

## Research Article

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# Numerical approach to the controllability of fractional order impulsive differential equations

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**Abstract:** In this manuscript, a numerical approach for the stronger concept of exact controllability (*total controllability*) is provided. The proposed control problem is a nonlinear fractional differential equation of order  $\alpha \in (1, 2]$  with non-instantaneous impulses in finite-dimensional spaces. Furthermore, the numerical controllability of an integro-differential equation is briefly discussed. The tool for studying includes the Laplace transform, the Mittag-Leffler matrix function and the iterative scheme. Finally, a few numerical illustrations are provided through MATLAB graphs.

**Keywords:** fractional differential equation, non-instantaneous impulses, total controllability, Mittag-Leffler matrix function

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## 1 Introduction

Sometimes, integer-order differential equation becomes inadequate to model some physical phenomena such as in anomalous diffusion. Such phenomena give rise to fractional order differential equations. The main advantage of studying fractional order systems is that they allow greater degrees of freedom in the model. Differential equations of fractional order appear more often in diverse areas of science and engineering, such as image processing, signal processing, bio-engineering, viscoelasticity, fluid flow and control theory [1–6]. For the fundamental understanding of fractional calculus and related numerical methods, one can refer to [7–11].

On the other hand, some phenomena are characterized by rapid changes. The first kind of changes takes place over a relatively short time compared to the overall duration of the entire process. Mathematical models in these cases are developed using impulsive differential equations. In the second kind, the changes are not negligibly short in duration and these changes begin impulsively at some points and remain active over certain time intervals. The mathematical model of these situations gives rise to a differential equation with non-instantaneous impulses.

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The study of non-instantaneous impulsive differential equations has significant applications in different areas, for example, in hemodynamical equilibrium and the theory of rocket combustion. An excellent application of non-instantaneous impulse is the introduction of insulin into the bloodstream. It produces an abrupt change in the bloodstream. The consequent absorption is a gradual process that remains active over a finite time span.

Recently, many researchers have shown their interest in existence, uniqueness of solutions, stability and controllability of impulsive problems with non-instantaneous impulses [12–16]. Hernández and Regan [17] studied mild and classical solutions for the impulsive differential equation with non-instantaneous impulses. Wang and Fečkan [18] have shown existence, uniqueness and stability of solutions of such a general class of first-order impulsive differential equations. Later, Muslim *et al.* [12] investigated existence, uniqueness of solutions and stability of second-order differential equations with non-instantaneous impulses. However, the controllability of the non-instantaneous impulsive control system is the less treated topic as compared to the existence and uniqueness of solutions.

In the setting of controllability, the control system is an interconnection of components forming a system configuration that will result in a desired system response. Controllability is one of the structural properties of dynamical systems. It provides the ability to move a system around entire configuration space using only certain feasible manipulations. It deals with whether or not the state of a state-space dynamic system can be controlled from the input. Many authors dealt with controllability problems that can be found in many recently published papers [19–22].

Recently, Wang *et al.* [19] discussed controllability of fractional non-instantaneous impulsive differential inclusions. However, Wang *et al.* achieved exact controllability by only applying control in the last subinterval of time. But, Wang *et al.* did not propose any computational scheme for the steering control. In this manuscript, the control is applied for each subinterval of time, due to which the concept of total controllability arises. Moreover, none of the research papers have so far discussed the numerical approach for the controllability of the non-instantaneous impulsive differential equation of order  $\alpha \in (1, 2]$ . Therefore, this manuscript is devoted to the study of numerical controllability for the following fractional order nonlinear differential equation with non-instantaneous impulses in a space  $\mathbb{R}^n$ :

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + Bw(t) + g(t, u(t)), & t \in (s_i, t_{i+1}], \quad i = 0, 1, \dots, m, \\ u(t) = \Psi_i^1(t, u(t_i^-)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ u'(t) = \Psi_i^2(t, u(t_i^-)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \quad (1.1)$$

where  $\alpha \in (1, 2]$  and  $u(t)$  is a state function with time interval  $0 = s_0 = t_0 < t_1 < s_1 < t_2, \dots, t_m < s_m < t_{m+1} = T < \infty$ . Let  $A$  be a coefficient matrix of system (1.1). The control function  $w(\cdot) \in L^2(J_1 := \bigcup_{i=0}^m [s_i, t_{i+1}], \mathbb{R}^m)$ . Let  $B$  be a bounded linear operator from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . Consider the state function  $u \in C((t_i, t_{i+1}], \mathbb{R}^n)$ ,  $i = 0, 1, \dots, m$ , and there exist  $u(t_i^-)$  and  $u(t_i^+)$ ,  $i = 1, 2, \dots, m$  with  $u(t_i^-) = u(t_i)$ . The functions  $\Psi_i^1(t, u(t_i^-))$  and  $\Psi_i^2(t, u(t_i^-))$  represent non-instantaneous impulses during the intervals  $(t_i, s_i]$ ,  $i = 1, 2, \dots, m$ .  $\Psi_i^1$ ,  $\Psi_i^2$  and  $g$  are suitable functions. These functions will be explained briefly in the subsequent sections.

The manuscript proceeds as follows. In Sections 1–3, the introduction, notations, results and required assumptions are given, which will be required for the later sections. In Section 4, controllability of problem (1.1) is investigated by the iterative scheme. Later, controllability of the integro-differential equation is briefly mentioned in Section 5. In Section 6, a few numerical examples are given to show the application of the obtained results.

## 2 Preliminaries and assumptions

In this section, some useful definitions related to fractional calculus are briefly reviewed. Also, some necessary properties concerned with the Mittag-Leffler function are discussed.

Let  $L^1(\mathbb{R}^+, \mathbb{R}^n)$  and  $C^1(\mathbb{R}^+, \mathbb{R}^n)$ ,  $\mathbb{R}^+ = (0, \infty)$  be the space of all integrable functions and continuously differentiable functions, respectively. Furthermore,  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space, where  $n \in \mathbb{N}$ .

**Definition 2.1.** [24] The Riemann-Liouville fractional integral of order  $\alpha > 0$  is defined by

$$J_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds,$$

where  $g(t) \in L^1(\mathbb{R}^+, \mathbb{R}^n)$  and  $\Gamma(\cdot)$  is the gamma function.

**Definition 2.2.** [24] If  $g(t) \in L^1(\mathbb{R}^+, \mathbb{R}^n)$ , then the Riemann-Liouville fractional derivative of order  $\alpha \in (1, 2]$  is defined by

$${}^{\text{RL}}D_t^\alpha g(t) = \frac{d^2}{dt^2} J_t^{2-\alpha} g(t),$$

where  ${}^{\text{RL}}D_t^\alpha g(t) \in L^1(\mathbb{R}^+, \mathbb{R}^n)$ .

