Research Article

Eddy Kwessi*, Geraldo de Souza, Ngalla Djitte, and Mariama Ndiaye

The special atom space and Haar wavelets in higher dimensions

https://doi.org/10.1515/dema-2020-0011
received September 9, 2019; accepted May 22, 2020

Abstract: In this note, we will revisit the special atom space introduced in the early 1980s by Geraldo De Souza and Richard O'Neil. In their introductory work and in later additions, the space was mostly studied on the real line. Interesting properties and connections to spaces such as Orlicz, Lipschitz, Lebesgue, and Lorentz spaces made these spaces ripe for exploration in higher dimensions. In this article, we extend this definition to the plane and space and show that almost all the interesting properties such as their Banach structure, Hölder's-type inequalities, and duality are preserved. In particular, dual spaces of special atom spaces are natural extension of Lipschitz and generalized Lipschitz spaces of functions in higher dimensions. We make the point that this extension could allow for the study of a wide range of problems including a connection that leads to what seems to be a new definition of Haar functions, Haar wavelets, and wavelets on the plane and on the space.

Keywords: analytic function, special atom, Haar wavelets, high dimension

MSC 2010: 42B05, 42B30, 30B50, 30E20

1 Introduction

In our case, we start by recalling the definition of the special atom space on the interval $J = [0, 1]$. The definitions over general interval $[a, b]$ and over $\mathbb{R}$ follow along similar lines. This is done for the sake of understanding the transition from $[0, 1]$ to $[0, 1] \times [0, 1]$ and from $\mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$. First, let us recall the definition of general atom proposed in [1].

Definition 1. Let $0 < p \le 1$ be a real number and J be an interval of $\mathbb{R}$. An atom is a function b defined on the interval J and satisfying

1. $|b(\xi)| \le \frac{1}{|J|^{1/p}}$;
2. $\int_{-\infty}^{\infty} \xi^k b(\xi)\mathrm{d}\xi = 0$, for $0 \le k \le \left[\frac{1}{p}\right] - 1$, where $[x]$ is the integer part of x.

From this definition, special atoms for $p \ge 1$ were introduced as:

Definition 2. A special atom of **type 1** is a function $b : J \to \mathbb{R}$ such that

$$b(t) = 1 \text{ on } J \text{ or } b(t) = \frac{1}{|I|^{1/p}}\{\chi_L(t) - \chi_R(t)\},$$

*** Corresponding author: Eddy Kwessi,** Department of Mathematics, Trinity University, San Antonio, TX 78212, USA, e-mail: ekwessi@trinity.edu
Geraldo de Souza: Department of Mathematics, Auburn University, Auburn, AL 36849, USA
Ngalla Djitte: Department of Mathematics, Gaston Berger University, St. Louis, Senegal, e-mail: ngalla.djitte@ugb.edu.sn
Mariama Ndiaye: Department of Mathematics, Gaston Berger University, St. Louis, Senegal, e-mail: mndiak@yahoo.fr

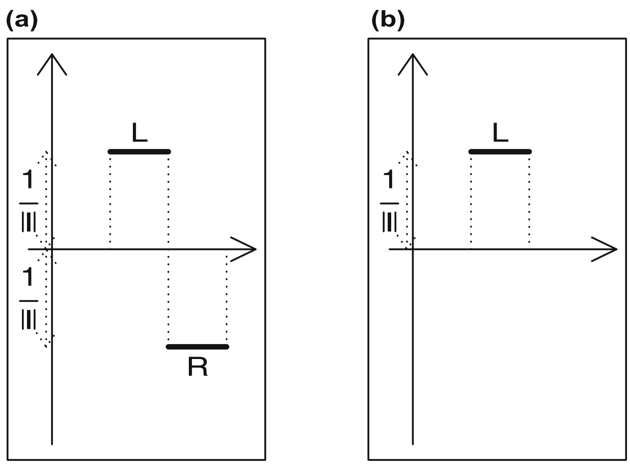


Figure 1: An illustration of the special atom of types 1 and 2 for $p = 1$. (a) Type 1, (b) Type 2.

where I is an interval contained in J, L and R are the halves of I such that $I = L \cup R$, and $|I|$ is the length of I (Figure 1a).

