\sum_{j=0}^m (-1)^j \binom{m}{j} \left\|Q^{\frac{1}{2}}A^{m-j}x\right\|^2 = 0$ for all $x \in \mathcal{H}$. Consider a $\lambda \in \sigma_p(S)$ such that $Ax = \lambda x$. Then, since Q is injective,

$$\sum_{j=0}^m (-1)^j \binom{m}{j} |\lambda|^{2(m-j)} \left\|Q^{\frac{1}{2}}x\right\|^2 = 0 \Leftrightarrow (1-|\lambda|^2)^m = 0 \Leftrightarrow |\lambda| = 1.$$

Since S^n is similar to A, $\sigma_p(S)^n = \sigma_p(S^n) = \sigma_p(A) \subseteq \partial(\mathbb{D})$. □

Proposition 2.2 is a generalization of some extant results. For example, if $d = \Delta$, $n = 1$ and $m = 2$, then $S^*\Delta^2_{S^*,S}(I)S = 0$ (i.e., S is 1-quasi 2-isometric) implies $\Delta^2_{A^*,A}(Q) = 0$ (where the operators A, Q are as defined in the proof of the proposition and the operator S is similar to A); if S is also left invertible, then $\Delta^2_{B^*,B}(I) = 0$ (i.e., B is 2-isometric) for some operator B similar to the operator S (cf. [10, Theorem 2.5]). In their considerations on the spectral properties of A-contractions, L. Suciu and N. Suciu [18] define an operator $S \in B(\mathcal{H})$ to be n-quasi isometric if $S^{*n}(S^*S-1)S^n = 0$. In our terminology, this equates to $S^{*n}\Delta_{S^*,S}(I)S^n = 0$ (equivalently, "S is n-quasi 1-isometric"). Thus, for n-quasi isometric operators S, S_1^n is isometric; indeed, since $S_1^{*n}(S_1^*S_1 - I_1)S_1^n = 0$, S_1 is isometric. Assume now that $n = 1$ and 0 is a normal eigenvalue of S (i.e., $S^{-1}(0) \subseteq S^{*-1}(0)$). Then $S = S_1 \oplus 0$ is a partial isometry (*cf.* [18, Theorem 3.12 and Corollary 3.13]). For a general n-quasi isometry S, $S = \begin{pmatrix} S_1 & S_0 \\ 0 & S_2 \end{pmatrix} \in B(\overline{S^n(\mathcal{H})} \oplus S^{*-n}(0))$, where S_1 is isometric and S_2 is n-nilpotent. Consequently, S has SVEP and hence [18, Theorem 4.6]: (i) $\sigma(S) = \overline{\sigma_a(S^*)}$. (ii) $\sigma(S) = \bar{\mathbb{D}}$, the closed unit disk, if S_1 is not invertible and $\sigma(S) \subseteq \partial\mathbb{D} \cup \{0\}$ if S_1 is invertible. In either case,

$\sigma_a(S) \subseteq \partial\mathbb{D} \cup \{0\}$. (iii) If λ, μ are two distinct non-zero eigenvalues of S, then $\lambda, \mu \in \sigma_p(S_1)$ and the corresponding eigenspaces are mutually orthogonal. Observe that if $n = 1$, then S_1 is isometric. If also $\|S\| \leq 1$, then $S^pS^{*p} = S_1^pS_1^{*p} + S_1^{p-1}S_0S_0^*S_1^{*(p-1)} \oplus 0$ is a contraction (thus: $S_1^pS_1^{*p} + S_1^{p-1}S_0S_0^*S_1^{*(p-1)} \leq I_1$). Consequently,

$$S^{*p}S^p = \begin{pmatrix} I_1 & S_1^{*p}S_0 \\ S_0^*S_1^{p-1} & S_0^*S_0 \end{pmatrix} \geq S^pS^{*p}$$

for all integers $p \geq 1$ [18, Theorem 3.3].

Let C be a conjugation of $\mathcal{H}$ (i.e., $C : \mathcal{H} \to \mathcal{H}$ is a conjugate-linear operator such that $C^2 = I$ and $\langle Cx, y\rangle = \langle Cy, x\rangle$ for all $x, y \in \mathcal{H}$). If one chooses $T = CS^*C$ in $d^m_{T,S}(I) = 0$, then

$$\Delta^m_{CS^*C,S}(I) = 0 \Leftrightarrow \Delta^m_{S^*,CSC}(I) = 0$$

defines the class of (m, C)-isometric operators and

$$\delta^m_{CS^*C,S}(I) = 0 \Leftrightarrow \delta^m_{S^*,CSC}(I) = 0$$

defines the class of (m, C)-*symmetric operators* [9,8]. It is known [9,8] that

$$d^m_{S^*,CSC}(I) = 0 \Leftrightarrow d^t_{S^*,CSC}(I) = 0 \text{ for all integers } t \geq m$$

and

$$d^m_{S^*,CSC}(I) = 0 \Leftrightarrow d^m_{S^{*p},CS^pC}(I) = 0 \text{ for all integers } p \geq 1.$$

It is clear that if $S^{*n}d^m_{S^*,CSC}(I)S^n = 0$, then $S \in B(\overline{S^n(\mathcal{H})} \oplus S^{*-n}(\mathcal{H}))$ has a representation

$$CS^nC = C\begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix}C = C\begin{pmatrix} U_1P_1 & X \\ 0 & 0 \end{pmatrix}C$$

(where the operator X is as defined above). In particular, if the conjugation $C : \overline{S^n(\mathcal{H})} \oplus S^{*-n}(\mathcal{H}) \to \overline{S^n(\mathcal{H})} \oplus S^{*-n}(\mathcal{H})$ has a representation $C = C_1 \oplus C_2$, then

$$d^m_{S_1^*,C_1S_1C_1}(I_1) = 0 \Rightarrow S^{*n}d^m_{S^{*p},CS^pC}(I)S^n = 0$$

for all integers $p \geq 1$. If, now, S satisfies the additional property that $CSCS = S^2$, then

$$S^{*n}d^m_{S^*,CSC}(I)S^n = 0 \Leftrightarrow S^{*n}d^m_{S^*,S}(I)S^n = 0$$

and Proposition 2.2 applies. In general, Proposition 2.2 seemingly does not extend to operators S satisfying $S^{*n}d^m_{S^*,CSC}(I)S^n = 0$. Define the operator $M \in B(\mathcal{H})$ by

$$M = \begin{pmatrix} U_1 & X \\ 0 & 0 \end{pmatrix}$$

(where U_1 and X are the operators defined above). The following proposition says that a result very similar to Proposition 2.2 holds in the case in which $[C, M] = 0$ and $C = C_1 \oplus C_2$.

Proposition 2.4. *Let $S \in B(\mathcal{H})$ be such that $S^{*n}d^m_{S^*,CSC}(I)S^n = 0$ (so that S is either n-quasi (m, C)-isometric or S is n-quasi (m, C) symmetric), where the conjugation $C = C_1 \oplus C_2 : \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0) \to \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$ satisfies $[C, M] = 0$.*

(i) *If $d = \Delta$, then there exist operators $Q, A \in B(\mathcal{H})$ such that $Q \geq 0$, $\Delta^m_{A^*,CAC}(Q) = 0 = \Delta^m_{CA^*C,A}(CQC)$ and S^n is similar to A.*
(ii) *If $d = \delta$ and the operator S is injective, then there exist operators $Q, A \in B(\mathcal{H})$ such that $Q \geq 0$, $\delta^m_{A^*,CAC}(Q) = 0 = \delta^m_{CA^*C,A}(CQC)$ and $\delta_{A,S^n}(P) = 0$.*
(iii) *If S is left invertible, then there exists an operator $B \in B(\mathcal{H})$ such that $d^m_{B^*,CBC}(I) = 0$ and S^n is similar to B.*

Proof. We start by observing that

$$S^{*n}d^m_{S^*,CSC}(I)S^n = 0 \Leftrightarrow S^{*n}d^m_{S^{*p},CS^pC}(I)S^n = 0$$

for all integers $p \geq 1$.