**Definition 2.3.** [24] The Caputo fractional derivative of order  $\alpha \in (1, 2]$  is defined by

$${}^{\text{C}}D_t^\alpha g(t) = J_t^{2-\alpha} \frac{d^2}{dt^2} g(t),$$

where  $g(t) \in L^1(\mathbb{R}^+, \mathbb{R}^n) \cap C^1(\mathbb{R}^+, \mathbb{R}^n)$ .

The Laplace transform of the Riemann-Liouville fractional derivatives is defined by

$$\mathcal{L}\{{}^{\text{RL}}D_t^\alpha g(t)\} = s^\alpha G(s) - \sum_{k=1}^n s^{k-1} g^{(\alpha-k)}(0^+).$$

The Laplace transform of the Caputo fractional derivatives is defined by

$$\mathcal{L}\{{}^{\text{C}}D_t^\alpha g(t)\} = s^\alpha G(s) - \sum_{k=1}^n s^{\alpha-k} g^{(k-1)}(0^+).$$

In particular, if  $\alpha \in (1, 2]$ , then

$$\mathcal{L}\{{}^{\text{C}}D_t^\alpha g(t)\} = s^\alpha G(s) - g(0^+) s^{\alpha-1} - g'(0^+) s^{\alpha-2}.$$

## 2.1 Mittag-Leffler function

The Mittag-Leffler function is a generalization of the exponential function, and it plays an important role in the solution of the fractional differential equations.

**Definition 2.4.** [23] A function of the complex variable  $z$  defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (2.1)$$

is called the one-parameter Mittag-Leffler function.

In particular, when  $\alpha = 1$ , we obtain

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

i.e., the classical exponential function.

An extension of the one-parameter Mittag-Leffler function is given by the following two-parameter function.

**Definition 2.5.** [23] A function of the complex variable  $z$  defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0 \quad (2.2)$$

is called the two-parameter Mittag-Leffler function.

For  $\beta = 1$ , from (2.2) we obtain (2.1). Moreover, we have the following identities:

$$\begin{aligned} E_{1,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z, \\ E_{0,1}(z) &= \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \\ E_{2,1}(-z^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+1)} = \cos z, \\ E_{2,2}(-z^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+2)} = \frac{\sin z}{z}. \end{aligned}$$

**Definition 2.6.** [23] A function of the matrix  $A \in \mathbb{R}^{n \times n}$  defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0$$

is called the two-parameter Mittag-Leffler matrix function.

Let us consider the following fractional impulsive differential equation of order  $\alpha \in (1, 2]$  without control term  $Bw(t)$ :

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + g(t), & t \in (s_i, t_{i+1}], \quad i = 0, 1, \dots, m, \\ u(t) = \Psi_i^1(t, u(t_i^-)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ u'(t) = \Psi_i^2(t, u(t_i^-)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \quad (2.3)$$

where  $u \in \mathbb{R}^n$  and  $A \in \mathbb{R}^{n \times n}$  is a coefficient matrix of system (2.3). Let  $g : J_1 \rightarrow \mathbb{R}^n$ ,  $J_1 = \bigcup_{i=0}^m [s_i, t_{i+1}]$  be a continuous function. The solution of system (2.3) is obtained by applying the Laplace transform method. It is given by the following integral equation [23,24]:

$$\begin{aligned} u(t) &= \Phi_0(t - s_i) \Psi_i^1(s_i, u(t_i^-)) + \Phi_1(t - s_i) \Psi_i^2(s_i, u(t_i^-)) \\ &\quad + \int_{s_i}^t \Phi(t - s) g(s) ds, \quad \forall t \in [s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m, \end{aligned} \quad (2.4)$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha), \quad \Phi_1(t) = tE_{\alpha,2}(At^\alpha), \quad \Phi(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$$

and

$$\Psi_i^1(0, \cdot) = u_0, \quad \Psi_i^2(0, \cdot) = v_0.$$

### 3 Controllability for the linear system

We consider the linear impulsive differential equation of order  $\alpha \in (1, 2]$  with control function  $u(t)$  as follows:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + Bw(t), & t \in (s_i, t_{i+1}], \quad i = 0, 1, \dots, m, \\ u(t) = \Psi_i^1(t, u(t_i^-)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ u'(t) = \Psi_i^2(t, u(t_i^-)), & t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \quad (3.1)$$

where  $u \in \mathbb{R}^n$  and control function  $w(\cdot) \in L^2(J_1 := \bigcup_{i=0}^m [s_i, t_{i+1}], \mathbb{R}^m)$ . Let  $B$  be a bounded linear operator from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . The solution of the linear impulsive system (3.1) is given by the following integral equation:

$$\begin{aligned} u(t) = & \Phi_0(t - s_i) \Psi_i^1(s_i, u(t_i^-)) + \Phi_1(t - s_i) \Psi_i^2(s_i, u(t_i^-)) \\ & + \int_{s_i}^t \Phi(t - s) B w(s) ds, \quad \forall t \in [s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m. \end{aligned} \quad (3.2)$$

**Definition 3.1.** (Exact controllability) [22] System (3.1) is said to be exactly controllable on  $[0, T]$ , if for the initial state  $u(0) \in \mathbb{R}^n$  and arbitrary final state  $u_f(t_{m+1}) \in \mathbb{R}^n$ , there exists a control  $w \in L^2(J_1, \mathbb{R}^m)$  such that solution (3.2) satisfies  $u(t_{m+1}) = u_f(t_{m+1})$ .

**Definition 3.2.** (Total controllability) [22] System (3.1) is said to be totally controllable on  $[0, T]$ , if for the initial state  $u(0) \in \mathbb{R}^n$  and arbitrary final state  $u_f(t_{i+1}) \in \mathbb{R}^n$  of each sub-interval  $[s_i, t_{i+1}]$ , there exists a control  $w \in L^2(J_1, \mathbb{R}^m)$  such that solution (3.2) satisfies  $u(t_{i+1}) = u_f(t_{i+1})$ , where  $i = 0, 1, \dots, m$ .

**Remark.** Total controllability  $\Rightarrow$  Exact controllability.

**Lemma 3.3.** [21,25] The linear system (3.1) is controllable on  $[0, T]$  iff the controllability Gramian

$$M_{s_i}^{t_{i+1}} = \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) B B^* \Phi^*(t_{i+1} - s) ds$$

is nonsingular, where  $s_0 = 0$ ,  $t_{i+1} = T$ ,  $i = 0, 1, 2, \dots, m$ .