Definition 3. A special atom of type 2 is a function $a : J \to \mathbb{R}$ such that

$$a(t) = \frac{1}{|I|^{1/p}}\{\chi_I(t)\},$$

where I is an interval contained in J (Figure 1b).

Definition 4. For $1 \le p < \infty$, the special atom space is defined as:

Type 1:

$$B^p = \left\{f : J \to \mathbb{R};\ f(t) = \sum_{n=0}^{\infty} \alpha_n b_n(t);\ \sum_{n=0}^{\infty} |\alpha_n| < \infty\right\},$$

where $b_n(t)$ are special atoms of type 1. B^p is endowed with the norm

$$\|f\|_{B^p} = \inf \sum_{n=0}^{\infty} |\alpha_n|,$$

where the infimum is taken over all representations of f.

Type 2:

$$C^p = \left\{f : J \to \mathbb{R};\ f(t) = \sum_{n=0}^{\infty} \beta_n a_n(t);\ \sum_{n=0}^{\infty} |\beta_n| < \infty\right\},$$

where $a_n(t)$ are special atoms of type 2. C^p is endowed with the norm

$$\|f\|_{C^p} = \inf \sum_{n=0}^{\infty} |\beta_n|,$$

where the infimum is taken over all representations of f.

Remark 1. For $p = 1$, it is worth noting that the space C^1 contains all simple functions. That is, if f is a simple function with $f(x) = \sum_{n=0}^{k} \alpha_n \chi_{I_n}(x)$, then $f \in C^1$. Also, every element in C^1 is the limit of a sequence of simple functions. Indeed $C^1 \equiv L^1$.

Theorem 1. *$(B^p, \|\cdot\|_{B^p})$ and $(C^p, \|\cdot\|_{C^p})$ are Banach spaces.*

Proof. The proof can be found in [2]. □

2 Motivation for the need of high dimension special atom spaces

The special atom was introduced by Geraldo de Souza in PhD thesis (see [2]) partly to answer one main criticism of atoms, in that, they are too general and so far their main application was to prove that the dual of the Hardy space H^1, unknown at the time, was indeed the Space of Bounded Means Oscillations. Unbeknownst to the community at the time was that special cases of the atomic decomposition of Hardy's space would prove very beneficial in addressing unsolved problems. For instance, the special atom space as introduced by Geraldo de Souza has for dual space the space of derivatives (in the sense of distributions) of functions belonging to the Zygmund space Λ_* defined on $J = [a - h, a + h]$ as

$$\Lambda_* = \left\{ f : J \to \mathbb{R} : \|f\|_{\Lambda_*} = \sup_{\substack{\xi \in J \\ h > 0}} \left| \frac{f(\xi + h) + f(\xi - h) - 2f(\xi)}{h} \right| < \infty \right\}.$$

This result led to a simple proof that the Hardy space H^1 indeed contains functions whose Fourier series diverge almost everywhere by observing that the Hardy space H^1 is a superspace of the special atom space, and such functions actually exist in the special atom spaces, see for instance [3]. Moreover, the special atom space B^1 is a Banach equivalent to the space of analytic functions F on the complex unit disc for which $F(z) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{e^{i\xi} + z}{e^{i\xi} - z} f(\xi) \mathrm{d}\xi$, where $\int_0^1 \int_0^{2\pi} |F'(z)| \mathrm{d}z < \infty$. This analytic characterization also led to the lacunary characterization of functions in B^p, with $1 \leq p < \infty$. The question that was later raised by Brett Wick (Washington University, St. Louis, USA) was whether such a characterization could be achieved in the bidisk or even tridisk. To entertain such a question, a rigorous definition of special atom spaces in higher dimension is needed that would also preserve the key properties of the underlying space. Another important aspect of the special atom spaces that needs special care in high dimension is that of its connections with the Lorentz spaces $L_{p,q}$. Indeed, one property of the special atom space (type 2) is that it is the atomic decomposition of the Lorentz space $L_{p,1}$, which facilitates the study of operators such as the composition and multiplication operators. It is still not clear how to deal with these operators in higher dimensions but with a rigorous definition of special atoms, this could be possible. The extension of the special atom space we propose in the sequel leads to a natural definition of Haar wavelets in higher dimensions. Indeed, Haar wavelets are still preferred by certain practitioners for their simplicity and relatively ease of use. They can prove very useful in physical problems like heat transfer where the solution can be found relatively fast even at low resolution. The definition we propose allows us to easily prove that the Haar system forms an orthonormal basis in $L^2(\mathbb{R}^2)$, and generally in $L^2(\mathbb{R}^k),\ k \geq 2$. In the last section, we provide some applications of these facts.

