Research Article

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Two modifications of the inertial Tseng extragradient method with self-adaptive step size for solving monotone variational inequality problems

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Abstract: In this work, we introduce two new inertial-type algorithms for solving variational inequality problems (VIPs) with monotone and Lipschitz continuous mappings in real Hilbert spaces. The first algorithm requires the computation of only one projection onto the feasible set per iteration while the second algorithm needs the computation of only one projection onto a half-space, and prior knowledge of the Lipschitz constant of the monotone mapping is not required in proving the strong convergence theorems for the two algorithms. Under some mild assumptions, we prove strong convergence results for the proposed algorithms to a solution of a VIP. Finally, we provide some numerical experiments to illustrate the efficiency and advantages of the proposed algorithms.

Keywords: extragradient method, inertial, monotone, variational inequality, Lipschitz-continuous

MSC 2020: 65K15, 47J25, 65J15, 90C33

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset in H. In this article, we consider the classical variational inequality problem (VIP), which is to find a point $x^\dagger \in C$ such that

$$\langle Ax^\dagger, y - x^\dagger\rangle \geq 0, \quad \forall\, y \in C, \tag{1}$$

where $A : H \to H$ is a given operator. The solution set of VIP (1) is denoted by $VI(C, A)$.

Variational inequality theory is an important tool in economics, engineering, mathematical programming, transportation, and in other fields (see, for example, [1–8]). Many numerical methods have been constructed for solving variational inequalities and related optimization problems, see [9–27] and references therein.

One of the most popular methods for solving the problem (VIP) is the extragradient method (EGM). This method was introduced by Korpelevich [28] in 1976 as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad n \geq 0, \end{cases} \tag{2}$$

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where $\lambda \in \left(0, \frac{1}{L}\right)$ and P_C denotes the metric projection from H onto C. The EGM was first introduced for solving saddle point problems, after which the method was further extended to VIPs in both the Euclidean spaces and Hilbert spaces. The convergence of the EGM only requires that the operator A is monotone and L-Lipschitz continuous. If the solution set $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by algorithm (2) converges weakly to an element in $VI(C, A)$.

In recent years, the EGM (2) has received great attention by many authors, who improved it in various ways (see, for instance, [9,17,29–32] and references therein). In order to obtain the strong convergence of the EGM in real Hilbert spaces, Maingé [33] proposed a modified version of the algorithm as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = t_n - \alpha_n F t_n, \end{cases} \tag{3}$$

where $A : H \to H$ is monotone on C and L-Lipschitz continuous on H and $F : H \to H$ is Lipschitz continuous and strongly monotone on C such that $VI(C, A) \neq \varnothing$. Maingé proved that if the parameters satisfy the conditions: $\lambda_n \in [a, b] \subset \left(0, \frac{1}{L}\right)$, $\alpha_n \in [0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ converges strongly to $x^\dagger \in VI(C, A)$, where $x^\dagger$ is the solution of the following VIP

$$\langle Ax^\dagger, y - x^\dagger \rangle \geq 0, \quad \forall\, y \in VI(C, F).$$

One of the drawbacks of the EGM and its modified version above is that in each iteration in the algorithm, two projections are made onto the closed convex set C. However, projections onto a general closed and convex set are not easily executed, a fact that might affect the efficiency and applicability of the method.

In order to overcome the aforementioned drawback, Censor et al. [9] presented the subgradient extragradient method, in which the second projection onto C is replaced by a projection onto a specific constructible half-space which can be easily calculated. Their algorithm is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H | \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \quad \forall\, n \geq 0, \end{cases} \tag{4}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$.

Tseng in [32] proposed another method for solving the VIP (1), which uses only one projection in each iteration. This method is known as the Tseng extragradient method (TEGM) and is presented as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \quad \forall\, n \geq 0, \end{cases} \tag{5}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$.

Another shortcoming of algorithms (2), (3), (4), and (5) is the choice of stepsize. The stepsize plays an essential role in the convergence properties of iterative methods. In the aforementioned algorithms, the stepsizes are defined to be dependent on the Lipschitz constant L of the monotone operator. In this case, a prior knowledge or estimate of the Lipschitz constant is required. However, in many cases, this parameter is unknown or difficult to estimate. Moreover, the stepsize defined by the constant is often very small and slows down the convergence rate of iterative methods. In practice, a larger stepsize can often be used and yields better numerical results.

Yang and Liu [34] inspired by the TEGM and the viscosity method with a simple step size proposed the following algorithm for solving VIP (1):

Algorithm 1.1.
Step 0. Take $\lambda_0 > 0$, $x_0 \in H$, $\mu \in (0, 1)$.
Step 1. Given the current iterate x_n, compute

$$y_n = P_C(x_n - \lambda_n F(x_n)).$$

If $x_n = y_n$, then stop: x_n is a solution. Otherwise, go to **Step 2**.
Step 2. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|F(x_n) - F(y_n)\|},\ \lambda_n\right\}, & \text{if } F(x_n) - F(y_n) \neq 0, \\ \lambda_n, & \text{otherwise,} \end{cases}$$

where $z_n = y_n + \lambda_n(F(x_n) - F(y_n))$. Set $n := n + 1$ and return to **Step 1**,

where $F : H \to H$ is monotone and Lipschitz continuous with constant $L > 0$, $f : H \to H$ is a strict contraction mapping with constant $\rho \in [0, 1)$, and $\{\alpha_n\} \subset (0, 1)$. They proved the strong convergence of the algorithm without any prior knowledge of the Lipschitz constant of the mapping.

Very recently, Thong et al. [35] introduced a new algorithm which was a combination of the modified TEGM and the viscosity method with inertial technique. The proposed algorithm is presented as follows.