(i). Case $d = \Delta$. Following the notation of the proof of Proposition 2.2, we have:

$$\begin{aligned}
S^{*n}d^m_{S^{*n},CS^nC}(I)S^n = 0 \;&\Leftrightarrow\; S^{*n}\left\{\sum_{j=0}^m (-1)^j\binom{m}{j}S^{*n(m-j)}CS^{n(m-j)}C\right\}S^n = 0\\
&\Leftrightarrow \sum_{j=0}^m (-1)^j\binom{m}{j}S^{*n(m-j+1)}CS^{n(m-j+1)}C = 0\\
&\Leftrightarrow \sum_{j=0}^m (-1)^j\binom{m}{j}\begin{pmatrix}P_1U_1^* & 0\\ X^* & 0\end{pmatrix}^{m-j+1}C\begin{pmatrix}U_1P_1 & X\\ 0 & 0\end{pmatrix}^{m-j+1}C = 0\\
&\Leftrightarrow \begin{pmatrix}P_1 & 0\\ 0 & I_2\end{pmatrix}\left\{\sum_{j=0}^m (-1)^j\binom{m}{j}\begin{pmatrix}U_1^*P_1 & 0\\ X^*P_1 & 0\end{pmatrix}^{m-j}\begin{pmatrix}U_1^* & 0\\ X^* & 0\end{pmatrix}\right.\\
&\qquad \times\left.\begin{pmatrix}C_1U_1C_1 & C_1XC_2\\ 0 & 0\end{pmatrix}\begin{pmatrix}C_1P_1U_1C_1 & C_1P_1XC_2\\ 0 & 0\end{pmatrix}^{m-j}\right\}\begin{pmatrix}C_1P_1C_1 & 0\\ 0 & I_2\end{pmatrix} = 0\\
&\Leftrightarrow \sum_{j=0}^m (-1)^j\binom{m}{j}\begin{pmatrix}U_1^*P_1 & 0\\ X^*P_1 & 0\end{pmatrix}^{m-j}\begin{pmatrix}I_1 & U_1^*C_1XC_2\\ X^*U_1C_1U_1 & X^*C_1XC_2\end{pmatrix}\begin{pmatrix}C_1P_1U_1C_1 & C_1P_1XC_2\\ 0 & 0\end{pmatrix}^{m-j} = 0.
\end{aligned}$$

By hypothesis, $[C, M] = 0$. Hence

$$C_1X = XC_2,\ [C_1, U_1] = 0 \quad\text{and}\quad \begin{pmatrix}I_1 & U_1^*C_1XC_2\\ X^*U_1C_1U_1 & X^*C_1XC_2\end{pmatrix} = \begin{pmatrix}I_1 & U_1^*X\\ X^*U_1 & X^*X\end{pmatrix} = Q$$

for some positive operator Q. Consequently,

$$\begin{aligned}
S^{*n}\Delta^m_{S^*,CSC}(I)S^n = 0 \;&\Rightarrow\; \sum_{j=0}^m (-1)^j\binom{m}{j}\begin{pmatrix}U_1^*P_1 & 0\\ X^*P_1 & 0\end{pmatrix}^{m-j}Q\begin{pmatrix}C_1P_1U_1C_1 & C_1P_1XC_2\\ 0 & 0\end{pmatrix}^{m-j} = 0\\
&\Rightarrow \Delta^m_{A^*,CAC}(Q) = 0 \Leftrightarrow \Delta^m_{CA^*C,A}(CQC) = 0,
\end{aligned}$$

where, as before, the operator A is defined by $A = \begin{pmatrix}P_1U_1 & P_1X\\ 0 & 0\end{pmatrix} = PS^nP^{-1}$.

(ii). Case $d = \delta$. The hypothesis S is injective implies $P \geq 0$ has a dense range. Using the same notation as above, we have:

$$\begin{aligned}
S^{*n}\delta^m_{S^*,CSC}(I)S^n = 0 \;&\Rightarrow\; \sum_{j=0}^m (-1)^j\binom{m}{j}S^{*n(m-j+1)}CS^{n(j+1)}C = 0\\
&\Leftrightarrow \begin{pmatrix}P_1 & 0\\ 0 & I_2\end{pmatrix}\left\{\sum_{j=0}^m (-1)^j\binom{m}{j}\begin{pmatrix}U_1^*P_1 & 0\\ X^*P_1 & 0\end{pmatrix}^{m-j}\begin{pmatrix}U_1^* & 0\\ X^* & 0\end{pmatrix}\right.\\
&\qquad \times\left.\begin{pmatrix}C_1U_1C_1 & C_1XC_2\\ 0 & 0\end{pmatrix}\begin{pmatrix}C_1P_1U_1C_1 & C_1P_1XC_2\\ 0 & 0\end{pmatrix}^{j}\right\}\begin{pmatrix}C_1P_1C_1 & 0\\ 0 & I_2\end{pmatrix} = 0\\
&\Leftrightarrow \sum_{j=0}^m (-1)^j\binom{m}{j}\begin{pmatrix}U_1^*P_1 & 0\\ X^*P_1 & 0\end{pmatrix}^{m-j}Q\begin{pmatrix}C_1P_1U_1C_1 & C_1P_1XC_2\\ 0 & 0\end{pmatrix}^{j} = 0\\
&\Rightarrow \sum_{j=0}^m (-1)^j\binom{m}{j}A^{*(m-j)}QCA^jC = 0\\
&\Leftrightarrow \delta^m_{A^*,CAC}(Q) = 0 \Leftrightarrow \delta^m_{CA^*C,A}(CQC) = 0.
\end{aligned}$$

It being evident that $\delta_{A,S^n}(P) = 0$, $P \geq 0$ a quasi-affinity, the proof is complete.

(iii). Arguing as in the proof of Proposition 2.2, it is seen that $P = P_1 \oplus I_2 > 0$ and $Q > 0$ are invertible; furthermore, $CQC = Q$. Since

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}A^{*a_j}QCA^{b_j}C = 0 \Leftrightarrow \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(Q^{-\frac{1}{2}}A^*Q^{\frac{1}{2}}\right)^{a_j}\left(CQ^{\frac{1}{2}}AQ^{-\frac{1}{2}}C\right)^{b_j} = 0$$

for all positive integers a_j and b_j, we have

$$d^m_{B^*,CBC}(I) = 0 \Leftrightarrow d^m_{CB^*C,B}(I) = 0; \quad B = Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}.$$

Clearly, $S^n = P^{-1}Q^{-\frac{1}{2}}BQ^{\frac{1}{2}}P$ is similar to B. □

3 The polaroid property

If $\Delta^m_{T,S}(I) = 0$ for some $S, T \in B(\mathcal{X})$ (i.e., if $S \in B(\mathcal{X})$ is left m-invertible by $T \in B(\mathcal{X})$), then $0 \notin \sigma_a(S)$ (for if $0 \in \sigma_a(S)$ and $\{x_n\} \subset \mathcal{X}$ is a sequence of unit vectors such that $\lim_{n\to\infty} Sx_n = 0$, then

$$\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|\Delta^m_{T,S}(I)x_n\| = \lim_{n\to\infty}\left\|\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j}x_n\right\| = 0$$

is a contradiction). Indeed, if $\lambda \in \sigma_a(S)$, and $\{x_n\} \subset X$ is a sequence of unit vectors such that $\lim_{n\to\infty}(S-\lambda)x_n = 0$, then

$$\begin{aligned}\lim_{n\to\infty}\Delta^m_{T,S}(I)x_n &= \lim_{n\to\infty}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^jx_n\right\} = \lim_{n\to\infty}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\lambda T)^{m-j}x_n\right\}\\ &= \lim_{n\to\infty}(1-\lambda T)^m x_n = 0 \Rightarrow \frac{1}{\lambda} \in \sigma_a(T).\end{aligned}$$

A similar argument, using this time the fact that

$$\Delta^m_{S^*,T^*}(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}S^{*(m-j)}T^{*(m-j)} = 0,$$

shows that $\lambda \in \sigma_{su}(T)$ implies $\frac{1}{\lambda} \in \sigma_{su}(S)$ for all non-zero λ. (Here $\sigma_{su}(\cdot)$ denotes the surjectivity spectrum.)