3 Extension to high dimensions

Now consider the square $J = [0, 1] \times [0, 1]$. We extend the definition of the function $b_n(t)$ to the plane as follows: consider an integer n, and real numbers a_n, b_n, h_n, and k_n such that $h_n, k_n > 0$ and $\lim \inf_{n \in \mathbb{N}} h_n k_n > 0$. Consider a sub-rectangle J_n of J as

$$J_n = [a_n - h_n, a_n + h_n] \times [b_n - k_n, b_n + k_n].$$

Definition 5. Let $1 \le p < \infty$ and consider

$$L_{n,1} = [a_n - h_n, a_n] \times [b_n - k_n, b_n], \quad L_{n,2} = [a_n - h_n, a_n] \times [b_n, b_n + k_n],$$
$$R_{n,1} = [a_n, a_n + h_n] \times [b_n - k_n, b_n), \quad R_{n,2} = [a_n, a_n + h_n] \times (b_n, b_n + k_n].$$

Let

$$L_n = L_{n,1} \cup R_{n,2} \text{ and } R_n = L_{n,2} \cup R_{n,1}.$$

We define the function $B_n(x, y)$ in the plane as:

$$B_n(x,y) = \frac{1}{|J_n|^{1/p}} \{\chi_{R_n}(x,y) - \chi_{L_n}(x,y)\}. \tag{3.1}$$

Remark 2.
1. Note that the subsets L_n and R_n of J_n are nonempty and disjoint so that $B_n(x, y)$ can be written as:
$$B_n(x,y) = \frac{1}{|J_n|^{1/p}} \{\chi_{L_{n,2}}(x,y) + \chi_{R_{n,1}}(x,y) - \chi_{L_{n,1}}(x,y) - \chi_{R_{n,2}}(x,y)\},$$
where $|\cdot|$ represents the Lebesgue measure or the area to be more precise.
2. We also note that by restricting $B_n(x, y)$ to the real line, we recover the definition (Definition 2) above, using the fact that B^1 is equivalent to the space of functions $f(x) = \sum_{n=0}^{\infty} \alpha_n b_n(x)$, where b_n's are special atoms of type 1, see [4].
3. We observe that the role of $1/p$ and by extension that of p is to extend the definition above to L^p spaces and as such, $|J|^{1/p}$ is a normalizing constant so that $\|B_n\|_{L^p} = 1$.

Figure 2 is an illustration of L_n and R_n in the plane.

In the next definition, we will drop the index n in $L_{n,i}$ and $R_{n,i}$, $i = 1, 2$ respectively, for the sake of clarity. Consider $U \subseteq J$, measurable such that $U = L \cup R$, with $L = L_1 \cup R_2$ and $R = L_2 \cup R_1$, for some sub-rectangles L_1, L_2, R_1, R_2 of U similar to those in Figure 2. Now we can define the special atom space on the plane. Let $J = [0, 1] \times [0, 1]$ and let a real number $1 \le p < \infty$.