Algorithm 1.2.
Initialization: Let $x_0, x_1 \in H$ be arbitrary.
Iterative steps: Calculate x_{n+1} as follows:
Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \lambda A w_n).$$

If $y_n = w_n$, then stop and y_n is a solution of the VIP. Otherwise, go to **Step 2**.
Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) z_n,$$

where $z_n = y_n - \lambda(Ay_n - Aw_n)$. Set $n := n + 1$ and go to **Step 1**,

where $A : H \to H$ is monotone and Lipschitz continuous with constant $L > 0$, $f : H \to H$ is a contraction mapping with contraction parameter, $\lambda \in \left(0, \frac{1}{L}\right)$, $\{\alpha_n\} \subset [0, \alpha)$ for some $\alpha > 0$ and $\{\beta_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$. Under certain mild assumptions, they proved that the proposed algorithm converges strongly to a solution of the VIP (1).

In this work, we propose iterative schemes to remedy the drawbacks highlighted above. Motivated by the works of Yang et al. [34] and Thong et al. [35] and the current research interest in this direction, we propose two new inertial-type algorithms for solving the VIP (1) based on the TEGM and Moudafi's viscosity scheme which does not require a prior knowledge of the Lipschitz constant of the monotone operator. The inertial term $\alpha_n(x_n - x_{n-1})$ introduced can be regarded as a procedure for speeding up the convergence properties (see, for example, [22,23,33,36–39]). The first algorithm requires the computation

of only one projection onto the feasible set per iteration while the second algorithm needs the computation of only one projection onto a half-space, which is easy to compute. Under some mild conditions, we prove strong convergence theorems for the algorithms without any prior knowledge of the Lipschitz constant of the monotone operator. Finally, we provide some numerical experiments to show the efficiency and advantages of the proposed algorithms. The numerical illustrations show that our proposed algorithms with inertial effects converge faster than the original algorithms without inertial effects.

2 Preliminaries

Let H be a real Hilbert, for a nonempty, closed, and convex subset C of H, the metric projection $P_C : H \to C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

It is known that P_C is nonexpansive. We denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Definition 2.1. A function $f : H \to \mathbb{R}$ is said to be weakly lower semicontinuous (w-lsc) at $x \in H$, if

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

holds for an arbitrary sequence $\{x_n\}_{n=0}^{\infty}$ in H satisfying $x_n \rightharpoonup x$.

Lemma 2.2. [40,41] *Let $\delta \in (0, 1)$, for $x, y \in H$, we have the following statements:*
(1) $|\langle x, y\rangle| \le \|x\|\|y\|$;
(2) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;
(3) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(4) $\|\delta x + (1 - \delta)y\|^2 = \delta\|x\|^2 + (1 - \delta)\|y\|^2 - \delta(1 - \delta)\|x - y\|^2$.

Lemma 2.3. [42] *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have*

$$z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \ge 0 \quad \text{for all } y \in C.$$

Lemma 2.4. [42] *Let C be a closed convex subset in a real Hilbert space H, and $x \in H$. Then,*
(1) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$ *for all* $y \in C$;
(2) $\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2$ *for all* $y \in C$.

Lemma 2.5. [43] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n, \quad \text{for all } n \ge 1,$$

if $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \ge 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [44] *If $A : C \to H$ is a continuous and monotone mapping, then x^* is a solution of (1) if and only if x^* is a solution of the following problem:*

$$\text{find} \quad x^* \in C \text{ such that } \langle Ay, y - x^*\rangle \ge 0, \quad \forall y \in C.$$

3 Main results

In this work, we consider the VIP (1) under the following assumptions:
(A1) The solution set of (1) denoted by $VI(C, A)$ is nonempty.
(A2) The mapping A is monotone, i.e.,

$$\langle Ax - Ay,\ x - y\rangle \geq 0, \quad \forall\, x, y \in H. \tag{6}$$

(A3) The mapping A is Lipschitz-continuous with constant $L > 0$, i.e., there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\| \ \forall\, x, \quad y \in H. \tag{7}$$

We take $f : H \to H$ to be a strict contraction mapping with contraction parameter $k \in [0, 1)$. Let $\{\alpha_n\} \subset [0, \alpha)$ for some $\alpha > 0$ and $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

$$\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty \text{ and } \lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0. \tag{8}$$

Now, the first algorithm is presented as follows.

Algorithm 3.1.
Step 0. Take $x_0, x_1 \in H$ arbitrarily, $\lambda_0 > 0, \mu \in (0, 1)$.
Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \lambda_n A w_n).$$

If $y_n = w_n$, then stop and y_n is the solution of the VIP (1). Otherwise, go to **Step 2**.
Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) z_n$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\}, & \text{if } Aw_n - Ay_n \neq 0, \\ \lambda_n, & \text{otherwise,} \end{cases}$$

where $z_n = y_n + \lambda_n(Aw_n - Ay_n)$. Set $n := n + 1$ and return to **Step 1**.

Lemma 3.2. *The sequence $\{\lambda_n\}$ generated by Algorithm 3.1 is monotonically decreasing with lower bound* $\min\left\{\frac{\mu}{L}, \lambda_0\right\}$.

Proof. Repeating the proof as in [34] and replacing $\{x_n\}$ by $\{w_n\}$, we obtain the desired result. □

Remark 3.3. It is clear that the limit of $\{\lambda_n\}$ exists, and we denote $\lambda = \lim_{n\to\infty}\lambda_n$. It then follows that $\lambda > 0$.