In order to prove the controllability for the nonlinear system (1.1), the following assumptions are taken:

(A1)  $g : J_1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $J_1 = \bigcup_{i=0}^m [s_i, t_{i+1}]$  is a continuous function and there exist positive constants  $\lambda_1$  and  $\lambda_2$  such that

$$\|g(t, u) - g(t, v)\| \leq \lambda_1 \|u - v\|$$

and  $\|g(t, u)\| \leq \lambda_2$ , for every  $u, v \in \mathbb{R}^n$ ,  $t \in J_1$ .

(A2) There exist positive constants  $C_{\Psi_i^1}$  and  $C_{\Psi_i^2}$ ,  $i = 1, 2, \dots, m$  such that

$$C_{\Psi_i^1} = \max_{t \in I_i} \|\Psi_i^1(t, \cdot)\| \text{ and } C_{\Psi_i^2} = \max_{t \in I_i} \|\Psi_i^2(t, \cdot)\|, \text{ where } I_i := [t_i, s_i].$$

(A3)  $\Psi_i^k \in C(I_i \times \mathbb{R}^n, \mathbb{R}^n)$  and there are positive constants  $L_{\Psi_i^k}$ ,  $i = 1, 2, \dots, m$ ,  $k = 1, 2$ , such that

$$\|\Psi_i^k(t, u) - \Psi_i^k(t, v)\| \leq L_{\Psi_i^k} \|u - v\|, \quad \forall t \in I_i \text{ and } u, v \in \mathbb{R}^n.$$

Let  $\mathcal{H} = PC([0, T], \mathbb{R}^n)$  be the space of piecewise continuous functions.

$PC([0, T], \mathbb{R}^n) := \{u : [0, T] \rightarrow \mathbb{R}^n : u \in C((t_k, t_{k+1}], \mathbb{R}^n), \quad k = 0, 1, \dots, m, \text{ and there exist } u(t_k^-) \text{ and } u(t_k^+), \quad k = 1, 2, \dots, m\}$ . It can be easily proved that  $PC([0, T], \mathbb{R}^n)$ , for all  $t \in [0, T]$ , is a Banach space endowed with the supremum norm. For the sake of notational convenience, let us define

$$K_{\Phi_0} = \sup_{t \in [0, T]} \|\Phi_0(t)\|; \quad K_{\Phi_1} = \sup_{t \in [0, T]} \|\Phi_1(t)\|;$$

$$K_{\Phi} = \sup_{t \in [0, T]} \|\Phi(t)\|; \quad K_M = \|(M_{S_i}^{t_{i+1}})^{-1}\|; \quad K_B = \|B\|.$$

**Lemma 3.4.** *If all the assumption (A2) is fulfilled, then the control function for problem (3.1) has an estimate  $\|w(t)\| \leq \delta$ ,  $\forall t \in [s_i, t_{i+1}]$ ,  $i = 0, 1, 2, \dots, m$ , where*

$$\delta = K_B K_{\Phi} K_M (\|u_f(t_{i+1})\| + K_{\Phi_0} C_{\Psi_i^1} + K_{\Phi_1} C_{\Psi_i^2}).$$

**Proof.** The control function for  $t \in [s_i, t_{i+1}]$ ,  $i = 0, 1, 2, \dots, m$ , is defined as follows:

$$w(t) = B^* \Phi^*(t_{i+1} - t) (M_{S_i}^{t_{i+1}})^{-1} [u_f(t_{i+1}) - \Phi_0(t_{i+1} - s_i) (\Psi_i^1(s_i, u(t_i^-))) - \Phi_1(t_{i+1} - s_i) (\Psi_i^2(s_i, u(t_i^-)))], \quad (3.3)$$

where  $u_f(t_{i+1})$  is the arbitrary final state of each sub-interval  $[s_i, t_{i+1}]$ ,  $i = 0, 1, \dots, m$ .

By solution (3.2), the final state at  $t = t_{i+1}$  is

$$\begin{aligned} u(t_{i+1}) &= \Phi_0(t_{i+1} - s_i) (\Psi_i^1(s_i, u(t_i^-))) + \Phi_1(t_{i+1} - s_i) (\Psi_i^2(s_i, u(t_i^-))) + \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) B w(s) ds \\ &= \Phi_0(t_{i+1} - s_i) (\Psi_i^1(s_i, u(t_i^-))) + \Phi_1(t_{i+1} - s_i) (\Psi_i^2(s_i, u(t_i^-))) + \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) B B^* \Phi^*(t_{i+1} - t) ds \\ &\quad \times \left\{ (M_{S_i}^{t_{i+1}})^{-1} [u_f(t_{i+1}) - \Phi_0(t_{i+1} - s_i) (\Psi_i^1(s_i, u(t_i^-))) - \Phi_1(t_{i+1} - s_i) (\Psi_i^2(s_i, u(t_i^-)))] \right\} \\ &= \Phi_0(t_{i+1} - s_i) (\Psi_i^1(s_i, u(t_i^-))) + \Phi_1(t_{i+1} - s_i) (\Psi_i^2(s_i, u(t_i^-))) \\ &\quad + (M_{S_i}^{t_{i+1}})^{-1} (M_{S_i}^{t_{i+1}})^{-1} [u_f(t_{i+1}) - \Phi_0(t_{i+1} - s_i) (\Psi_i^1(s_i, u(t_i^-))) - \Phi_1(t_{i+1} - s_i) (\Psi_i^2(s_i, u(t_i^-)))] \\ &= u_f(t_{i+1}). \end{aligned}$$

Hence, control function (3.3) is suitable for problem (3.1), for every  $t \in [s_i, t_{i+1}]$  and  $i = 0, 1, 2, \dots, m$ . The estimate of control function  $w(t)$  is given by:

$$\begin{aligned} \|w(t)\| &\leq \|B^* \Phi^*(t_{i+1} - t) (M_{S_i}^{t_{i+1}})^{-1}\| [\|u_f(t_{i+1})\| + \|\Phi_0(t_{i+1} - s_i) \Psi_i^1(s_i, u(t_i^-))\| + \|\Phi_1(t_{i+1} - s_i) \Psi_i^2(s_i, u(t_i^-))\|] \\ &\leq K_B K_{\Phi} K_M (\|u_f(t_{i+1})\| + K_{\Phi_0} C_{\Psi_i^1} + K_{\Phi_1} C_{\Psi_i^2}). \end{aligned}$$

Hence, the required estimate for control (3.3) is obtained.  $\square$

## 4 Controllability for the nonlinear system

Steering of a dynamical control system from an arbitrary initial state to an arbitrary final state on each sub-interval  $[t_k, t_{k+1}]$  using the set of admissible controls is called a totally controllable system. In this section, total controllability of system (1.1) is investigated through the iterative scheme.