Definition 6. We define a special atom of **Type 1** on J as:

$$B_n(x,y) = \frac{1}{|J_n|^{1/p}} \{\chi_{R_n}(x,y) - \chi_{L_n}(x,y)\}, \tag{3.2}$$

where $|J_n| = 4h_n k_n$. Note that the subsets L_n and R_n are nonempty and disjoints so that

$$B_n(x,y) = \frac{1}{|J_n|^{1/p}} \{\chi_{L_{n,2}}(x,y) + \chi_{R_{n,1}}(x,y) - \chi_{L_{n,1}}(x,y) - \chi_{R_{n,2}}(x,y)\}.$$

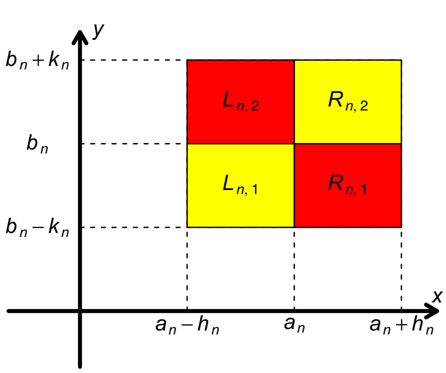


Figure 2: An illustration of L_n and R_n in the plane.

Definition 7. We define the special atom of **Type 2** on J as

$$A_n(x,y) = \frac{1}{|J_n|^{1/p}} \chi_{J_n}(x,y).$$

Definition 8. The special atom space on J (or $\mathbb{R}^2$) is defined as:

Type 1:

$$B^p = \left\{ f : J \to \mathbb{R};\ f(x,y) = \sum_{n=0}^{\infty} \alpha_n B_n(x,y);\ \sum_{n=0}^{\infty} |\alpha_n| < \infty \right\},$$

where J_n's are subsets in J, α_n are real numbers, $B_n(x,y)$ are special atoms of type 1. B^p is endowed with the norm:

$$\|f\|_{B^p} = \inf \sum_{n=0}^{\infty} |\alpha_n|,$$

where the infimum is taken over all representations of f.

Type 2:

$$C^p = \left\{ f : J \to \mathbb{R}; f(x,y) = \sum_{n=0}^{\infty} \beta_n A_n(x,y);\ \sum_{n=0}^{\infty} |\beta_n| < \infty \right\},$$

where $A_n(x,y)$ are special atoms of type 2 and β_n are real numbers. C^p is endowed with the norm

$$\|f\|_{C^p} = \inf \sum_{n=0}^{\infty} |\beta_n|,$$

where the infimum is taken over all representations of f.

In the next definition, U and V represent measurable subsets of J, where $|\cdot|$ is the Lebesgue measure.

Definition 9. *Let* $1 \le p < \infty$. *We define the following auxiliary spaces.*

$$D^p = \left\{ f : J \to \mathbb{R};\ f \in L^1(J),\ \|f\|_{D^p} = \sup_{\substack{U \subseteq J \\ 0<|U|<1}} \frac{1}{|U|^{1/p}} \left| \iint_U f(x,y)\,\mathrm{d}x\mathrm{d}y \right| < \infty \right\},$$

$$E^p = \left\{ f : J \to \mathbb{R};\ f \in L^1(J),\ \|f\|_{E^p} = \sup_{\substack{V \subseteq J \\ V = L \cup R \\ 0<|V|<1}} \frac{1}{|V|^{1/p}} \left| \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y - \iint_L f(x,y)\,\mathrm{d}x\mathrm{d}y \right| < \infty \right\}.$$

These spaces will be shown later to have interesting connections to one another. We note that B^p and C^p are special atomic decomposition spaces in the plane, whereas the auxiliary spaces D^p and E^p are their dual counterparts. We have the following results.

Theorem 2. *For* $1 \le p < \infty$, $(C^p, \|\cdot\|_{C^p})$, $(B^p, \|\cdot\|_{B^p})$, $(D^p, \|\cdot\|_{D^p})$, *and* $(E^p, \|\cdot\|_{E^p})$ *are all Banach spaces.*

Proof. The proof is similar to the one defined on the real line, see for instance [4], and will be omitted here for the sake of brevity. □

Remark 3. Recall that the Lipchitz space of order $0 < \alpha < 1$ is defined as:

$$\mathrm{Lip}_{\alpha} = \left\{ f : J \to \mathbb{R} : \|f\|_{\mathrm{Lip}_\alpha} = \sup_{\substack{h>0 \\ \xi \in J}} \frac{|f(\xi + h) - f(\xi)|}{h^{\alpha}} < \infty \right\}.$$