Now, we prove the boundedness of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.*

Proof. Suppose $p \in VI(C, A)$. Then, by the definition of $\{z_n\}$ and using Lemma 2.2, we obtain

$$\begin{aligned} \|z_n - p\|^2 &= \|y_n - \lambda_n(Ay_n - Aw_n) - p\|^2 \\ &= \|y_n - p\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \end{aligned} \tag{9}$$

$$
\begin{aligned}
&= \|y_n - w_n\|^2 + \|w_n - p\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle.
\end{aligned}
$$

Recalling that $y_n = P_C(w_n - \lambda_n Aw_n)$, by Lemma 2.3, we obtain

$$\langle y_n - w_n + \lambda_n Aw_n, y_n - p\rangle \le 0,$$

which is equivalent to

$$\langle y_n - w_n, y_n - p\rangle \le -\lambda_n\langle Aw_n, y_n - p\rangle. \tag{10}$$

Combining (9) and (10), we have

$$
\begin{aligned}
\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n\langle Aw_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&\le \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n\rangle \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n\rangle \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Ap\rangle - 2\lambda_n\langle y_n - p, Ap\rangle \\
&\le \|w_n - p\|^2 - \left(1 - \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)||y_n - w_n||^2.
\end{aligned} \tag{11}
$$

Now, consider the limit

$$\lim_{n\to\infty}\left(1 - \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0. \tag{12}$$

Hence, there exists $N \ge 0$ such that $\forall\ n \ge N$, we have that $1 - \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2} > 0$. Thus, it follows that $\forall\ n \ge N$, we have

$$\|z_n - p\| \le \|w_n - p\|. \tag{13}$$

From the definition of $\{w_n\}$, we obtain

$$\|w_n - p\| = \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \le \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\| = \|x_n - p\| + \beta_n\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|. \tag{14}$$

From the condition $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$, it follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1, \quad \forall\, n \ge 1. \tag{15}$$

Hence, combining (13), (14) and (15), we have

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \beta_n M_1. \tag{16}$$

From the definition of $\{x_n\}$, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1 - \beta_n)z_n - p\| \\
&= \|\beta_n(f(x_n) - p) + (1 - \beta_n)(z_n - p)\| \\
&\le \beta_n\|f(x_n) - p\| + (1 - \beta_n)\|z_n - p\| \\
&\le \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|z_n - p\| \\
&\le \beta_n k\|x_n - p\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|z_n - p\|.
\end{aligned} \tag{17}
$$

Substituting (16) into (17), we obtain

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-(1-k)\beta_n)\|x_n-p\|+\beta_n M_1+\beta_n\|f(p)-p\|\\ &= (1-(1-k)\beta_n)\|x_n-p\|+(1-k)\beta_n\frac{M_1+\|f(p)-p\|}{1-k}\\ &\leq \max\left\{\|x_n-p\|, \frac{M_1+\|f(p)-p\|}{1-k}\right\}\\ &\vdots\\ &\leq \max\left\{\|x_N-p\|, \frac{M_1+\|f(p)-p\|}{1-k}\right\}.\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. It also follows that $\{z_n\}$, $\{f(x_n)\}$, $\{w_n\}$, and $\{y_n\}$ are bounded. □

Lemma 3.5. *Assume that $\{w_n\}$ and $\{y_n\}$ are sequences generated by Algorithm (3.1) such that $\lim_{n\to\infty}\|w_n-y_n\|=0$. If $\{w_{n_k}\}$ converges weakly to some $z_0\in H$, then $z_0\in VI(C,A)$.*

Proof. By the hypotheses of the lemma, we have that $y_{n_k}\rightharpoonup z_0$ and $z_0\in C$. Since A is monotone, then by the definition of y_{n_k} and by applying Lemma 2.3, we obtain

$$\langle y_{n_k}-w_{n_k}+\lambda_{n_k}Aw_{n_k},\ z-y_{n_k}\rangle\geq 0,\quad \forall z\in C.$$

This implies that

$$\begin{aligned}0 &\leq \langle y_{n_k}-w_{n_k},\ z-y_{n_k}\rangle+\lambda_{n_k}\langle Aw_{n_k},\ z-y_{n_k}\rangle\\ &= \langle y_{n_k}-w_{n_k},\ z-y_{n_k}\rangle+\lambda_{n_k}\langle Aw_{n_k},\ z-w_{n_k}\rangle+\lambda_{n_k}\langle Aw_{n_k},\ w_{n_k}-y_{n_k}\rangle\\ &\leq \langle y_{n_k}-w_{n_k},\ z-y_{n_k}\rangle+\lambda_{n_k}\langle Az,\ z-w_{n_k}\rangle+\lambda_{n_k}\langle Aw_{n_k},\ w_{n_k}-y_{n_k}\rangle.\end{aligned}$$

Letting $k\to\infty$, applying the facts that $\lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0$, $\{y_{n_k}\}$ is bounded and $\lim_{k\to\infty}\lambda_{n_k}=\lambda>0$, we have

$$\langle Az,\ z-z_0\rangle\geq 0,\quad \forall z\in C.$$

Applying Lemma 2.6, we have that $z_0\in VI(C,A)$. □

Lemma 3.6. *Let $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, and μ be as defined in Algorithm 3.1 and $M_4>0$ be some constant. Then, the following inequality holds:*

$$(1-\beta_n)\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\beta_n M_4,\tag{18}$$

where $x^*\in VI(C,A)$.

Proof. Let $x^*\in VI(C,A)$, then by the definition of $\{x_n\}$ and using Lemma 2.2, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n f(x_n)+(1-\beta_n)z_n-x^*\|^2\\ &= \beta_n\|f(x_n)-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2-\beta_n(1-\beta_n)\|f(x_n)-z_n\|^2\\ &\leq \beta_n\|f(x_n)-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2\\ &\leq \beta_n(\|f(x_n)-f(x^*)\|+\|f(x^*)-x^*\|)^2+(1-\beta_n)\|z_n-x^*\|^2\\ &\leq \beta_n(k\|x_n-x^*\|+\|f(x^*)-x^*\|)^2+(1-\beta_n)\|z_n-x^*\|^2\\ &\leq \beta_n(\|x_n-x^*\|+\|f(x^*)-x^*\|)^2+(1-\beta_n)\|z_n-x^*\|^2\\ &= \beta_n\|x_n-x^*\|^2+\beta_n(2\|x_n-x^*\|\cdot\|f(x^*)-x^*\|+\|f(x^*)-x^*\|^2)+(1-\beta_n)\|z_n-x^*\|^2\\ &\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2+\beta_n M_2,\end{aligned}\tag{19}$$

for some $M_2>0$. Substituting (11) into (19), we get

$$\|x_{n+1}-x^*\|^2 \le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|w_n-x^*\|^2 - (1-\beta_n)\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2 + \beta_n M_2. \tag{20}$$