If we assume S, $\Delta^m_{T,S}(I) = 0$, to be a contraction satisfying $\sigma(S) = \bar{\mathbb{D}}$, then iso$\sigma(S) = \emptyset$ and S is (vacuously) polaroid. If, instead, we assume that S is an invertible contraction with spectrum a subset of the boundary $\partial\mathbb{D}$ of the unit disk $\mathbb{D}$, then S is normaloid (i.e., $\|S\| = r(S)$) and $\sigma(S)$ consists of the peripheral spectrum $(= \{\lambda : |\lambda| = r(S)\})$ of S. The normaloid property of S implies that asc$(S-\lambda) \le 1$ and $\dim(\mathcal{X}\backslash(S-\lambda)(\mathcal{X})) > 0$ [14, Proposition 54.2]. Thus, if the range $(S-\lambda)^d(\mathcal{X})$ is closed for some integer $d \ge 1$, then $(S-\lambda)(\mathcal{X})$ is closed [15, Proposition 4.10.4] and asc$(S-\lambda) \le 1$, i.e., λ is a left pole of S. Since λ is a boundary point of the spectrum, λ is indeed a pole of S. Conclusion: "A necessary and sufficient condition for a point $\lambda \in \sigma(S)$ to be a pole of S for a given left m-invertible contraction S (i.e., a contraction S such that $\Delta^m_{T,S}(I) = 0$ for some $T \in B(\mathcal{X})$) with $\sigma(S) \subseteq \partial\mathbb{D}$ is that $(S-\lambda)(\mathcal{X})$ is closed."

The hypothesis that S is a left m-invertible contraction (resp., T is a right m-invertible contraction), even that S is an invertible contraction (resp., T is an invertible contraction), is not sufficient for S to be polaroid. For example, the operator $S = (I+Q)^{-1}$, I the identity operator and Q the Volterra integration operator, is invertible with $\sigma(S) = \{1\}$ and $\|S\| = 1$ [19, solution 190, page 302]. Since $(I+Q)^{-1} - I = -Q(I+Q)^{-1} = -(I+Q)^{-1}Q$ and $\|((I+Q)^{-1} - I)^n)\|^{\frac{1}{n}} \le \|(I+Q)^{-1}\|\|Q^n\|^{\frac{1}{n}}$ converges to 0 as $n \to \infty$, S is not polaroid. Again, if we let $T = (I+Q)^{-1}$ and $S = I + Q$, then S is not polaroid. A sufficient condition for an operator S, $\Delta^m_{T,S}(I) = 0$, to be polaroid is that both S, T are power bounded. We recall: $A \in B(\mathcal{X})$ is power bounded if there exists a positive scalar M such that $\sup_{n\in\mathbb{N}}\|A^n\| < M$.

Theorem 3.1. *If $S, T \in B(\mathcal{X})$ satisfy $\Delta^m_{T,S}(I) = 0$ for some integer $m \geq 1$, then a sufficient condition for S to be polaroid is that S, T are power bounded.*

Proof. If S, T are power bounded, then there exist scalars M_1, M_2 such that

$$\sup_{n\in\mathbb{N}} \|S^n\| < M_1, \quad \sup_{n\in\mathbb{N}} \|T^n\| < M_2$$

(and hence $r(S) = r(T) = 1$). This, in view of the fact that ($0 \notin \sigma_a(S)$ and) $\left\{\frac{1}{\lambda} : 0 \neq \lambda \in \sigma_a(S)\right\} \subseteq \sigma_a(T)$ implies $\sigma_a(S) \subseteq \partial\mathbb{D}$. Hence,

$$\sigma(S) = \bar{\mathbb{D}} \text{ if } S \text{ is not invertible and } \sigma(S) \subseteq \partial\mathbb{D} \text{ if } S \text{ is invertible}.$$

Trivially, S is polaroid in the case in which $\sigma(S) = \bar{\mathbb{D}}$. Assume hence that S is invertible (so that $\sigma(S) \subseteq \partial\mathbb{D}$). Since $\Delta^m_{T,S}(I) = 0$ implies $\Delta^m_{T^p,S^p}(I) = 0$ for all integers $p \geq 1$, we have upon defining the operator C_p by

$$C_p = (-1)^{m+1}\left\{\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}T^{p(m-j)}S^{p(m-j-1)}\right\}$$

that

$$C_pS^p = I, \quad \text{all integers } p \geq 1.$$

Evidently, the operator S^p is left invertible by C_p for all integers $p \geq 1$, and

$$\|C_p\| \leq \left\{1 + \binom{m}{1} + \cdots + \binom{m}{m-2} + \binom{m}{m-1}\right\}M_1M_2 < 2^mM_1M_2 = M$$

for all integers $p \geq 1$. Thus, for all $x \in \mathcal{X}$ and integers $p \geq 1$,

$$\|x\| = \|C_pS^px\| \leq M\|S^px\| \Leftrightarrow \left(\frac{1}{M}\right)\|x\| \leq \|S^px\|.$$

Since already

$$\|S^px\| \leq \|S^p\|\|x\| \leq M_1\|x\|$$

for all $x \in \mathcal{X}$, it follows that S is similar to an invertible isometry (on an equivalent Banach space). (This is well known – see, for example, [20].) The proof now follows, since invertible isometries are polaroid and the polaroid property is preserved by similarities. □

Power bounded m-isometric operators satisfy the property that they are isometric – see [6, Theorem 2.4] and [21, Theorem 2.4]. Hence:

Corollary 3.2. *Power bounded m-isometric Banach space operators, i.e. power bounded operators $S \in B(\mathcal{X})$ such that $\Delta^m_{S^*,S}(I) = 0$, are polaroid.*

The Power bounded hypothesis on S may be dropped in the case in which $\Delta^2_{S^*,S}(I) = 0$ (i.e., the operator S is 2-isometric), for the reason that invertible 2-isometries are isometries: *2-isometric Banach space operators are polaroid.* Corollary 3.2 extends to operators $S \in B(\mathcal{H})$ satisfying $\Delta^m_{S^*,CSC}(I)$ for some conjugation C (i.e., to (m, C)-isometries $S \in B(\mathcal{H})$). Observe that if S is power bounded, then so is CSC and $\sigma_a(CSC) = \overline{\sigma_a(S)}$ (= complex conjugate of $\sigma_a(S)$) for every conjugation C. Hence:

Corollary 3.3. *Power bounded (m, C)-isometries $\in B(\mathcal{H})$ are polaroid.*

Extension to n-quasi left m-invertible operators. Theorem 3.1 extends to n-quasi left m-invertible operators $S \in B(\mathcal{H})$,

$$S^{*n}\Delta^m_{T,S}(I)S^n = S^{*n}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}S^{m-j}\right\}S^n = 0,$$

such that $[S, T^*] = ST^* - T^*S = 0$. Letting S and T^* have the upper triangular representations (2), it is seen that ($\sigma(S) = \sigma(S_1) \cup \{0\}$, $S_1^n(\mathcal{H}) = \overline{S_1^n\mathcal{H}}$, $[S_1, T_1^*] = 0$ and)

$$\Delta^m_{T_1,S_1}(I_1) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T_1^{m-j}S_1^{m-j} = 0$$

(so that S_1 is left m-invertible by T_1). Recall from the previous section that $S^n = \begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix}$, where S_1^n is left m-invertible by T_1^n. Since S and T power bounded imply S_1^n and T_1^n are power bounded, S_1^n (therefore, S_1) is polaroid. Hence:

Theorem 3.4. *Power bounded operators $S, T \in B(\mathcal{H})$ satisfying $S^{*n}\Delta^m_{T,S}(I)S^n = 0$ such that $[S, T^*] = 0$ are polaroid.*

Proof. Since $S = \begin{pmatrix} S_1 & S_0 \\ 0 & S_2 \end{pmatrix}$, where S_1 is polaroid and S_2 is n-nilpotent, the proof follows from the inequalities that $\text{asc}(S-\lambda) \le \text{asc}(S_1-\lambda) + \text{asc}(S_2-\lambda)$ and $\text{dsc}(S-\lambda) \le \text{dsc}(S_1-\lambda) + \text{dsc}(S_2-\lambda)$ for all complex λ [16, exercise 7, page 293]. □

Remark 3.5. Theorem 3.4 has an n-quasi m-isometric and an n-quasi (m, C)-isometric analogue, namely:

Power bounded n-quasi m-isometric operators $S \in B(\mathcal{H})$, $S^{*n}\Delta^m_{S^*,S}(I)S^n = 0$, and power bounded n-quasi (m, C)-isometric operators $S \in B(\mathcal{H})$, $S^{*n}\Delta_{S^*,CSC}(I)S^n = 0$, such that $C = C_1 \oplus C_2$ are polaroid.

In particular, 1-quasi 2-isometries are polaroid [10]: This follows since operators S such that $S^*\Delta^2_{S^*,S}(I)S = 0$ have a representation $\begin{pmatrix} S_1 & X \\ 0 & 0 \end{pmatrix}$, where the operator S_1 (satisfying $\Delta^2_{S_1^*,S_1}(I_1) = 0$) is polaroid. Observe here that either $\sigma(S) = \bar{\mathbb{D}}$ or $\sigma(S) \subseteq \partial\mathbb{D} \cup \{0\}$.