Recall that the generalized Lipchitz space of order $0 < \alpha \leq 2$ is defined as:

$$\Lambda_{\alpha} = \left\{ f : J \to \mathbb{R} : \|f\|_{\Lambda_\alpha} = \sup_{\substack{h>0 \\ \xi \in J}} \frac{|f(\xi + h) - f(\xi + h) - 2f(\xi)|}{(2h)^{\alpha}} < \infty \right\}.$$

Note that for $0 < \alpha < 1$, the spaces Lip_α and Λ_α are the same and for $\alpha = 1$, Λ_α is the Zygmund space $\Lambda_\star$. Suppose $J = [0, 1] \times [0, 1]$ and let $g \in D^p$ such that $g(x, y) = f'(x)$ for some differentiable function f defined on $[0, 1]$. Let $U = [\xi, \xi + h] \times [0, 1] \subseteq J$. Then, $|U| = h$ and

$$\|g\|_{D^p} = \sup_{\substack{U \subseteq J \\ 0<|U|<1}} \frac{1}{|U|^{\frac{1}{p}}} \left| \iint_U g(x, y)\,\mathrm{d}x\mathrm{d}y \right| = \sup_{\substack{h>0 \\ \xi \in J}} \frac{1}{h^{1/p}} \left| \int_{\xi}^{\xi+h} f'(x)\,\mathrm{d}x \right| = \sup_{\substack{h>0 \\ \xi \in J}} \frac{|f(\xi + h) - f(\xi)|}{h^{1/p}} = \|f\|_{\mathrm{Lip}^{1/p}}.$$

Suppose $J = [0, 1] \times [0, 1]$ and let $g \in E^p$ such that $g(x, y) = f'(x)$ for some differentiable function f defined on $[0, 1]$.

Let $V = [\xi - h, \xi + h] \times [0, 1] = [\xi - h, \xi) \times [0, 1] \cup [\xi, \xi + h] \times [0, 1] \subseteq J$. Then, $|V| = 2h$. Let $R = [\xi, \xi + h] \times [0, 1]$ and $L = [\xi - h, \xi) \times [0, 1]$. Then,

$$\begin{aligned}\|g\|_{E^p} &= \sup_{\substack{V \subseteq J \\ V = L \cup R \\ 0<|V|<1}} \frac{1}{|V|^{\frac{1}{p}}} \left| \iint_R g(x, y)\,\mathrm{d}x\mathrm{d}y - \iint_L g(x, y)\,\mathrm{d}x\mathrm{d}y \right| = \sup_{\substack{h>0 \\ \xi \in J}} \frac{1}{h^{1/p}} \left| \int_{\xi}^{\xi+h} f'(x)\,\mathrm{d}x - \int_{\xi-h}^{\xi} f'(x)\,\mathrm{d}x \right| \\ &= \sup_{\substack{h>0 \\ \xi \in J}} \frac{|f(\xi + h) + f(\xi - h) - 2f(\xi)|}{(2h)^{1/p}} = \|f\|_{\Lambda_{1/p}}.\end{aligned}$$

The above equalities show that the spaces D^p and E^p are, respectively, natural extensions of Lipschitz spaces Lip_α of order $0 < \alpha < 1$ and generalized Lipschitz spaces Λ_α of order $0 < \alpha < 2$ of functions in higher dimensions.

In the sequel, we will show the consequence of this extension to higher dimensions. We will in particular connect this extension to the proper definition of Haar systems and Haar wavelets.