From (16), we obtain

$$\|w_n-x^*\|^2 \le (\|x_n-x^*\|+\beta_n M_1)^2 = \|x_n-x^*\|^2 + \beta_n(2M_1\|x_n-x^*\|+\beta_n M_1^2) \le \|x_n-x^*\|^2 + \beta_n M_3, \tag{21}$$

for some $M_3 > 0$. Combining (20) and (21), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|x_n-x^*\|^2 + \beta_n M_3 - (1-\beta_n)\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2 + \beta_n M_2\\ &= \|x_n-x^*\|^2 + \beta_n M_3 - (1-\beta_n)\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2 + \beta_n M_2.\end{aligned}$$

Hence, we have that

$$(1-\beta_n)\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2 \le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \beta_n M_4,$$

where $M_4 := M_2 + M_3$. □

Now, we prove the convergence of Algorithm 3.1.

Theorem 3.7. *Assume that (A1), (A2), and (A3) hold and the sequence $\{\alpha_n\}$ is chosen such that it satisfies (8). Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to an element $x^* \in VI(C,A)$, where $x^* = P_{VI(C,A)} \circ f(x^*)$.*

Proof. Let $x^* \in VI(C,A)$, then using (13) and Lemma 2.2, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)z_n - x^*\|^2\\ &= \|\beta_n(f(x_n)-f(x^*)) + (1-\beta_n)(z_n-x^*) + \beta_n(f(x^*)-x^*)\|^2\\ &\le \|\beta_n(f(x_n)-f(x^*)) + (1-\beta_n)(z_n-x^*)\|^2 + 2\beta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\ &= \beta_n\|(f(x_n)-f(x^*)\|^2 + (1-\beta_n)\|(z_n-x^*)\|^2 - \beta_n(1-\beta_n)\|(f(x_n)-f(x^*)) - (z_n-x^*)^2\|\\ &\quad + 2\beta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\ &\le \beta_n\|(f(x_n)-f(x^*)\|^2 + (1-\beta_n)\|(z_n-x^*)\|^2 + 2\beta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\ &\le \beta_n k^2\|x_n-x^*\|^2 + (1-\beta_n)\|(z_n-x^*)\|^2 + 2\beta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\ &\le \beta_n k\|x_n-x^*\|^2 + (1-\beta_n)\|(w_n-x^*)\|^2 + 2\beta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle.\end{aligned} \tag{22}$$

By the definition of $\{w_n\}$ and using Lemma 2.2, we have

$$\begin{aligned}\|w_n-x^*\|^2 &= \|x_n+\alpha_n(x_n-x_{n-1})-x^*\|^2\\ &= \|x_n-x^*\|^2 + 2\alpha_n\langle x_n-x^*, x_n-x_{n-1}\rangle + \alpha_n^2\|x_n-x_{n-1}\|^2\\ &\le \|x_n-x^*\|^2 + 2\alpha_n\|x_n-x^*\|\cdot\|x_n-x_{n-1}\| + \alpha_n^2\|x_n-x_{n-1}\|^2.\end{aligned} \tag{23}$$

Combining (22) and (23), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-(1-k)\beta_n)\|x_n-x^*\|^2 + 2\alpha_n\|x_n-x^*\|\cdot\|x_n-x_{n-1}\| + \alpha_n^2\|x_n-x_{n-1}\|^2 + 2\beta_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\ &= (1-(1-k)\beta_n)\|x_n-x^*\|^2 + (1-k)\beta_n\cdot\frac{2}{1-k}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\\ &\quad + \alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\| + \alpha_n\|x_n-x_{n-1}\|)\\ &\le (1-(1-k)\beta_n)\|x_n-x^*\|^2 + (1-k)\beta_n\cdot\frac{2}{1-k}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + 3M\alpha_n\|x_n-x_{n-1}\|\\ &= (1-(1-k)\beta_n)\|x_n-x^*\|^2 + (1-k)\beta_n\left\{\frac{2}{1-k}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \frac{3M}{1-k}\cdot\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|\right\},\end{aligned} \tag{24}$$

where $M := \sup_{n\in N}\{\|x_n-x^*\|, \alpha_n\|x_n-x_{n-1}\|\} > 0$.

Next, we claim that the sequence $\{\|x_n - x^*\|\}$ converges to zero. In order to establish this, by Lemma 2.5, it suffices to show that $\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{n_k+1} - x^*\rangle \leq 0$ for every subsequence $\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \geq 0.$$

Now, suppose that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \geq 0.$$

Then, it follows that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2) = \liminf_{k\to\infty}\{(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|)(\|x_{n_k+1} - x^*\| + \|x_{n_k} - x^*\|)\} \geq 0.$$

Then from Lemma 3.6, and using the facts that $\lim_{k\to\infty}\left(1 - \lambda_{n_k}^2 \frac{\mu^2}{\lambda_{n_k+1}^2}\right) = 1 - \mu^2 > 0$ and $\lim_{k\to\infty}\beta_{n_k} = 0$, we obtain

$$\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = 0. \tag{25}$$

From (25), we get

$$\begin{aligned}\|z_{n_k} - w_{n_k}\| &= \|y_{n_k} + \lambda_{n_k}(Aw_{n_k} - Ay_{n_k}) - w_{n_k}\| \\ &\leq \|y_{n_k} - w_{n_k}\| + \lambda_{n_k}\|Aw_{n_k} - Ay_{n_k}\| \\ &\leq \|y_{n_k} - w_{n_k}\| + \lambda_{n_k} \times \frac{\mu}{\lambda_{n_k+1}}\|w_{n_k} - y_{n_k}\| \\ &= \left(1 + \lambda_{n_k} \times \frac{\mu}{\lambda_{n_k+1}}\right)\|y_{n_k} - w_{n_k}\| \to 0.\end{aligned} \tag{26}$$