It is easily seen that for an m-symmetric operator $S \in B(\mathcal{H})$, $\delta^m_{S^*,CSC}(I) = 0$, $\sigma_a(S) = \overline{\sigma_a(CSC)}$ and $\lambda \in \sigma_a(S) \Rightarrow \lambda \in \sigma_a(CS^*C) = \sigma_{su}(S)$. (Recall: $\sigma_{su}(S)$ = the surjectivity spectrum of S.) Hence, $\sigma(S) = \sigma_a(S) \cup \sigma_{su}(S) \subseteq \sigma_{su}(S) \subseteq \sigma(S)$, i.e., $\overline{\sigma(S)} = \sigma(CSC) = \overline{\sigma_a(S)} = \overline{\sigma_{su}(S)}$. The argument of the proof of Theorem 3.4 implies that *if the left invertible operator $S \in B(\mathcal{H})$ is n-quasi m-symmetric, $S^{*n}\delta^m_{S^*,CSC}(I)S^n = 0$, and $C = C_1 \oplus C_2$, then S is power bounded implies that if S_1^n is polaroid, then (S^n, therefore) S is polaroid.*

For m-self-adjoint operators $S \in B(\mathcal{H})$, $\delta^m_{S^*,S}(I) = 0$, it is seen that if λ is an eigenvalue of S with an eigenvector x and $\bar{\mu}$ is an eigenvalue of S^* with an eigenvector y, then $(\lambda - \bar{\mu})xy = 0$. Hence, the eigenvalues of an m-self-adjoint operator are real. Since λ is a pole of S implies λ is an eigenvalue of S, the poles of S are all real. Consider now a left invertible n-quasi m-self-adjoint operator $S \in B(\mathcal{H})$, $S^{*n}\delta^m_{S^*,S}(I)S^n = 0$. Then, follow an argument similar to that above, *S is polaroid if the left invertible m-self-adjoint operator S_1^n is polaroid, and this happens if and only if the isolated points of the intersection of $\sigma(S_1)$ with the real line consist of the poles of S_1.*

Self-adjoint Riesz Idempotents. Restricting ourselves to operator $S, T \in B(\mathcal{H})$ for which $S^{*n}\Delta^m_{T,S}(I)S^n = 0$ (i.e., n-quasi left m-invertible operators in $B(\mathcal{H})$) for which $[S, T^*] = 0$, in the following we consider conditions guaranteeing the self-adjointness of the Riesz idempotents P_λ attached with the poles $\lambda \in \text{iso}\sigma(S)$ of S. It is clear from the above that if a point $\lambda \neq 0$ is a pole of S, then S has a matrix representation

$$S = \begin{pmatrix} \lambda & X_1 & Y_1 \\ 0 & S_{11} & Y_2 \\ 0 & 0 & S_2 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = (S_1 - \lambda)^{-1}(0) \oplus (S_1 - \lambda)(\mathcal{H}) \oplus S^{*-n}(0)$. If $x = (x_1, x_2, x_3) \in (S - \lambda)^{-1}(0)$, then (necessarily) $x_3 = x_2 = 0$. Hence $x \in (S - \lambda)^{-1}(0)$ if and only if $x = (x_1, 0, 0)$. Consider now $(S - \lambda)^{*-1}(0)$. Since $(S - \lambda)^{-1}(0) \subseteq (S - \lambda)^{*-1}(0)$ if and only if $X_1^* x_1 = 0 = Y_1^* x_1$,

$$(S - \lambda)^{-1}(0) \subseteq (S - \lambda)^{*-1}(0) \Leftrightarrow (S - \lambda)^*(S - \lambda)^{-1}(0) \subseteq \{0\}.$$

Evidently, if $(S - \lambda)^*(S - \lambda)^{-1}(0) \subset \{0\}$, then $(S - \lambda)^{-1}(0) \subseteq (S - \lambda)^{*-1}(0)$. The point λ being a simple pole of S, if $(S - \lambda)^*(S - \lambda)^{-1}(0) \subseteq \{0\}$, then $((S - \lambda)(\mathcal{H})$ is closed and)

$$\begin{aligned}\mathcal{H} &= (S - \lambda)^{-1}(0) \oplus (S - \lambda)(\mathcal{H}) = P_\lambda \mathcal{H} \oplus (I - P_\lambda)\mathcal{H} = (S - \lambda)^{*-1}(0) \oplus (S - \lambda)^{*-1}(0)^\perp \\ &= (S - \lambda)^{-1}(0) \oplus (S - \lambda)^{-1}(0)^\perp = P_\lambda \mathcal{H} \oplus P_\lambda \mathcal{H}^\perp \Rightarrow P_\lambda \mathcal{H}^\perp = P_\lambda^{-1}\mathcal{H} = (I - P_\lambda)\mathcal{H},\end{aligned}$$

i.e., P_λ is self-adjoint.

Consider now the case in which $\lambda = 0$ is a pole of S. Then $P_\lambda \mathcal{H} = S^{-n}(0)$ and S^n has a triangulation

$$S^n = \begin{pmatrix} S_1^n & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{S^n \mathcal{H}} \\ S^{*-n}(0) \end{pmatrix},$$

where S_1 is invertible (since $0 \in \operatorname{iso}\sigma(S^n)$ implies $0 \notin \sigma(S_1^n)$). Since $x = (x_1, x_2) \in S^{-n}(0)$ if and only if $x = (-S_1^{-n} X x_2, x_2)$, $S^{-n}(0) \subseteq S^{*-n}(0)$ if and only if $X x_2 = 0$, i.e., if and only if $S^n(S^{*-n}(0)) = \{0\}$ (and then $S^{-n}(0) = S^{*-n}(0)$). Arguing as above, it now follows that the projection P_0 is self-adjoint if and only if $S^n : S^{*-n}(0) \to \{0\}$. We have proved:

Proposition 3.6. *Given an n-quasi left m-invertible operator $S \in B(\mathcal{H})$ such that $[S, T^*] = 0$, the Riesz projection P_λ corresponding to a pole $\lambda \neq 0$ (resp., $\lambda = 0$) of S is self-adjoint if and only if $(S - \lambda)^*$: $(S - \lambda)^{-1}(0) \to \{0\}$ (resp., $S^n : S^{*-n}(0) \to \{0\}$).*

Remark 3.7. It is immediate from the above that if $S \in B(\mathcal{H})$ is a 1-quasi 2-isometry, then the Riesz projection P_λ corresponding to a pole $\lambda \neq 0$ (resp., $\lambda = 0$) is self-adjoint if and only if $(S - \lambda)^*$: $(S - \lambda)^{-1}(0) \to \{0\}$ (resp., $S : S^{*-1}(0) \to \{0\}$); cf. [10, Theorems 2.7 and 2.8].

4 Products

Let $S_i, T_i \in B(\mathcal{X})$, $i = 1, 2$, be such that $[S_1, S_2] = 0 = [T_1, T_2]$ and $d^{n_i}_{T_i,S_i}(I) = 0$. Then

$$\Delta^n_{T_1T_2,S_1S_2} = (L_{T_1}L_{T_2}R_{S_1}R_{S_2} - I)^n = \{L_{T_1}(L_{T_2}R_{S_2} - I)R_{S_1} + (L_{T_1}R_{S_1} - I)\}^n = \sum_{j=0}^n \binom{n}{j} \Delta^{n-j}_{T_2,S_2} L^{n-j}_{T_1} R^{n-j}_{S_1} \Delta^j_{T_1,S_1}$$

implies

$$\Delta^n_{T_1T_2,S_1S_2}(I) = \sum_{j=0}^n \binom{n}{j} T_1^{n-j} \Delta^{n-j}_{T_2,S_2}(I) S_1^{n-j} \Delta^j_{T_1,S_1}(I)$$

and

$$\delta^n_{T_1T_2,S_1S_2} = (L_{T_1}L_{T_2} - R_{S_1}R_{S_2})^n = \{L_{T_2}(L_{T_1} - R_{S_1}) + (L_{T_2} - R_{S_2})R_{S_1}\}^n = \sum_{j=0}^n \binom{n}{j} L^{n-j}_{T_2} \delta^{n-j}_{T_1,S_1} \delta^j_{T_2,S_2} R^j_{S_1}$$

implies

$$\delta^n_{T_1T_2,S_1S_2}(I) = \sum_{j=0}^n \binom{n}{j} T_2^{n-j} \delta^{n-j}_{T_1,S_1}(I) \delta^j_{T_2,S_2}(I) S_1^j.$$