**Theorem 4.1.** *If all the assumptions (A1)–(A3) are satisfied and the linear system (3.1) is controllable, then the nonlinear system (1.1) is totally controllable on  $[0, T]$ .*

**Proof.** In order to prove the controllability results, we adopt the successive approximation technique. Let us define the iterative scheme as follows:

$$\begin{aligned}
u_0(t) &= u_0 \\
u_{n+1}(t) &= \Phi_0(t - s_i) \Psi_i^1(s_i, u_n(t_i^-)) + \Phi_1(t - s_i) \Psi_i^2(s_i, u_n(t_i^-)) + \int_{s_i}^t \Phi(t - s) B w_n(s) ds \\
&\quad + \int_{s_i}^t \Phi(t - s) g(s, u_n(s)) ds, \quad \forall t \in [s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
w_n(t) &= B^* \Phi^*(t_{i+1} - t) (M_{s_i}^{t_{i+1}})^{-1} \left[ u_f(t_{i+1}) - \Phi_0(t_{i+1} - s_i) \Psi_i^1(s_i, u_n(t_i^-)) \right. \\
&\quad \left. - \Phi_1(t_{i+1} - s_i) \Psi_i^2(s_i, u_n(t_i^-)) - \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) g(s, u_n(s)) ds \right]
\end{aligned} \tag{4.2}$$

and  $n = 0, 1, 2, \dots$

Since the initial vector  $u_0$  is given, the sequence  $\{u_n(t)\}$  can be easily obtained. We will show that the sequence  $\{u_n(t)\}$  is Cauchy in  $\mathcal{H}$ . Moreover, we observe that

$$\begin{aligned}
\|w_n(t)\| &\leq \|B^* \Phi^*(t_{i+1} - t) (M_{s_i}^{t_{i+1}})^{-1}\| \left[ \|u_f(t_{i+1})\| + \|\Phi_0(t_{i+1} - s_i) \Psi_i^1(s_i, u_n(t_i^-))\| \right. \\
&\quad \left. + \|\Phi_1(t_{i+1} - s_i) \Psi_i^2(s_i, u_n(t_i^-))\| + \int_{s_i}^{t_{i+1}} \|\Phi(t_{i+1} - s)\| \|g(s, u_n(s))\| ds \right] \\
&\leq K_B K_\Phi K_M (\|u_f(t_{i+1})\| + K_{\Phi_0} C_{\Psi_i^1} + K_{\Phi_1} C_{\Psi_i^2} + \lambda_2 K_\Phi T)
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\|w_n(t) - w_{n-1}(t)\| &\leq \|B^* \Phi^*(t_{i+1} - t) (M_{s_i}^{t_{i+1}})^{-1}\| \left[ \|\Phi_0(t_{i+1} - s_i) \Psi_i^1(s_i, u_n(t_i^-)) - \Psi_i^1(s_i, u_{n-1}(t_i^-))\| \right. \\
&\quad + \|\Phi_1(t_{i+1} - s_i) \Psi_i^2(s_i, u_n(t_i^-)) - \Psi_i^2(s_i, u_{n-1}(t_i^-))\| \\
&\quad \left. + \int_{s_i}^{t_{i+1}} \|\Phi(t_{i+1} - s)\| \|g(s, u_n(s)) - g(s, u_{n-1}(s))\| ds \right] \\
&\leq K_B K_\Phi K_M [K_{\Phi_0} L_{\Psi_i^1} + K_{\Phi_1} L_{\Psi_i^2} + \lambda_1 K_\Phi T] \|u_n(t) - u_{n-1}(t)\| \\
&\leq \Omega \|u_n(t) - u_{n-1}(t)\|,
\end{aligned} \tag{4.4}$$

where  $\Omega = K_B K_\Phi K_M [K_{\Phi_0} L_{\Psi_i^1} + K_{\Phi_1} L_{\Psi_i^2} + \lambda_1 K_\Phi T]$ . Furthermore, we have

$$\begin{aligned}
\|u_{n+1}(t) - u_n(t)\| &\leq \|\Phi_0(t - s_i) \Psi_i^1(s_i, u_n(t_i^-)) - \Psi_i^1(s_i, u_{n-1}(t_i^-))\| \\
&\quad + \|\Phi_1(t - s_i) \Psi_i^2(s_i, u_n(t_i^-)) - \Psi_i^2(s_i, u_{n-1}(t_i^-))\| \\
&\quad + \int_{s_i}^t \|\Phi(t - s)\| \|B\| \|w_n(s) - w_{n-1}(s)\| ds \\
&\quad + \int_{s_i}^t \|\Phi(t - s)\| \|g(s, u_n(s)) - g(s, u_{n-1}(s))\| ds \\
&\leq [K_{\Phi_0} L_{\Psi_i^1} + K_{\Phi_1} L_{\Psi_i^2} + K_B K_\Phi \Omega T + \lambda_1 K_\Phi T] \|u_n(t) - u_{n-1}(t)\| \\
&\leq \Theta \|u_n(t) - u_{n-1}(t)\|,
\end{aligned} \tag{4.5}$$

where  $\Theta = [K_{\Phi_0}L_{\Psi_i^1} + K_{\Phi_1}L_{\Psi_i^2} + K_B K_{\Phi} \Omega T + \lambda_1 K_{\Phi} T]$ . Moreover, it can be observed that

$$\begin{aligned} \|u_1(t) - u_0(t)\| &\leq \|\Phi_0(t - s_i)\| \|\Psi_i^1(s_i, u_0(t_i^-))\| + \|\Phi_1(t - s_i)\| \|\Psi_i^2(s_i, u_0(t_i^-))\| + \|u_0\| \\ &\quad + \int_{s_i}^t \|\Phi(t - s)\| \|B\| \|w_0(s)\| ds + \int_{s_i}^t \|\Phi(t - s)\| \|g(s, u_0(s))\| ds \\ &\leq [K_{\Phi_0} C_{\Psi_i^1} + K_{\Phi_1} C_{\Psi_i^2} + \|u_0\| + (K_B K_{\Phi} \Omega T + \lambda_1 K_{\Phi} T)] \\ &\leq [K_{\Phi_0} C_{\Psi_i^1} + K_{\Phi_1} C_{\Psi_i^2} + \|u_0\| + (K_B K_{\Phi} \Omega + \lambda_1 K_{\Phi}) T] \\ &\leq KT, K > 0. \end{aligned} \quad (4.6)$$