Also, we have that

$$\|x_{n_k+1} - z_{n_k}\| = \|\beta_{n_k}f(x_{n_k}) + (1 - \beta_{n_k})z_{n_k} - z_{n_k}\| = \beta_{n_k}\|f(x_{n_k}) - z_{n_k}\| \to 0 \tag{27}$$

and

$$\|x_{n_k} - w_{n_k}\| = \|x_{n_k} - [x_{n_k} + \alpha_{n_k}(x_{n_k} - x_{n_k-1})]\| = \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| = \beta_{n_k}.\frac{\alpha_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{28}$$

Applying (26), (27), and (28), we obtain

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0. \tag{29}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ that converges weakly to some $z_0 \in H$, such that

$$\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle = \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_{k_j}} - x^*\rangle = \langle f(x^*) - x^*, z_0 - x^*\rangle. \tag{30}$$

By Lemma 3.5, and (25) and (28), we have $z_0 \in VI(C, A)$. Since the solution set $VI(C, A)$ is a closed, convex subset and f is a strict contraction, the mapping $P_{VI(C,A)} \circ f$ is a contraction mapping. Hence, by the Banach contraction mapping principle, there exists a unique element $x^* \in VI(C, A)$ such that $x^* = P_{VI(C,A)} \circ f(x^*)$. By Lemma 2.3, we have

$$\langle f(x^*) - x^*,\ z - x^*\rangle \leq 0, \quad \forall z \in VI(C, A). \tag{31}$$

Hence, it follows from (31) that

$$\limsup_{k\to\infty}\langle f(x^*) - x^*,\ x_{n_k} - x^*\rangle = \langle f(x^*) - x^*,\ z_0 - x^*\rangle \leq 0. \tag{32}$$

Combining (29) and (32), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k+1}-x^*\rangle &\le \limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k+1}-x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k}-x^*\rangle \\ &= \langle f(x^*)-x^*,\ z_0-x^*\rangle \le 0.\end{aligned} \tag{33}$$

Thus, by (33), $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, (24) and Lemma 2.5, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$ as required. □

We next propose our second algorithm. Suppose C is a nonempty convex set which satisfies the following conditions:

(B1) The set C is given by

$$C = \{x \in H : h(x) \le 0\},$$

where $h : H \to \mathbb{R}$ is a convex and subdifferentiable function on C.

(B2) h is weakly lower semicontinuous.

(B3) For any $x \in H$, at least one subgradient $\xi \in \partial h(x)$ can be calculated, where $\partial h(x)$ is defined as follows:

$$\partial h(x) = \{z \in H : h(u) \ge h(x) + \langle u - x, z\rangle,\ \ \forall u \in H\}.$$

In addition, $\partial h(x)$ is bounded on bounded sets.

(B4) Define the set C_n by the following half-space:

$$C_n = \{x \in H : h(w_n) + \langle \xi_n, x - w_n\rangle \le 0\},$$

where $\xi_n \in \partial h(w_n)$. By the definition of the subgradient, it is clear that $C \subseteq C_n$.

We now present the following algorithm using the half-space defined above.

Let $f : H \to H$ be a strict contraction mapping with contraction parameter $k \in [0, 1)$. Let $\{\alpha_n\} \subset [0, \alpha)$ for some $\alpha > 0$ and $\{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

$$\lim_{n\to\infty}\beta_n = 0, \quad \sum_{n=1}^{\infty}\beta_n = \infty \text{ and } \lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0.$$

Let $\{x_n\}$ be a sequence generated by the following iterative process.

Algorithm 3.8.

Step 0. Take $x_0, x_1 \in H$ arbitrarily, $\lambda_0 > 0$, $\mu \in (0, 1)$.

Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_{C_n}(w_n - \lambda_n A w_n).$$

If $y_n = w_n$, then stop and y_n is the solution of the VIP (1). Otherwise, go to **Step 2**.

Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)z_n$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \lambda_n\right\}, & \text{if } Aw_n - Ay_n \ne 0, \\ \lambda_n, & \text{otherwise,}\end{cases}$$

where $z_n = y_n + \lambda_n(Aw_n - Ay_n)$. Set $n := n + 1$ and return to **Step 1**.

It is easy to extend Lemmas 3.2, 3.4, and 3.6 for Algorithms 3.1–3.8.

Lemma 3.9. *The sequence $\{\lambda_n\}$ generated by Algorithm 3.8 is monotonically decreasing with lower bound* $\min\left\{\frac{\mu}{L}, \lambda_0\right\}$.

Lemma 3.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.8. Then, the sequence $\{x_n\}$ is bounded.*

Lemma 3.11. *Let $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, and μ be as defined in Algorithm 3.8 and $M_5 > 0$ be some constant. Then, the following inequality holds:*

$$(1-\beta_n)\left(1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2 \le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \beta_n M_5, \tag{34}$$

where $x^* \in VI(C,A)$.