4 Main results

4.1 Relationship between B^p and C^p

Theorem 3. *For a real number $1 \leq p < \infty$, the space B^p is continuously contained in C^p. Moreover, for $f \in B^p$*

$$\|f\|_{C^p} \leq 4\|f\|_{B^p}.$$

Proof. Let $f \in B^p$. Then, f has the atomic decomposition $f(x, y) = \sum_{n=0}^{\infty} \beta_n B_n(x, y)$ with $\sum_{n=0}^{\infty} |\beta_n| < \infty$. For a given sub-rectangle J_n of J, we have that

$$\begin{aligned} B_n(x,y) &= \frac{1}{|J_n|^{1/p}}\{\chi_{R_n}(x,y) - \chi_{L_n}(x,y)\} \\ &= \frac{1}{|J_n|^{1/p}}\{\chi_{L_{n,2}}(x,y) + \chi_{R_{n,1}}(x,y) - \chi_{L_{n,1}}(x,y) - \chi_{R_{n,2}}(x,y)\} \\ &= \left|\frac{L_{n,2}}{J_n}\right|^{1/p} \cdot \frac{1}{|L_{n,2}|^{1/p}}\chi_{L_{n,2}}(x,y) + \left|\frac{R_{n,1}}{J_n}\right|^{1/p} \cdot \frac{1}{|R_{n,1}|^{1/p}}\chi_{R_{n,1}}(x,y) \\ &\quad - \left|\frac{L_{n,1}}{J_n}\right|^{1/p} \cdot \frac{1}{|L_{n,1}|^{1/p}}\chi_{L_{n,1}}(x,y) - \left|\frac{R_{n,2}}{J_n}\right|^{1/p} \cdot \frac{1}{|R_{n,2}|^{1/p}}\chi_{R_{n,2}}(x,y). \end{aligned}$$

Put

$$\begin{aligned} A_{1,n}(x,y) &= \frac{1}{|L_{n,2}|^{1/p}}\chi_{L_{n,2}}(x,y), \quad A_{2,n}(x,y) = \frac{1}{|R_{n,1}|^{1/p}}\chi_{R_{n,1}}(x,y), \\ A_{3,n}(x,y) &= \frac{1}{|L_{n,1}|^{1/p}}\chi_{L_{n,1}}(x,y), \quad A_{4,n}(x,y) = \frac{1}{|R_{n,2}|^{1/p}}\chi_{R_{n,2}}(x,y). \end{aligned}$$

Likewise, put

$$K_{1,n} = \left|\frac{L_{n,2}}{J_n}\right|^{1/p}, \quad K_{2,n} = \left|\frac{R_{n,1}}{J_n}\right|^{1/p}, \quad K_{3,n} = \left|\frac{L_{n,1}}{J_n}\right|^{1/p}, \quad K_{4,n} = \left|\frac{R_{n,2}}{J_n}\right|^{1/p}.$$

It follows that

$$f(x,y) = f_1(x,y,\beta) + f_2(x,y) - f_3(x,y) - f_4(x,y), \quad \text{where } f_i(x,y) = \sum_{n=0}^{\infty} \beta_n K_{i,n} A_{i,n}(x,y).$$

With this notation, and considering the fact that C^p is a linear space, we can conclude that $f \in C^p$. Moreover, since the sub-rectangles $L_{n,i}$ and $R_{n,i}$ for $i = 1, 2$ are contained in J_n, we have that $K_{i,n} \le 1$, which implies that $|K_{i,n}|\,|\beta_n| \le |\beta_n|$. Hence,

$$\|f_i\|_{C^p} = \inf \sum_{n=0}^{\infty} |\beta_n||K_{i,n}| \le \inf \sum_{n=0}^{\infty} |\beta_n| = \|f\|_{B^p}.$$

It follows that

$$\|f\|_{C^p} = \|f_1 + f_2 - f_3 - f_3\|_{C^p} \le \|f_1\|_{C^p} + \|f_2\|_{C^p} + \|f_3\|_{C^p} + \|f_4\|_{C^p} \le 4\|f\|_{B^p}. \qquad \square$$

Remark 4. It is important to note that the constant 4 in the theorem above is sharp which comes from the proof. As for the inclusion of C^p into B^p, at this point, we can only conjecture that it may be true as well.

4.2 Relationship with the Lebesgue spaces L^∞ and L^p

Theorem 4. *Consider* $1 \le p < \infty$.