Letting $n = m_1 + m_2 - 1$, since $d^j_{T_2,S_2}(I) = 0$ for all $j \geq m_2$ and $d^{m_1+m_2-1-j}_{T_1 S_1}(I) = 0$ for all $j \leq m_2 - 1$ (implies $m_1 + m_2 - 1 - j \geq m$), we have:

Lemma 4.1. *If $S_i, T_i \in B(\mathcal{X})$, $i = 1, 2$, are such that $[S_1, S_2] = 0 = [T_1, T_2]$ and $d^{m_i}_{T_i,S_i}(I) = 0$, then $d^{m_1+m_2-1}_{T_1T_2,S_1S_2}(I) = 0$.*

The following theorem is an $n(S)$-quasi $[m, d]$-version of these results. (Recall here that the operators $S_1, T_1 \in B(\mathcal{X})$ *are $n(S)$-quasi $[m, d]$-intertwined for an operator $S \in B(\mathcal{H})$ if $S^{*n}d^m_{T_1,S_1}(I)S^n = 0$.*)

Theorem 4.2. *If $S^{*n}d^{m_i}_{T_i,S_i}(I)S^n = 0$, $i = 1, 2$, for some operators $S, S_1, S_2, T_1, T_2 \in B(\mathcal{H})$ such that $[S, S_i] = 0 = [S, T_i^*]$ and $[S_1, S_2] = 0 = [T_1, T_2]$, then $S^{*n}d^{m_1+m_2-1}_{T_1T_2,S_1S_2}(I)S^n = 0$ (i.e., T_1T_2 and S_1S_2 are $n(S)$-quasi $[m_1 + m_2 - 1, d]$-intertwined).*

Proof. The hypotheses imply that the operators S, S_i and T_i^* have the upper triangular matrix representations

$$S = \begin{pmatrix} S_{01} & S_{00} \\ 0 & S_{02} \end{pmatrix}, \quad S_i = \begin{pmatrix} S_{i1} & S_{i0} \\ 0 & S_{i2} \end{pmatrix}, \quad T_i^* = \begin{pmatrix} T_{i1}^* & T_{i0}^* \\ 0 & T_{i2}^* \end{pmatrix}; \quad i = 1, 2,$$

with respect to the decomposition $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$ of $\mathcal{H}$. The hypothesis $S^{*n}d^{m_1}_{T_1,S_1}(I)S^n = 0$ implies $d^{m_1}_{T_{11},S_{11}}(I_1) = 0$ and the hypothesis $S^{*n}d^{m_2}_{T_2,S_2}(I)S^n = 0$ implies $d^{m_2}_{T_{21},S_{21}}(I_1) = 0$. Hence, since the hypothesis $[S_1, S_2] = 0 = [T_1, T_2]$ implies $[S_{11}, S_{21}] = 0 = [T_{11}, T_{21}]$, Lemma 4.1 implies $d^{m_1+m_2-1}_{T_{11}T_{21},S_{11}S_{21}}(I_1) = 0$. Finally, since

$$\sum_{j=0}^{m_1+m_2-1} (-1)^j \binom{m_1 + m_2 - 1}{j} (T_1T_2)^{m_1+m_2-1-j}(S_1S_2)^{m_1+m_2-1-j} = \begin{pmatrix} 0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix}$$

for some operators Z_i $(i = 1, 2, 3)$, and $S^n = \begin{pmatrix} S_{01}^n & X \\ 0 & 0 \end{pmatrix}$ for some operator X, with respect to $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$,

$$S^{*n}d^{m_1+m_2-1}_{T_1T_2,S_1S_2}(I)S^n = 0,$$

i.e., T_1T_2 and S_1S_2 are $n(S)$-quasi $[m_1 + m_2 - 1, d]$-intertwined. □

Remark 4.3.

(i) Recall that T is a strict left m-inverse of S if $\Delta^m_{S,T}(I) = 0$ but $\Delta^{m-1}_{S,T}(I) \neq 0$ [2,3]. Letting $m_1 = 1$ in $\Delta^{m_1}_{T_1,S_1}(I) = 0$ (so that T_1 is a left 1-inverse of S_1, i.e., $T_1S_1 = I$), it follows that T_1T_2 is a strict left m_2-inverse of S_1S_2 if and only if $\Delta^{m_2-1}_{T_2,S_2}(I) \neq 0$ [3, Theorem 13], i.e., if and only if T_2 is a strict left m_2-inverse of S_2. Theorem 4.2 does not extend to $n(S)$-quasi strict $[m_1 + m_2 - 1, d]$-intertwinings. Thus, given T_1 an $n(S)$-quasi left 1-inverse of S_1 (i.e., $S^{*n}\Delta_{T_1,S_1}(I)S^n = 0$) and T_2 a strict left m-inverse of S_2 (i.e., $\Delta^m_{T_2,S_2}(I) = 0$ and $\Delta^{m-1}_{T_2,S_2}(I) \neq 0$), T_1T_2 may not be an $n(S)$-quasi strict left m-inverse of S_1S_2. To see this, consider operators S_i and T_i satisfying the commutativity hypotheses of Theorem 4.2 such that T_{11} is left 1-inverse of S_{11}, T_{21} is a left $(m - 1)$-inverse of S_{21} and T_{22} is a strict left m-inverse of S_{22}. Define S_i and T_i by

$$S_1 = S_{11} \oplus I, S_2 = S_{21} \oplus S_{22}, T_1 = T_{11} \oplus I \quad \text{and} \quad T_2 = T_{21} \oplus T_{22}$$

(with respect to the decomposition $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$ of $\mathcal{H}$). Then T_1 is an $n(S)$-quasi left 1-inverse of S_1, T_2 is a strict left m-inverse of S_2 and T_1T_2 is not an $n(S)$-quasi strict left m-inverse of S_1S_2.

(ii) Trivially, one may replace $n(S)$-quasi by $n(S_iS)$-quasi, $i = 1, 2$, in the conclusion of Theorem 4.2.

Given Hilbert spaces $\mathcal{H}_i$, $i = 1, 2$, let $H_1 \bar{\otimes} \mathcal{H}_2$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and, for $A_i \in B(\mathcal{H}_i)$, $i = 1, 2$, let $A_1 \otimes A_2 \in B(H_1 \bar{\otimes} \mathcal{H}_2)$ denote the tensor product of A_1 and A_2. Theorem 4.2 applies to tensor products of n-quasi left m-invertible, m-isometric and (m, C)-isometric operators. Let A_i, B_i $(i = 1, 2)$ and S, T be operators in $B(\mathcal{H})$.

Corollary 4.4. *If* $A_1^{*n} d^{m_1}_{B_1,A_1}(I) A_1^n = 0 = d^{m_2}_{B_2,A_2}(I)$ *and* $[A_1, B_1^*] = 0$, *then* $(A_1 \otimes A_2)^{*n} d^{m_1+m_2-1}_{B_1\otimes B_2, A_1\otimes A_2}(I\otimes I)(A_1 \otimes A_2)^n = 0$.

Proof. Define the operators S, S_i and T_i, $i = 1, 2$, by

$$S = S_1 = A_1 \otimes I, \quad T_1 = B_1 \otimes I, \quad S_2 = I \otimes A_2 \quad \text{and} \quad T_2 = I \otimes B_2.$$

Then, since $[A_1, B_1^*] = 0$,

$$[S_1, S_2] = 0 = [T_1, T_2] \quad \text{and} \quad [S, T_i^*] = 0 = [S, S_i]$$

($i = 1, 2$). Theorem 4.2 applies to prove

$$(A_1^* \otimes I)^n d^{m_1+m_2-1}_{B_1\otimes B_2, A_1\otimes A_2}(I\otimes I)(A_1 \otimes I)^n = 0.$$

Multiplying by $(I \otimes B_1^*)^n$ on the left and by $(I \otimes B_1)^n$ on the right, the proof follows. □

Translated to (m, C)-isometric operators, Theorem 3.1 and Corollary 3.2 imply the following.