By using inequality (4.6) and the method of induction, the estimate for inequality (4.5) is as follows:

$$\|u_{n+1}(t) - u_n(t)\| \leq K \Theta^n \frac{T^{n+1}}{n!}. \quad (4.7)$$

The right-hand side in the aforementioned estimate (4.7) can be made arbitrarily small by choosing sufficiently large value of  $n$ . This implies that  $\{u_n(t)\}$  is a Cauchy sequence in  $\mathcal{H}$ . Since  $\mathcal{H}$  is a Banach space, the sequence  $\{u_n(t)\}$  converges uniformly to a continuous function  $u(t)$  on  $[0, T]$ . It is followed by taking limit as  $n \rightarrow \infty$  on both sides of (4.1) and (4.2). Thus, we have

$$\begin{aligned} u(t) &= \Phi_0(t - s_i) \Psi_i^1(s_i, u(t_i^-)) + \Phi_1(t - s_i) \Psi_i^2(s_i, u(t_i^-)) + \int_{s_i}^t \Phi(t - s) B w(s) ds \\ &\quad + \int_{s_i}^t \Phi(t - s) g(s, u(s)) ds, \quad \forall t \in [s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} w(t) &= B^* \Phi^*(t_{i+1} - t) \left( M_{s_i}^{t_{i+1}} \right)^{-1} \left[ u_f(t_{i+1}) - \Phi_0(t_{i+1} - s_i) \Psi_i^1(s_i, u(t_i^-)) \right. \\ &\quad \left. - \Phi_1(t_{i+1} - s_i) \Psi_i^2(s_i, u(t_i^-)) - \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) g(s, u(s)) ds \right]. \end{aligned} \quad (4.9)$$

Since, the control  $u(t)$  steers the system from the initial state  $u_0$  to final state  $u_f(t_{i+1})$  at time  $t_{i+1}$ , system (1.1) is said to be totally controllable on  $[0, T]$ .  $\square$

## 5 Controllability for an integro-differential equation

In this section, a control system represented by an integro-differential equation in space  $\mathbb{R}^n$  is considered as follows:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + Bw(t) + g(t, u(t)) + \int_0^t \xi(t - s) h(s, u(s)) ds, & t \in \bigcup_{i=0}^m (s_i, t_{i+1}], \\ u(t) = \Psi_i^1(t, u(t_i^-)), & t \in \bigcup_{i=1}^m (t_i, s_i], \\ u'(t) = \Psi_i^2(t, u(t_i^-)), & t \in \bigcup_{i=1}^m (t_i, s_i], \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \quad (5.1)$$



where  $\alpha \in (1, 2]$  and  $u(t)$  is a state function with time interval  $0 = s_0 = t_0 < t_1 < s_1 < t_2, \dots, t_m < s_m < t_{m+1} = T < \infty$ . Let  $A \in \mathbb{R}^{n \times n}$  be a coefficient matrix of system (5.1). The control function  $w(\cdot) \in L^2(J_1, \mathbb{R}^m)$  and  $B : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a bounded linear operator.

In order to prove the controllability of the integro-differential Eq. (5.1), the following conditions are required:

- (A4) The real-valued function  $\xi$  is piece-wise continuous on  $[0, T]$  and there exists a positive constant  $Y$  such that  $Y = \int_0^T |\xi(s)| ds$ .
- (A5)  $h : J_1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $J_1 = \bigcup_{i=0}^m [s_i, t_{i+1}]$  is a continuous function and there exist positive constants  $\lambda_3$  and  $\lambda_4$  such that

$$\|h(t, u) - h(t, v)\| \leq \lambda_3 \|u - v\|$$

$$\text{and } \|h(t, u)\| \leq \lambda_4, \text{ for every } u, v \in \mathbb{R}^n, \quad t \in J_1.$$

**Theorem 5.1.** *If all the assumptions (A1)–(A5) are satisfied and the linear system (3.1) is controllable, then the nonlinear integro-differential system (5.1) is totally controllable on  $[0, T]$ .*

**Proof.** Let us define an iterative scheme for the integro-differential system (5.1) as follows:

$$\begin{aligned} u_0(t) &= u_0 \\ u_{n+1}(t) &= \Phi_0(t - s_i) \Psi_i^1(s_i, u_n(t_i^-)) + \Phi_1(t - s_i) \Psi_i^2(s_i, u_n(t_i^-)) + \int_{s_i}^t \Phi(t - s) B w_n(s) ds \\ &\quad + \int_{s_i}^t \Phi(t - s) g(s, u_n(s)) ds + \int_{s_i}^t \Phi(t - s) \left( \int_0^s \xi(s - \eta) h(\eta, u_n(\eta)) d\eta \right) ds, \\ &\quad \forall t \in [s_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m, \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} w_n(t) &= B^* \Phi^*(t_{i+1} - t) (M_{s_i}^{t_{i+1}})^{-1} [u_f(t_{i+1}) - \Phi_0(t_{i+1} - s_i) \Psi_i^1(s_i, u_n(t_i^-)) \\ &\quad - \Phi_1(t_{i+1} - s_i) \Psi_i^2(s_i, u_n(t_i^-)) - \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) g(s, u_n(s)) ds \\ &\quad - \int_{s_i}^{t_{i+1}} \Phi(t_{i+1} - s) \left( \int_0^s \xi(s - \eta) h(\eta, u_n(\eta)) d\eta \right) ds] \end{aligned} \quad (5.3)$$

and  $n = 0, 1, 2, \dots$

Furthermore, the proof is similar to Theorem 4.1. Therefore, it is omitted.  $\square$

## 6 Application

In this section, we will consider the forced string problem and apply the results obtained in the previous section. The Mittag-Leffler matrix function will be evaluated by using Roberto Garrappa's MATLAB algorithm.

**Example 1.** Let us consider the linear fractional order impulsive system without control

$$\begin{cases} {}^C D_t^{5/4} u_1(t) = u_1(t) + u_2(t), & t \in (0, 1] \cup \left(\frac{3}{2}, 2\right], \\ {}^C D_t^{5/4} u_2(t) = 2u_1(t) + 3u_2(t), & t \in (0, 1] \cup \left(\frac{3}{2}, 2\right], \\ u(t) = \begin{bmatrix} \sin(t) \\ 3t + 2 \end{bmatrix}, & t \in \left(1, \frac{3}{2}\right], \\ u'(t) = \begin{bmatrix} \cos(t) \\ 3 \end{bmatrix}, & t \in \left(1, \frac{3}{2}\right] \end{cases} \quad (6.1)$$

with initial conditions  $\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} u'_1(0) \\ u'_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .

Comparing (6.1) with (1.1), we have  $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ ,  $\Psi_1^1 = \begin{bmatrix} \sin(t) \\ 3t + 2 \end{bmatrix}$ ,  $\Psi_1^2 = \begin{bmatrix} \cos(t) \\ 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and  $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ .

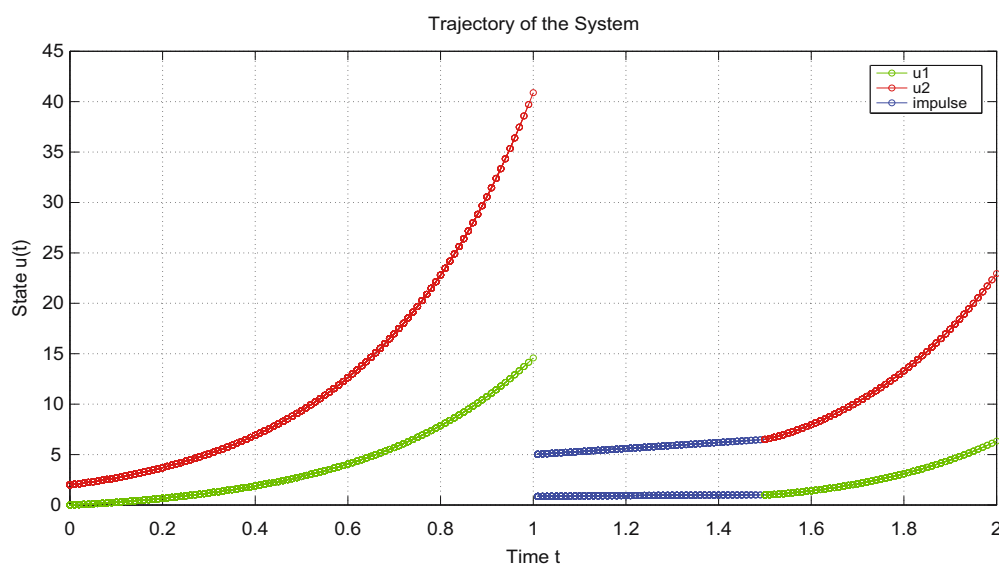
Let the final state  $u_f(2) = \begin{bmatrix} u_1(2) \\ u_2(2) \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$  (Figure 1).

After introducing the control parameter  $w(t)$  in (6.1), we have the impulsive linear control system as follows (Figure 2):

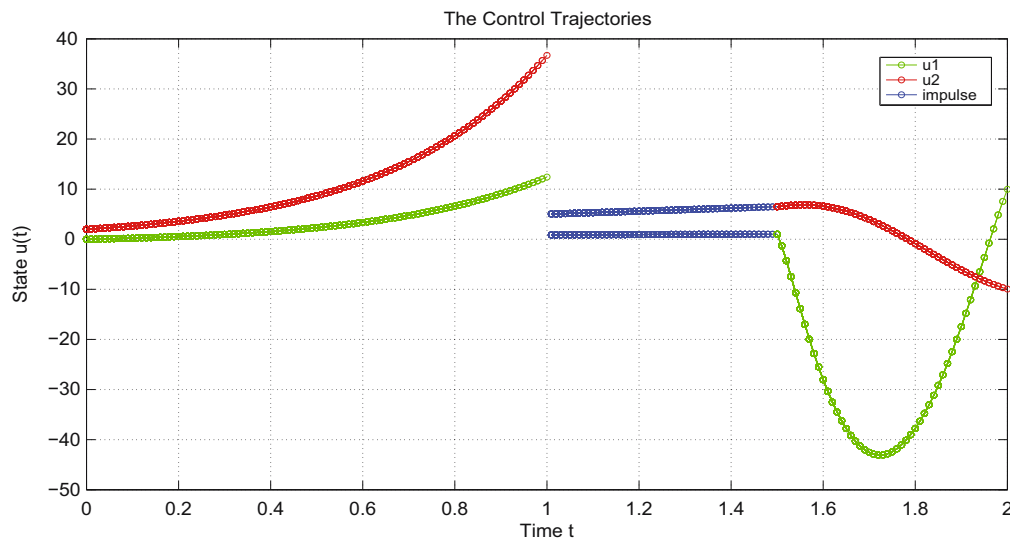
$$\begin{cases} {}^C D_t^{5/4} u_1(t) = u_1(t) + u_2(t) + w(t), & t \in (0, 1] \cup \left(\frac{3}{2}, 2\right], \\ {}^C D_t^{5/4} u_2(t) = 2u_1(t) + 3u_2(t), & t \in (0, 1] \cup \left(\frac{3}{2}, 2\right], \\ u(t) = \begin{bmatrix} \sin(t) \\ 3t + 2 \end{bmatrix}, & t \in \left(1, \frac{3}{2}\right], \\ u'(t) = \begin{bmatrix} \cos(t) \\ 3 \end{bmatrix}, & t \in \left(1, \frac{3}{2}\right]. \end{cases} \quad (6.2)$$

Here,  $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . The controllability Gramian matrix for system (6.2) is

$$M_{\frac{3}{2}}^2 = \begin{bmatrix} 0.4227 & 0.1655 \\ 0.1655 & 0.0796 \end{bmatrix}.$$



**Figure 1:** The trajectory of the impulsive linear system (6.1) starts from the initial state  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  and does not reach the final state  $\begin{bmatrix} 10 \\ -10 \end{bmatrix}$  on  $[0, 2]$ .



**Figure 2:** The trajectory of the impulsive linear system (6.2) starts from the initial state  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  and reaches the final state  $\begin{bmatrix} 10 \\ -10 \end{bmatrix}$  on  $[0, 2]$ .