Lemma 3.12. *Assume that $\{w_n\}$ and $\{y_n\}$ are sequences generated by Algorithm (3.8) such that $\lim_{n\to\infty}\|w_n - y_n\| = 0$. If $\{w_{n_j}\}$ converges weakly to some $\hat{x} \in H$ as $j \to \infty$, then $\hat{x} \in VI(C,A)$.*

Proof. Since $w_{n_j} \rightharpoonup \hat{x}$, it follows that $y_{n_j} \rightharpoonup \hat{x}$ as $j \to \infty$. Since $y_{n_j} \in C_{n_j}$, by the definition of C_n, we get

$$h(w_{n_j}) + \langle \xi_{n_j}, y_{n_j} - w_{n_j}\rangle \le 0.$$

Since $\{x_n\}$ is bounded by Lemma 3.10, then $\{w_n\}$ and $\{y_n\}$ are also bounded, and by condition (B3) there exists a constant $M > 0$ such that $\|\xi_{n_j}\| \le M \quad \text{for all} \quad j \ge 0$. So $h(w_{n_j}) \le M\|w_{n_j} - y_{n_j}\| \to 0$ as $j \to \infty$, and this in turn implies that $\liminf_{j\to\infty} h(w_{n_j}) \le 0$. Using condition (B2), we have $h(\hat{x}) \le \liminf_{j\to\infty} h(w_{n_j}) \le 0$. This means that $\hat{x} \in C$. From Lemma 2.3, we obtain

$$\langle y_{n_j} - w_{n_j} + \lambda_{n_j} A w_{n_j},\ z - y_{n_j}\rangle \ge 0, \quad \forall\, z \in C \subseteq C_{n_j}.$$

Since A is monotone, we have

$$\begin{aligned} 0 &\le \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ z - y_{n_j}\rangle \\ &= \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ z - w_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ w_{n_j} - y_{n_j}\rangle \\ &\le \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle A z,\ z - w_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ w_{n_j} - y_{n_j}\rangle. \end{aligned}$$

Letting $j \to \infty$, and since $\lim_{j\to\infty}\|y_{n_j} - w_{n_j}\| = 0$, we have

$$\langle Az,\ z - \hat{x}\rangle \ge 0, \quad \forall\, z \in C.$$

Applying Lemma 2.6, we have that $\hat{x} \in VI(C,A)$. □

Now, we prove the convergence theorem for Algorithm 3.8.

Theorem 3.13. *Assume that (A1), (A2), (A3), (B1), and (B2) hold. Let the sequence $\{\alpha_n\}$ be chosen such that it satisfies (8). Then, the sequence $\{x_n\}$ generated by Algorithm 3.8 converges strongly to an element $\hat{x} \in VI(C,A)$, where $\hat{x} = P_{VI(C,A)} \circ f(\hat{x})$.*

Proof. From (24), we have

$$\|x_{n+1}-x^*\|^2 \le (1-(1-k)\beta_n)\|x_n - x^*\|^2 + (1-k)\beta_n\left\{\frac{2}{1-k}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle + \frac{3M}{1-k}\cdot\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|\right\}, \tag{35}$$

where $M := \sup_{n\in N}\{\|x_n - x^*\|, \alpha_n\|x_n - x_{n-1}\|\} > 0$.

We claim that the sequence $\{\|x_n - x^*\|\}$ converges to zero. In order to establish this, by Lemma 2.5, it suffices to show that $\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{n_k+1} - x^*\rangle \le 0$ for every subsequence $\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \ge 0.$$

Suppose that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \ge 0.$$

By applying Lemma 3.11 and following similar argument as in Theorem 3.7 we have

$$\lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0,\quad \lim_{k\to\infty}\|z_{n_k}-w_{n_k}\|=0,\quad \lim_{k\to\infty}\|x_{n_k}-w_{n_k}\|=0,\quad \lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{36}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ that converges weakly to some $z_0\in H$, such that

$$\limsup_{k\to\infty}\langle f(x^*)-x^*, x_{n_k}-x^*\rangle=\lim_{j\to\infty}\langle f(x^*)-x^*, x_{n_{k_j}}-x^*\rangle=\langle f(x^*)-x^*, z_0-x^*\rangle. \tag{37}$$

By Lemma 3.12 and (36), we have $z_0\in VI(C,A)$. Since the solution set $VI(C,A)$ is a closed, convex subset and f is a strict contraction, the mapping $P_{VI(C,A)}\circ f$ is a contraction mapping. By the Banach contraction mapping principle, there exists a unique element $x^*\in VI(C,A)$ such that $x^*=P_{VI(C,A)}\circ f(x^*)$. Applying Lemma 2.3, we have

$$\langle f(x^*)-x^*,\ z-x^*\rangle\le 0,\quad \forall z\in VI(C,A). \tag{38}$$

Therefore, we have that

$$\limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k}-x^*\rangle=\langle f(x^*)-x^*,\ z_0-x^*\rangle\le 0. \tag{39}$$

From (36) and (39), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k+1}-x^*\rangle &\le \limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k+1}-x_{n_k}\rangle+\limsup_{k\to\infty}\langle f(x^*)-x^*,\ x_{n_k}-x^*\rangle\\ &=\langle f(x^*)-x^*,\ z_0-x^*\rangle\le 0.\end{aligned} \tag{40}$$

Hence, by (40), $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|=0$, (35), and applying Lemma 2.5, we have $\lim_{n\to\infty}\|x_n-x^*\|=0$, which is the required result. □

4 Numerical experiments

In this section, we present some numerical examples to demonstrate the efficiency of our algorithms in comparison with Algorithms 1.1 and 1.2 in the literature. All numerical computations were carried out using Matlab 2016(b) on an HP personal computer, 8-Gb RAM.