Corollary 4.5. *Given conjugations C and D, if:*

(*i*) *$S, T \in B(\mathcal{H})$ are commuting operators such that $S^{*n}\Delta^{m_1}_{S^*,CSC}(I)S^n = 0 = \Delta^{m_2}_{T^*,DTD}(I)$, $[S, CSC] = 0 = [S, DTD]$ and $[T, CSC] = 0 = [DTD, CSC]$, then*

$$(ST)^{*n}\Delta^{m_1+m_2-1}_{S^*T^*,CSCDTD}(I)(ST)^n = (ST)^{*n}\left\{\sum_{j=0}^{m_1+m_2-1}(-1)^j\binom{m_1+m_2-1}{j}(ST)^{*(m_1+m_2-1-j)}(CSCDTD)^{m_1+m_2-1-j}\right\}(ST)^n = 0.$$

In particular, if $C = D$, then

$$(ST)^{*n}\Delta^{m_1+m_2-1}_{S^*T^*,CSTC}(I)(ST)^n = 0$$

(i.e., ST is n-quasi $(m_1 + m_2 - 1, C)$-isometric).

(*ii*) *$A^{*n}\Delta^{m_1}_{A^*,CAC}(I)A^n = 0 = B^{*n}\Delta^{m_2}_{B^*,DBD}(I)B^n$ and $[A, CAC] = 0$, then*

$$(A\otimes B)^{*n}\Delta^{m_1+m_2-1}_{A^*\otimes B^*,(CAC\otimes DBD)}(I\otimes I)(A\otimes B)^n = 0$$

(i.e., $A \otimes B$ is n-quasi $(m_1 + m_2 - 1, C \otimes D)$-isometric).

Proof. (i) If we define S_i and T_i, $i = 1, 2$, by $S_1 = CSC$, $S_2 = DTD$, $T_1 = S^*$ and $T_2 = T^*$, then S, S_i and T_i ($i = 1, 2$) satisfy the hypotheses of Theorem 4.2. Hence, the proof of (i). The proof of (ii) is evident. □

Corollary 4.4 generalizes [11, Theorem 2.3] (proved for the case $n = 0$ and $C = D$), and Corollaries 2.1, 3.5 and Proposition 3.5 (proved for the cases $n = 2, 3$ of part (ii) of our Corollary 4.4) of [11].

Corollary 4.5 takes the following simpler form for m-isometries.

Corollary 4.6. *Given operators $S, T \in B(\mathcal{H})$ such that $S^{*n}\Delta^{m_1}_{S^*,S}(I)S^n = 0 = \Delta^{m_2}_{T^*,T}(I)$ (i.e., S is n-quasi m_1-isometric and T is m_2-isometric):*

(*i*) *if $[S, T] = 0$, then $(ST)^{*n}\Delta^{m_1+m_2-1}_{S^*T^*,ST}(I)(ST)^n = 0$ (i.e., ST is n-quasi $(m_1 + m_2 - 1)$-isometric);*

(*ii*) *$(S\otimes T)^{*n}\Delta^{m_1+m_2-1}_{S^*\otimes T^*,S\otimes T}(I\otimes I)(S\otimes T)^n = 0$ (i.e., $S \otimes T$ is n-quasi $(m_1 + m_2 - 1)$-isometric).*

A version of Corollary 4.6 holds for m-self-adjoint and m-symmetric operators.

Corollary 4.7. *Let $S, T \in B(\mathcal{H})$ satisfy $[S, T] = 0$ and let C be a conjugation of $\mathcal{H}$. If:*

(*i*) *S is n-quasi m_1-self-adjoint and T is m_2 self-adjoint, then ST is n-quasi $(m_1 + m_2 - 1)$-self-adjoint (i.e., $S^{*n}\delta^{m_1}_{S^*,S}(I)S^n) = 0 = \delta^{m_2}_{T^*,T}(I) = 0$ implies $(ST)^{*n}\delta^{m_1+m_2-1}_{S^*T^*,ST}(I)(ST)^n = 0$);*

(ii) *S is n-quasi m_1-symmetric with the symmetry implemented by the conjugation C, T is m_2-symmetric with the symmetry implemented by the conjugation C and* $[S, CSC] = 0$, *then ST is n-quasi $m_1 + m_2 - 1$-symmetric with the symmetry implemented by the conjugation C (i.e.,* $S^{*n}\delta^{m_1}_{S^*,CMC}(I)S^n = 0 = \delta^{m_2}_{T^*,CTC}(I) = 0$ *and* $[S, CSC] = 0$ *implies* $(ST)^{*n}\delta^{m_1+m_2-1}_{S^*T^*,CSTC}(I)(ST)^n)$;

(iii) *S is n-quasi m_1-self-adjoint and T is m_2-self-adjoint, then $S \otimes T$ is n-quasi $(m_1 + m_2 - 1)$-self-adjoint;*

(iv) *S is n-quasi m_1-symmetric and T is m_2-symmetric (with the symmetry implemented by the conjugation C for S and T), then $S \otimes T$ is n-quasi $(m_1 + m_2 - 1)$-symmetric (with the symmetry implemented by the conjugation C).*

5 Perturbation by nilpotents

Gu [3, Theorem 2] proves that if $T \in B(\mathcal{X})$ is a left (right) m-inverse of $S \in B(\mathcal{X})$ and $N \in B(\mathcal{X})$ is an n-nilpotent which commutes with T, then $T + N$ is a left (resp., right) $(m + n - 1)$-inverse of S. Consequently, *If T is a left m-inverse of S, N_1 is an n_1-nilpotent which commutes with T and N_2 is an n_2-nilpotent which commutes with S, then $T + N_1$ is a left $(m + n_1 + n_2 - 2)$-inverse of $S + N_2$.* Translated to m-isometric (and (m, C)-isometric) operators S, this implies: *If $N \in B(\mathcal{H})$ is an n-nilpotent operator which commutes with S, then $S + N$ is an $(m + 2n - 2)$-isometric* [5] *(resp., $(m + 2n - 2, C)$-isometric* [9]) *operator.* A similar result holds for m-self-adjoint and (m, C)-symmetric operators [8,7]. In the following, we consider perturbation by commuting nilpotents of operators $S, T \in B(\mathcal{X})$ satisfying $d^m_{T,S}(I) = 0$, and using an elementary argument we prove:

Theorem 5.1. *If $d^m_{T,S}(I) = 0$ and $N \in B(\mathcal{X})$ is an n-nilpotent operator satisfying $[S, N] = 0$, then $d^{m+n-1}_{T,S+N}(I) = 0$.*

Proof. We start by proving that

$$\begin{aligned} \Delta^p_{T,S+N}(I) &= \sum_{j=0}^{p} \binom{p}{p-j} T^j \Delta^{p-j}_{T,S}(I) N^j \quad \text{and} \\ \delta^p_{T,S+N}(I) &= \sum_{j=0}^{p} (-1)^j \binom{p}{j} \delta^{p-j}_{T,S}(I) N^j. \end{aligned}$$

The proof is by induction. Both the equalities being true for $p = 1$, assume their validity for some $k > 1$. Then

$$\begin{aligned} \Delta^{k+1}_{T,S+N}(I) &= \Delta_{T,S}(\Delta^k_{T,S+N}(I)) + T\Delta^k_{T,S+N}(I)N \\ &= \Delta^{k+1}_{T,S}(I) + \left\{\binom{k}{k} + \binom{k}{k-1}\right\} T\Delta^k_{T,S}N + \left\{\binom{k}{k-1} + \binom{k}{k-2}\right\} T^2\Delta^{k-1}_{T,S}N^2 \\ &\quad \cdots + \left\{\binom{k}{1} + \binom{k}{0}\right\} T^k\Delta_{T,S}N^k + \binom{k}{0} T^{k+1}N^{k+1} \\ &= \sum_{j=0}^{k+1} \binom{k+1}{j} T^j \Delta^{k+1-j}_{T,S}(I) N^j \end{aligned}$$

and

$$\begin{aligned} \delta^{k+1}_{T,S+N}(I) &= \delta_{T,S}(\delta^k_{T,S+N}(I)) - \delta^k_{T,S+N}(I)N \\ &= \delta^{k+1}_{T,S}(I) + \left\{(-1)\binom{k}{0} - \binom{k}{1}\right\} \delta^k_{T,S}N + \left\{(-1)^2\binom{k}{2} - (-1)\binom{k}{1}\right\} \delta^{k-1}_{T,S}N^2 \\ &\quad \cdots + \left\{(-1)^k\binom{k}{k} - (-1)^{k-1}\binom{k}{k-1}\right\} \delta_{T,S}N^k - (-1)^k\binom{k}{k} N^{k+1} \\ &= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \delta^{k+1-j}_{T,S}(I) N^j. \end{aligned}$$