It is clear that Gramian matrix  $M_{\frac{3}{2}}^2$  is nonsingular. Therefore, system (6.2) is exactly controllable on  $[0, 2]$ . The control which steers the initial state  $u(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  of system (6.2) to the arbitrary desired final state  $u_f(2) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$  during  $[0, 2]$  is given by

$$w(t) = B^* \Phi^*(2-t) \left( M_{\frac{3}{2}}^2 \right)^{-1} \left[ u_f(2) - \Phi_0 \left( 2 - \frac{3}{2} \right) \Psi_1^1 \left( \frac{3}{2}, u(1) \right) - \Phi_1 \left( 2 - \frac{3}{2} \right) \Psi_1^2 \left( \frac{3}{2}, u(1) \right) \right].$$

**Example 2.** Consider the nonlinear fractional order impulsive system without control

$$\begin{cases} {}^C D_t^{5/4} u_1(t) = u_1(t) + u_2(t) + \cos u_1(t), & t \in (0, 1] \cup \left( \frac{3}{2}, 2 \right], \\ {}^C D_t^{5/4} u_2(t) = 2u_1(t) + 3u_2(t) + \sin u_2(t), & t \in (0, 1] \cup \left( \frac{3}{2}, 2 \right], \\ u(t) = \begin{bmatrix} 5 \cos(t) \\ t - 5 \end{bmatrix}, & t \in \left( 1, \frac{3}{2} \right], \\ u'(t) = \begin{bmatrix} -5 \sin(t) \\ 1 \end{bmatrix}, & t \in \left( 1, \frac{3}{2} \right] \end{cases} \quad (6.3)$$

with initial conditions  $\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$  and  $\begin{bmatrix} u'_1(0) \\ u'_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

Comparing (6.3) with (1.1), we have  $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ ,  $\Psi_1^1 = \begin{bmatrix} 5 \cos(t) \\ t - 5 \end{bmatrix}$ ,  $\Psi_1^2 = \begin{bmatrix} -5 \sin(t) \\ 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $g(t, u(t)) = \begin{bmatrix} \cos u_1(t) \\ \sin u_2(t) \end{bmatrix}$ . Let the final state  $u_f(2) = \begin{bmatrix} u_1(2) \\ u_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$  (Figure 3).

After introducing the control parameter  $w(t)$  in (6.1), we have the impulsive nonlinear control system as follows (Figure 4):

$$\begin{cases} {}^C D_t^{5/4} u_1(t) = u_1(t) + u_2(t) + \cos u_1(t) + w_1(t), & t \in (0, 1] \cup \left(\frac{3}{2}, 2\right], \\ {}^C D_t^{5/4} u_2(t) = 2u_1(t) + 3u_2(t) + \sin u_2(t) + w_2(t), & t \in (0, 1] \cup \left(\frac{3}{2}, 2\right], \\ u(t) = \begin{bmatrix} 5 \cos(t) \\ t - 5 \end{bmatrix}, & t \in \left(1, \frac{3}{2}\right], \\ u'(t) = \begin{bmatrix} -5 \sin(t) \\ 1 \end{bmatrix}, & t \in \left(1, \frac{3}{2}\right]. \end{cases} \quad (6.4)$$

Here,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The controllability Gramian matrix for system (6.4) is

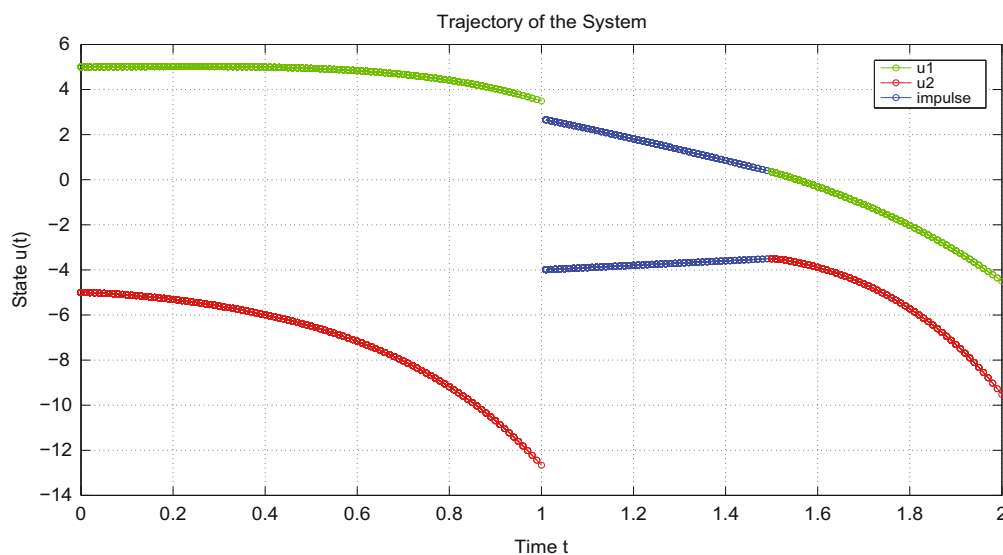
$$M_{\frac{3}{2}}^2 = \begin{bmatrix} 0.6081 & 0.9160 \\ 0.9160 & 1.4029 \end{bmatrix}.$$

It is clear that Gramian matrix  $M_{\frac{3}{2}}^2$  is nonsingular. Therefore, under assumptions (A1)–(A3), the nonlinear system (6.4) is exactly controllable on  $[0, 2]$  (Figure 5).

But, in order to get total controllability of the nonlinear system (6.4), we have the Gramian matrix

$$M = \begin{cases} M_0^1, & t \in [0, 1], \\ M_{\frac{3}{2}}^2, & t \in \left[\frac{3}{2}, 2\right], \end{cases} \quad (6.5)$$

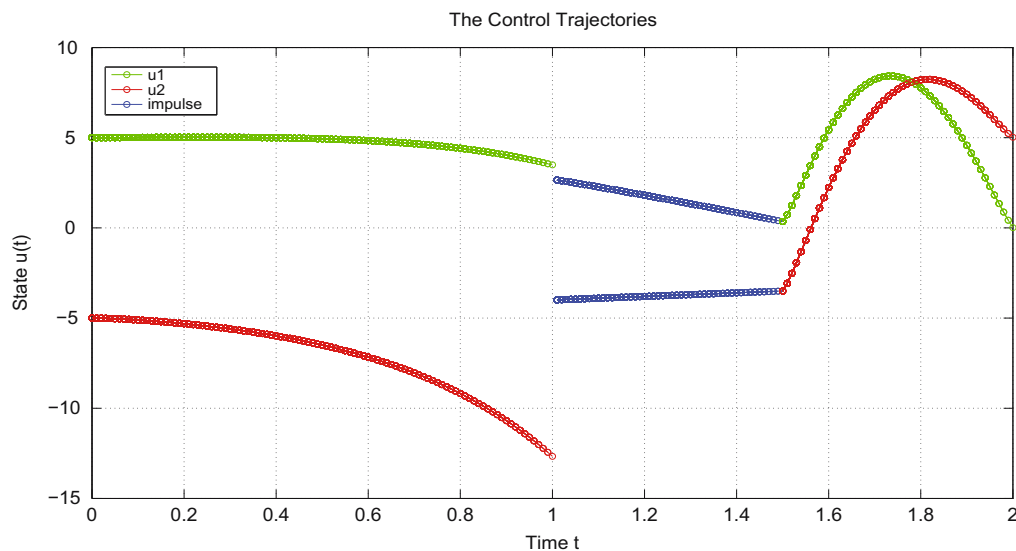
where  $M_0^1 = \begin{bmatrix} 7.2941 & 15.6108 \\ 15.6108 & 33.8369 \end{bmatrix}$  and  $M_{\frac{3}{2}}^2 = \begin{bmatrix} 0.6081 & 0.9160 \\ 0.9160 & 1.4029 \end{bmatrix}$ . It is clear that both matrices  $M_0^1$  and  $M_{\frac{3}{2}}^2$  are nonsingular. Moreover, the nonlinear function  $g(t, u(t))$  and non-instantaneous impulses  $\Psi_1^d(t, \cdot)$  and  $\Psi_1^2(t, \cdot)$  satisfy conditions (A1)–(A3). Hence, by Theorem 4.1, the nonlinear impulsive fractional order system (6.4) is totally controllable (Figure 6).