We choose $\lambda_0=0.9$, $\beta_n=\frac{1}{n+1}$, $f(x)=\frac{x}{5}$, and $\mu=0.6$ and use $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|}<\varepsilon$ as a stopping criterion to terminate the algorithm in each example. The projection onto the feasible set C is computed using the function "*fmincon*" in the optimization tool box. We take $\theta=0.6$ and choose the sequence $\{\alpha_n\}$ such that

$$\alpha_n=\begin{cases}\min\left\{\theta,\dfrac{\beta_n^2}{\|x_n-x_{n-1}\|}\right\} & \text{if } x_n\ne x_{n-1},\\ \theta & \text{otherwise.}\end{cases}$$

Problem 1.
The first test (also considered in [34]) is a classical example for which the usual gradient method does not converge. The feasible set is $C=R^m$ and $A(x)=Mx$, where M is a square $m\times m$ matrix given by

$$a_{i,j}=\begin{cases}-1, & \text{if } j=m+1-i \text{ and } j>i,\\ 1 & \text{if } j=m+1-i \text{ and } j<1,\\ 0, & \text{otherwise.}\end{cases}$$

For even m, the zero vector $x^*=(0,\ \ldots,\ 0)$ is the solution of Problem 1. In this example, we take (Case I:) $m=4$, (Case II:) $m=20$, and (Case III:) $m=100$ and $\varepsilon=10^{-4}$. The initial points are generated randomly using $x_0=\text{rand}(m,1)$ and $x_1=10\times\text{rand}(m,1)$. The numerical results are summarized in Table 1 and Figure 1.

Problem 2.
Suppose that $H = L^2([0,1])$ with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t, \quad \forall\, x, y \in H$$

and the induced norm

$$\|x\| := \left(\int_0^1 |x(t)|^2\mathrm{d}t\right), \quad \forall\, x \in H.$$

Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball and define the operator $A : C \to H$ by

$$(Ax)(t) = \max\{0, x(t)\}.$$

It can be easily verified that A is 1-Lipschitz continuous and monotone on C. With these given C and A, the solution set of the VIP (1) is given by

$$VI(C, A) = \{0\} \neq \varnothing.$$

It is known that

$$P_C(x) = \begin{cases} \dfrac{x}{\|x\|_{L^2}} & \text{if } \|x\|_{L^2} > 1, \\ x & \text{if } \|x\|_{L^2} \leq 1. \end{cases}$$

Table 1: Comparison between Algorithms 3.1, 1.1, and 1.2 for Problem 1

Dimension	Algorithm 3.1		Algorithm 1.1		Algorithm 1.2	
	Iter.	CPU time (s)	Iter.	CPU time (s)	Iter.	CPU time (s)
$m = 4$	23	6.0523	59	16.5546	29	7.6139
$m = 20$	22	7.4086	58	18.7391	29	8.7219
$m = 100$	24	15.9526	63	35.6063	31	18.4151

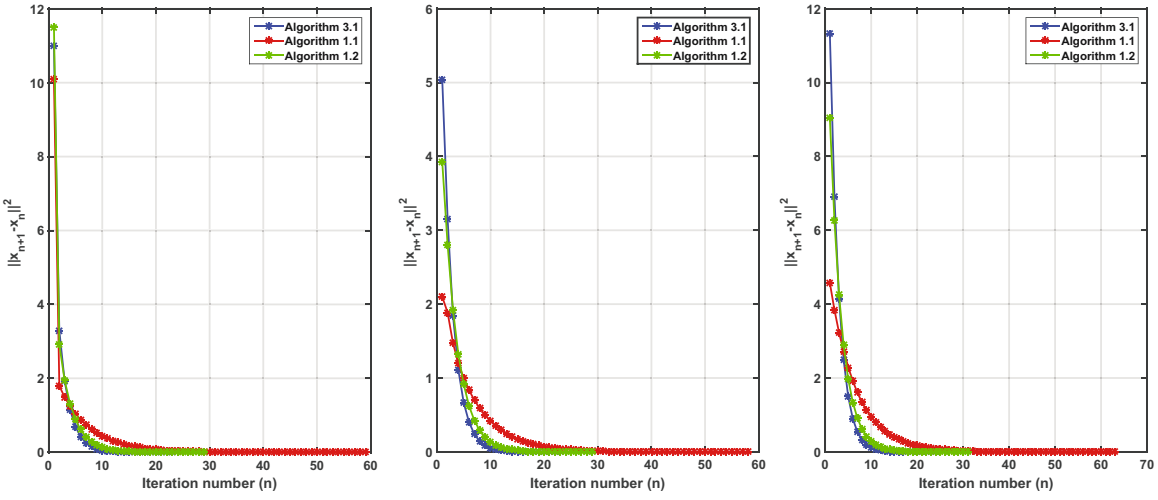

Figure 1: Problem 1, left: $m = 4$; middle: $m = 20$; and right: $m = 100$.

We test the algorithms for three different starting points and use $\varepsilon = 10^{-3}$ as a stopping criterion. The numerical results are summarized in Table 2 and Figure 2.

Case I: $x_0 = t^{\frac{3}{5}}$, $x_1 = t^3 + 5t^2 - 1$;
Case II: $x_0 = \exp(-5t)$, $x_1 = (2t^2 - 1)5$;
Case III: $x_0 = 5\sin(2\pi t)$, $x_1 = \cos(2t)4$.

Problem 3.
Next, we consider the Kojima-Shindo nonlinear complementarity problem, where $n = 4$ and the mapping A is defined by

$$A(x_1, x_2, x_3, x_4) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}.$$

It is known that A is Lipschitz continuous [45]. The feasible set is $C = \{x \in \mathbf{R}_+^4 \mid x_1 + x_2 + x_3 + x_4 = 4\}$. We choose the starting points: Case I: $x_0 = (1, 2, 0, 1)'$; $x_1 = (1, 1, 1, 1)'$ and Case II: $x_0 = (2, 0, 0, 2)'$; $x_1 = (1, 0, 1, 2)'$. For all the starting points, we have two tests: with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-6}$. The results are summarized in Tables 3–4 and Figures 3–4.