Recall now that $d^m_{T,S}(I) = 0$ implies $d^t_{T,S}(I) = 0$ for all integers $t \geq m$. Hence, since $N^j = 0$ for all $j \geq n$, $d^p_{T,S+N}(I) = 0$ for all p such that $p - n + 1 \geq m$ (in particular, if $p = m + n - 1$). □

Trivially, $d^m_{T,S}(I) = 0$ if and only if $d^m_{S^*,T^*}(I) = 0$ (where we have used I to denote the identity of both $B(\mathcal{X})$ and $B(\mathcal{X}^*)$). Hence:

Corollary 5.2. *If $d^m_{T,S}(I) = 0$ and $N_i \in B(\mathcal{X})$ $(i = 1, 2)$ are n_i-nilpotent operators satisfying $[S, N_1] = 0 = [T, N_2]$, then $d^{m+n_1+n_2-2}_{T+N_2,S+N_1}(I) = 0$.*

For perturbation by commuting nilpotents of n-quasi $[m, d]$-operators (i.e., operators $S, T \in B(\mathcal{H})$ such that $S^{*n}d^m_{T,S}(I)S^n = 0$), we have the following.

Theorem 5.3. *Suppose that $S^{*n}d^m_{T,S}(I)S^n = 0$ for some operators $S, T \in B(\mathcal{H})$ and integers $m, n \geq 1$. If $N_i \in B(\mathcal{H})$, $i = 1, 2$, are n_i-nilpotent operators such that $[S, N_1] = 0 = [S, T^*]$ and $[N_2, T] = 0 = [N_2^*, S]$, then*

$$(S^* + N_1^*)^{n+n_1-1}d^{m+n_1+n_2-2}_{T+N_2,S+N_1}(I)(S + N_1)^{n+n_1-1} = 0.$$

Proof. Letting S and T^* have the upper triangular representations (3) of Section 2, it follows from the hypotheses that N_1 and N_2 have the upper triangular representations

$$N_1 = \begin{pmatrix} N_{11} & N_{10} \\ 0 & N_{12} \end{pmatrix} \quad \text{and} \quad N_2^* = \begin{pmatrix} N_{21}^* & N_{20}^* \\ 0 & N_{22}^* \end{pmatrix}$$

(with respect to the decomposition $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$), where

$$N_{11}^{n_1} = N_{12}^{n_1} = 0 = N_{21}^{n_2} = N_{22}^{n_2} \text{ and } [N_{11}, S_1] = 0 = [N_{21}, T_1].$$

The hypothesis $S^{*n}d^m_{T,S}(I)S^n = 0$ implies $d^m_{T_1,S_1}(I_1) = 0$. Hence,

$$d^{m+n_1+n_2-2}_{T_1+N_{21},S_1+N_{11}}(I_1) = 0.$$

This, since

$$(S + N_1)^{n+n_1-1} = \begin{pmatrix} (S_1 + N_{11})^{n+n_1-1} & Z \\ 0 & 0 \end{pmatrix}$$

(for some operator Z) and

$$d^{m+n_1+n_2-2}_{T+N_2,S+N_1}(I) = \begin{pmatrix} 0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix}$$

for some operators Z_i $(i = 1, 2, 3)$, implies

$$(S^* + N_1^*)^{n+n_1-1}d^{m+n_1+n_2-2}_{T+N_2,S+N_1}(I)(S + N_1)^{n+n_1-1} = 0.$$

This completes the proof. □

More can be said in the case in which $T = S^*$ (i.e., when S is n-quasi m-isometric [11]).

Corollary 5.4. *Given an operator $S \in B(\mathcal{H})$ such that $S^{*n}\Delta^m_{S^*,S}(I)S^n = 0$, let $N \in B(\mathcal{H})$ be an n_1-nilpotent operator such that $[S, N] = 0$. Then:*

(i) *$S^{*(n+n_1-1)}\Delta^{m+2n_1-2}_{S^*+N^*,S+N}(I)(S + N)^{n+n_1-1} = 0$ (i.e., $(S + N)$ is an $(n + n_1 - 1)$-quasi $(m + 2n_1 - 2)$-isometric operator).*

(ii) *If $S_1 = S|_{\overline{S^n(\mathcal{H})}}$ has a dense range (or, S_1^* has SVEP at 0), then $(S + N)^{n+n_1-1}$ is similar to the operator $(S_1 + N_1)^{n+n_1-1} \oplus 0$.*

(*iii*) *There exists a positive operator* Q *and an operator* A *similar to* S^{n+n_1-1} *such that* $\Delta^{m+2n_1-2}_{A^*,A}(Q) = 0$ (*i.e.,* A *is* $(m+2n_1-2, Q)$*-isometric. Furthermore, if* S *is also left invertible, then* S^{n+n_1-1} *is similar to an* $(m+2n_1-2)$*-isometric operator.*

Proof. The proof of (i) follows from Theorem 5.3. To prove (ii), we start by observing that if we let $N = \begin{pmatrix} N_1 & N_0 \\ 0 & N_2 \end{pmatrix}$ (with respect to the decomposition $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$), then ($N_1^n = N_2^n = 0$ and $S_2^n = 0$ in the corresponding representation (3) for S)

$$(S+N)^{n+n_1-1} = \begin{pmatrix} (S_1+N_1)^{n+n_1-1} & X \\ 0 & 0 \end{pmatrix}$$

for some operator X. The operators S_1 and N_1 commute, and S_1 is left invertible (since S_1 is left m-invertible). Hence, since $\sigma_a(S_1+N_1) \subseteq \sigma_a(S_1) + \sigma_a(N_1) = \sigma_a(S_1)$, S_1+N_1 is left invertible. Define the operator $E \in B(\mathcal{H})$ by $E = \begin{pmatrix} (S_1+N_1)^{n+n_1-1} & X \\ 0 & 1 \end{pmatrix}$; then (since either of the hypotheses S_1 has a dense range and S_1^* has SVEP at 0 implies) E is invertible with

$$E^{-1} = \begin{pmatrix} (S_1+N_1)^{-(n+n_1-1)} & -(S_1+N_1)^{-(n+n_1-1)}X \\ 0 & 1 \end{pmatrix}.$$

If we now define $A \in B(\mathcal{H})$ by $A = (S_1+N_1)^{n+n_1-1} \oplus 0$, then $(S+N)^{n+n_1-1} = E^{-1}AE$. To prove (iii), we start by observing from the proof of Theorem 5.3 that the current hypotheses imply $(S_1+N_1)^p$ is $(m+2n_1-2)$-isometric and $(S+N)^p$ is $(n+n_1-1)$-quasi $(m+2n_1-2)$-isometric for all integers $p \geq 1$. Choose $p = n+n_1-1$ and let $(S_1+N_1)^{n+n_1-1}$ have the polar decomposition $(S_1+N_1)^{n+n_1-1} = U_1P_1$ (so that U_1 is an isometry and P_1 is positive invertible). Let $(S+N)^{n+n_1-1} = \begin{pmatrix} U_1P_1 & X \\ 0 & 0 \end{pmatrix}$ and argue as in the proof of Proposition 2.2. Then, upon defining $Q \geq 0$ as in the proof of Proposition 2.2 and letting $m+2n_1-2-j = t$,

$$(S+N)^{*(n+n_1-1)} \left\{ \sum_{j=0}^{m+2n_1-2} (-1)^j \binom{m+2n_1-2}{j} (S+N)^{*t(n+n_1-1)} (S+N)^{t(n+n_1-1)} \right\} (S+N)^{n+n_1-1} = 0$$
$$\Leftrightarrow \sum_{j=0}^{m+2n_1-2} (-1)^j \binom{m+2n_1-2}{j} \begin{pmatrix} U_1^*P_1 & 0 \\ X^*P_1 & 0 \end{pmatrix}^t Q \begin{pmatrix} P_1U_1 & P_1X \\ 0 & 0 \end{pmatrix}^t = 0.$$

Now define the operator A by $A = \begin{pmatrix} P_1U_1 & P_1X \\ 0 & 0 \end{pmatrix}$. Then A is $(m+2n_1-2, Q)$-isometric and $S^{n+n_1-1} = P^{-1}AP$, where $P = P_1 \oplus I_2$. To complete the proof, assume now that S is left invertible. Then P and Q are invertible positive operators, $B = Q^{\frac{1}{2}}AQ^{\frac{-1}{2}}$ is $(m+2n_1-2)$-isometric and $S^{n+n_1-1} = E^{-1}BE$, $E = Q^{\frac{1}{2}}P$. □

The corresponding result for n-quasi (m, C)-isometries S, $S^{*n}\Delta^m_{S^*,CSC}(I)S^n = 0$, such that $C = C_1 \oplus C_2$ (with respect to the decomposition $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$) is the following. Define the operator M (as before) by $M = \begin{pmatrix} U_1 & X \\ 0 & 0 \end{pmatrix}$, where the isometry U_1 and the operator X are as in the polar decomposition (above) of S^{n+n_1-1}.