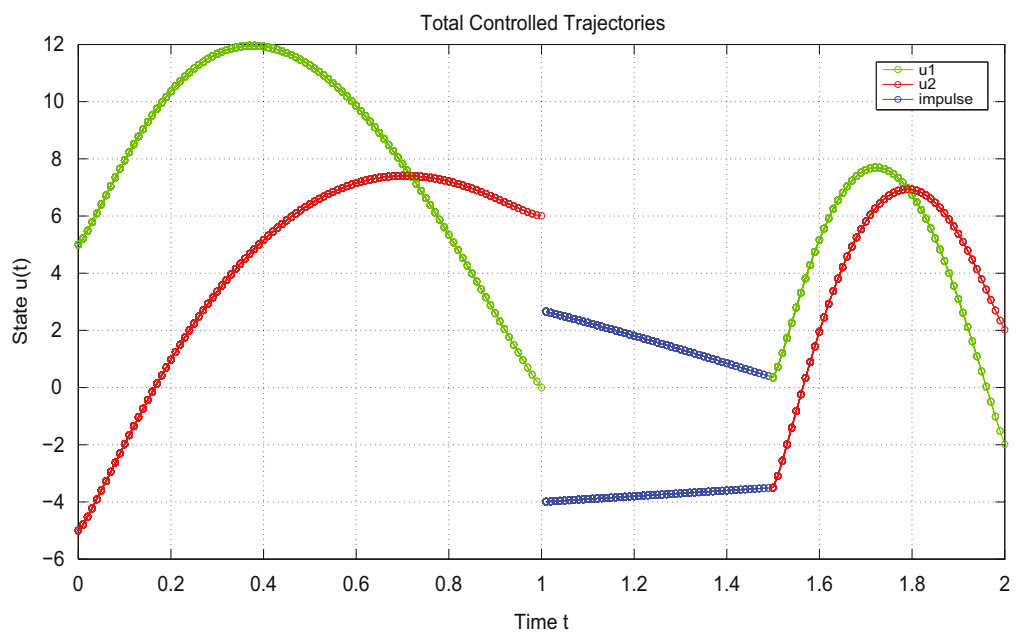
**Figure 3:** The trajectory of the impulsive linear system (6.3) starts from the initial state  $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$  and does not reach the final state  $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$  on  $[0, 2]$ .

## 7 Conclusion

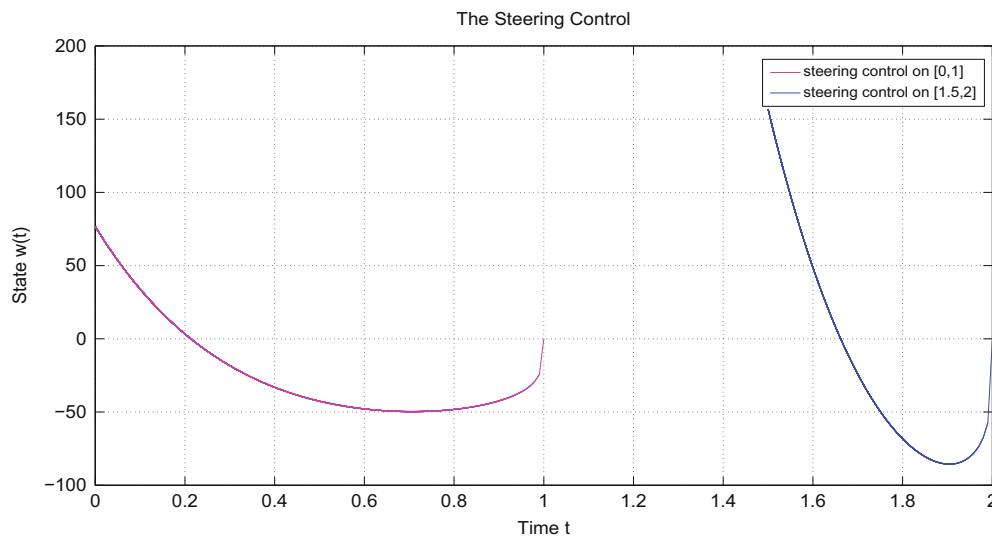
In this manuscript, the total controllability of fractional order nonlinear differential equations with non-instantaneous impulses is investigated through the iterative scheme. The total controllability conditions for the nonlinear system are examined by imposing that the linear system is controllable and the nonlinear function satisfies some suitable assumptions. The computation of controlled state and steering control for the linear and nonlinear fractional order impulsive system is proposed by using the Mittag-Leffler matrix



**Figure 4:** The trajectory of the impulsive nonlinear system (6.4) starts from the initial state  $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$  and reaches the final state  $\begin{bmatrix} 10 \\ -10 \end{bmatrix}$  on  $[0, 2]$ .



**Figure 5:** The trajectory of the nonlinear system (6.4) starts from the initial state  $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$  and reaches the final states  $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$  in the intervals  $[0, 1]$  and  $[\frac{3}{2}, 2]$ , respectively.



**Figure 6:** The trajectory of the nonlinear system (6.4) steers the initial state  $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$  to the final states  $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$  in the intervals  $[0, 1]$  and  $\left[\frac{3}{2}, 2\right]$ , respectively.

function and the Gramian matrix. In the future, for a better understanding of controllability, the computational scheme can be applied to the nonlinear control problems in finite dimensional spaces.

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