Table 2: Comparison between Algorithms 3.1, 1.1, and 1.2 for Problem 2

	Algorithm 3.1		Algorithm 1.1		Algorithm 1.2	
	Iter.	CPU time (s)	Iter.	CPU time (s)	Iter.	CPU time (s)
Case I	7	1.7756	6	1.9701	11	5.6289
Case II	7	5.0223	7	5.2426	11	7.0342
Case III	8	3.1035	8	3.1066	11	8.0202

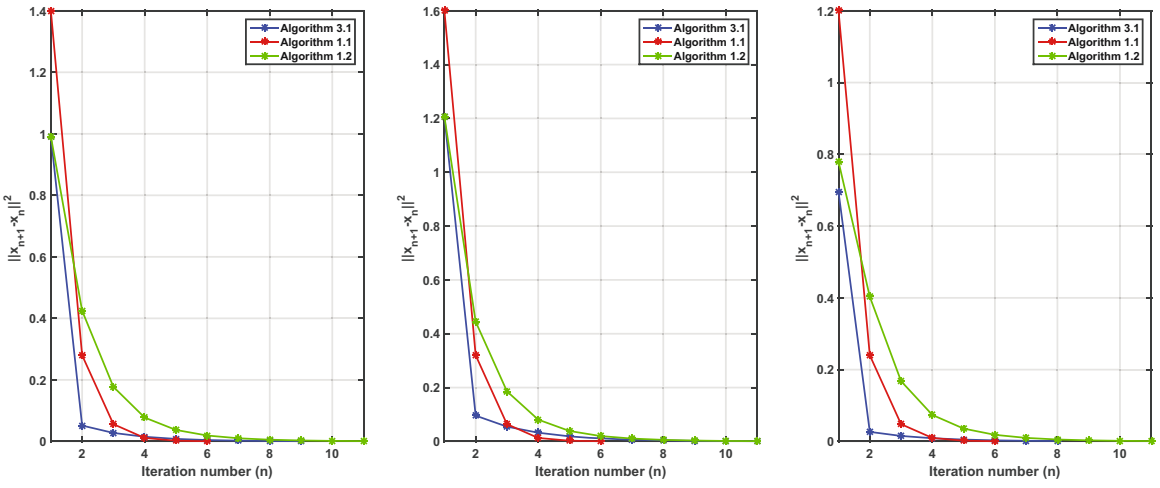

Figure 2: Problem 1, left: Case I; middle: Case II; and right: Case III.

Table 3: Comparison between Algorithms 3.1, 1.1, and 1.2 for Problem 3, $\varepsilon = 10^{-3}$

	Algorithm 3.1		Algorithm 1.1		Algorithm 1.2	
	Iter.	CPU time (s)	Iter.	CPU time (s)	Iter.	CPU time (s)
Case I	15	4.4316	35	9.5863	18	4.8669
Case II	18	5.5598	36	10.1659	23	6.3572

Table 4: Comparison between Algorithms 3.1, 1.1, and 1.2 for Problem 3, $\varepsilon = 10^{-6}$

	Algorithm 3.1		Algorithm 1.1		Algorithm 1.2	
	Iter.	CPU time (s)	Iter.	CPU time (s)	Iter.	CPU time (s)
Case I	31	9.5930	75	22.0722	41	10.6315
Case II	31	9.5526	75	21.7023	41	11.5232

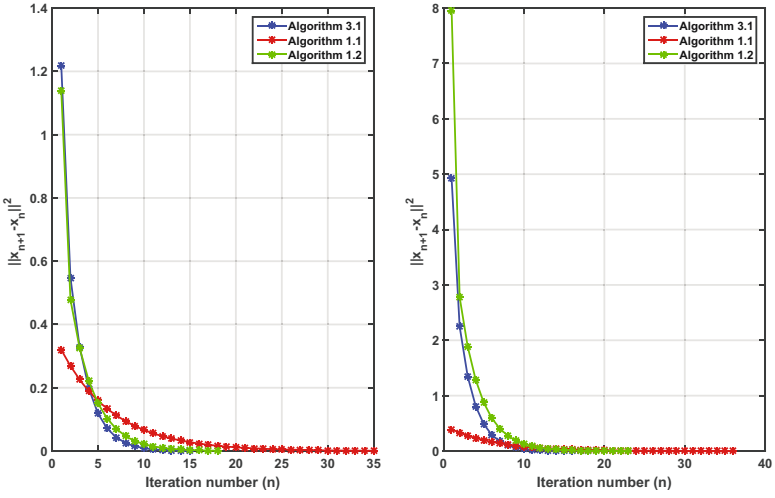

Figure 3: Problem 1, left: Case I; right: Case II.

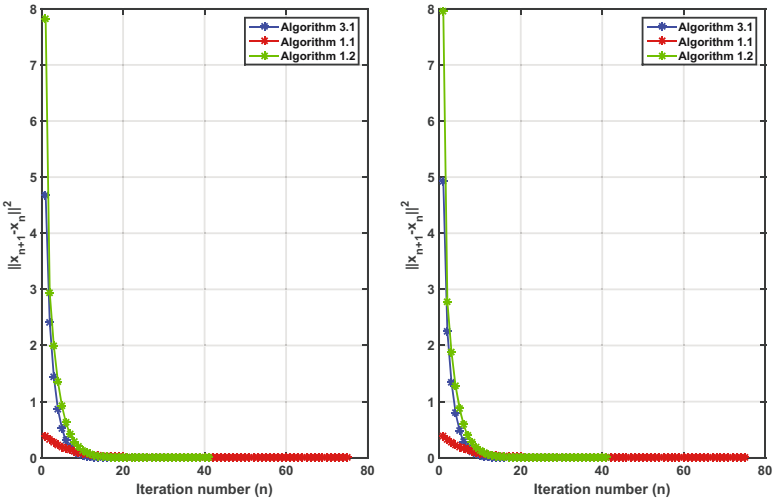

Figure 4: Problem 1, left: Case I; right: Case II.

5 Conclusion

In this article, we introduce two modifications of the inertial TEGM with self-adaptive step size for solving monotone VIPs. The algorithms were constructed in such a way that only one projection onto the feasible set C was made in each iteration. The results obtained improve many known results in this direction in the literature.

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