Corollary 5.5. *Let* $S \in B(\mathcal{H})$ *be an* n*-quasi* $[m, C]$*-isometry such that* $C = C_1 \oplus C_2$ *with respect to the decomposition* $\mathcal{H} = \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0)$. *If* $N \in B(\mathcal{H})$ *is an* n_1*-nilpotent operator such that* $[S, N] = 0$, *then:*

(*i*) $S+N$ *is* $(n+n_1-1)$*-quasi* $(m+2n-1, C)$*-isometric.*

(*ii*) $(S+N)^{n+n_1-1}$ *is similar to* $(S_1+N_1)^{n+n_1-1} \oplus 0$, $S_1 = S|_{\overline{S^n(\mathcal{H})}}$ *and* $N_1 = N|_{\overline{S^n(\mathcal{H})}}$, *whenever* S_1 *has a dense range (or* S_1^* *has SVEP at 0).*

(*iii*) *If also* $[C, M] = 0$, *then* $(S+N)^{n+n_1-1}$ *is similar to an* $(m+2n_1-2, C)$*-isometry.*

Proof. The hypothesis

$$\begin{aligned} S^{*n}\Delta^m_{S^*,CSC}(I)S^n = 0 &\Rightarrow \Delta^m_{S_1^*,C_1S_1C_1}(I_1) = 0 \\ &\Leftrightarrow \Delta^m_{C_1S_1^*C_1,S_1}(I_1) = 0 \Rightarrow \Delta^{m+n_1-1}_{C_1S_1^*C_1,S_1+N_1}(I_1) = 0 \\ &\Leftrightarrow \Delta^{m+n-1}_{C_1(S_1+N_1)^*C_1,S_1}(I_1) = 0 \Rightarrow \Delta^{m+2n_1-2}_{C_1(S_1+N_1)^*C_1,S_1+N_1}(I_1) = 0 \\ &\Leftrightarrow \Delta^{m+2n_1-2}_{S_1^*+N_1^*,C_1(S_1+N_1)C_1}(I_1) = 0 \\ &\Rightarrow (S+N)^{*(n+n_1-1)}\Delta^{m+2n_2-2}_{S^*+N^*,C(S+N)C}(I)S^{n+n_1-1} = 0. \end{aligned}$$

This proves (i). The proof of (ii) follows from the proof of Corollary 5.4, and the proof of (iii) follows from the argument of the proof of Corollary 5.4 and Proposition 2.4 applied to

$$\begin{aligned} &(S+N)^{*(n+n_1-1)}\Delta^{m+2n_2-2}_{S^*+N^*,C(S+N)C}(I)S^{n+n_1-1} = 0 \\ &\Rightarrow \sum_{j=0}^{m+2n_1-2}(-1)^j\binom{m+2n_1-2}{j}\begin{pmatrix} U_1^*P_1 & 0 \\ X^*P_1 & 0\end{pmatrix}^t\begin{pmatrix} I_1 & U_1^*C_1XC_2 \\ X^*U_1C_1U_1 & X^*C_1XC_2\end{pmatrix}\begin{pmatrix} C_1P_1U_1C_1 & C_1P_1XC_2 \\ 0 & 0\end{pmatrix}^t = 0, \end{aligned}$$

where $t = m + 2n_1 - 2 - j$. This completes the proof. □

We remark in closing that Corollaries 5.4 and 5.5 have an m-self-adjoint and m-symmetric operator version. For example, if $S \in B(\mathcal{H})$ satisfies $S^{*n}\delta^m_{S^*,S}(I)S^n = 0$ and $N \in B(\mathcal{H})$ is an n_1-nilpotent which commutes with S, then:

(i) $S_1 + N_1$, where $S_1 = S|_{\overline{S^n(\mathcal{H})}}$ and $N_1 = N|_{\overline{S^n(\mathcal{H})}}$, satisfies $\delta^{m+2n_1-2}_{S_1^*+N_1^*,S_1+N_1}(I_1) = 0$;

(ii) $(S^* + N^*)^{n+n_1-1}\delta^{m+2n_1-2}_{S^*+N^*,S+N}(I)(S+N)^{n+n_1-1} = 0$;

(iii) if S is also left invertible, then $(S+N)^{n+n_1-1}$ is similar to an $(m + 2n_1 - 2)$-self-adjoint operator.

We leave the proof of the above, and the formulation of the corresponding result for m-symmetric operators (for which $C = C_1 \oplus C_2 : \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0) \to \overline{S^n(\mathcal{H})} \oplus S^{*-n}(0))$) to the reader.

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References

[1] O. A. M. Sid Ahmed, *Some properties of m-isometries and m-invertible operators in Banach spaces*, Acta Math. Sci. Ser. B English Ed. **32** (2012), 520–530.
[2] B. P. Duggal and V. Müller, *Tensor product of left n-invertible operators*, Studia Math. **215** (2013), no. 2, 113–125.
[3] C. Gu, *Structure of left n-invertible operators and their applications*, Studia Math. **226** (2015), 189–211.
[4] J. Agler and M. Stankus, *m-isometric transformations of Hilbert space I*, Integr. Equat. Oper. Theory **21** (1995), 383–420.
[5] T. Bermûdez, A. Martinón, V. Müller, and J. N. Noda, *Perturbation of m-isometries by nilpotent operators*, Abstr. Appl. Anal. **2014** (2014), 745479, DOI: 10.1155/2014/745479.
[6] B. P. Duggal, *Tensor product of n-isometries III*, Funct. Anal. Approx. Comput. **4** (2012), no. 2, 61–67.
[7] T. Le, *Algebraic properties of operator roots of polynomials*, J. Math. Anal. Appl. **421** (2015), no. 2, 1238–1246.
[8] M. Chō, E. Ko, and J. E. Lee, *On (m,C)-isometric operators*, Complex Anal. Oper. Theory **10** (2016), 1679–1694, DOI: 10.1007/s11785-016-0549-0.
[9] M. Chō, J. E. Lee, and H. Motoyoshi, *On [m,C]-isometric operators*, Filomat **31** (2017), no. 7, 2073–2080, DOI: 10.2307/26194944.
[10] S. Mecheri and S. M. Patel, *On quasi 2-isometric operators*, Linear Multilinear Algebra **66** (2018), no. 5, 1019–1025, DOI: 10.1080/03081087.2017.1335283.
[11] O. A. M. Sid Ahmed, M. Chō, and J. E. Lee, *On n-quasi-(m,C)-isometric operators*, Linear Multilinear Algebra **68** (2020), no. 5, 1001–1020, DOI: 10.1080/03081087.2018.1524437.

[12] I. H. Kim, *On* (p,k)*-quasihyponormal operators*, Math. Inequal. Appl. **7** (2004), 629–638.
[13] P. Aiena, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic Publishers, New York, Boston, Dordrecht, London, Moscow, 2004.
[14] H. G. Heuser, *Functional Analysis*, John Wiley and Sons, Chicester, New York, Brisbane, Toronto, Singapore, 1982.
[15] K. B. Laursen and M. N. Neumann, *Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.
[16] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, John Wiley and Sons, New York, 1980.
[17] B. P. Duggal, *Hereditarily normaloid operators*, Extracta Math. **20** (2005), no. 2, 203–217.
[18] L. Suciu and N. Suciu, *Ergodic conditions and spectral properties for A-contractions*, Opscula Math. **28** (2008), 195–216.
[19] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd edn, Springer, New York, 1982.
[20] D. Koehler and P. Rosenthal, *On isometries of normed linear spaces*, Studia Math. **36** (1970), 213–216.
[21] B. P. Duggal and C. S. Kubrusly, *Power bounded left* m*-invertible operators*, Linear Multilinear Algebra (2019), DOI: 10.1080/03081087.2019.1604623.