- *The Lebesgue space* L^∞ *is continuously contained in* D^p *and* E^p. *That is,*
  1. $L^\infty \subseteq D^p$ *and* $\|g\|_{D^p} \le \|g\|_{L^\infty}$.
  2. $L^\infty \subseteq E^p$ *and* $\|g\|_{E^p} \le 2\|g\|_{L^\infty}$.
- *The space* B^p *is continuously contained in the Lebesgue space* $L^p(J)$. *That is,* $B^p \subseteq L^p$ *and* $\|g\|_{L^p} \le \|g\|_{B^p}$.

Proof. Let $g \in L^\infty$. Then, we know that $|g(x,y)| \le \|g\|_\infty := \|g\|_{L^\infty}$. Therefore, given $U \subseteq J$

$$\left|\iint_U g(x,y)\mathrm{d}x\mathrm{d}y\right| \le \iint_U \|g\|_\infty \mathrm{d}x\mathrm{d}y = |U|\|g\|_\infty.$$

Hence, multiplying both sides of the above inequality by $1/|U|^{1/p}$ and taking the supremum over all $U \subseteq J$ such that $|U| \le 1$, we have:

$$\sup_{\substack{U\subseteq J\\ 0<|U|<1}} \frac{1}{|U|^{1/p}}\left|\iint_U g(x,y)\mathrm{d}x\mathrm{d}y\right| \le \|g\|_\infty.$$

In other words, $g \in D^p$ and $\|g\|_{D^p} \le \|g\|_{L^\infty}$. The proof that $L^\infty \subseteq E^p$ follows along the same lines.

Now let $f \in B^p$ such that $f(x,y) = \sum_{n=0}^\infty \alpha_n B_n(x,y)$ with $\sum_{n=0}^\infty |\alpha_n| < \infty$. Then, $\|f\|_{L^p} = \|\sum_{n=0}^\infty \alpha_n B_n\|_{L^p}$. We observe that by definition, $\|B_n\|_{L^p} = 1$. Hence, given an integer $N > 0$, we have

$$\left\|\sum_{n=0}^N \alpha_n B_n\right\|_{L^p} \le \sum_{n=0}^N |\alpha_n|\|B_n\|_{L^p} = \sum_{n=0}^n |\alpha_n| \le \sum_{n=0}^\infty |\alpha_n|.$$

Using the continuity of the norm in L^p, it follows that:

$$\|f\|_{L^p} = \left\|\sum_{n=0}^\infty \alpha_n B_n(x,y)\right\|_{L^p} = \left\|\lim_{N\to\infty}\sum_{n=0}^N \alpha_n B_n\right\|_{L^p} = \lim_{N\to\infty}\left\|\sum_{n=0}^N \alpha_n B_n\right\|_{L^p} \le \sum_{n=0}^\infty |\alpha_n|.$$

Taking the infimum of ever all representations of f, we have:

$$\|f\|_{L^p} \le \|f\|_{B^p}.$$

□

We observe that from the same token that C^p is also a subspace of L^p. The proof is similar to the one above and will not be given for the sake of brevity.

4.3 Hölder's-type inequalities

Theorem 5. *Let $1 < p < \infty$ be a real number.*

1. *If $g \in D^p$ and $f \in C^p$, then*

$$\left|\iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y\right| \le \|f\|_{C^p}\|g\|_{D^p}.$$

2. *If $g \in E^p$ and $f \in B^p$, then*

$$\left|\iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y\right| \le \|f\|_{B^p}\|g\|_{E^p}.$$

Proof. Let $1 < p < \infty$ be a real number, let $f \in C^p$ such that $f(x,y) = \sum_{n=0}^\infty \alpha_n A_n(x,y)$ with $\sum_{n=0}^\infty |\alpha_n| < \infty$. Consider $g \in D^p$. We know from the above result that both $f, g \in L^1$. Since $\sum_{n=0}^\infty |\alpha_n| < \infty$, $\alpha = \sup_{n\in\mathbb{N}}|\alpha_n|$ exists. Let $l = \liminf_{n\in\mathbb{N}} |J_n|$. Let $F_n(x,y) = \alpha_n A_n(x,y)g(x,y)$. Then, for all $n \in \mathbb{N}$ and $(x,y) \in J$,

$$|F_n(x,y)| = \left|\alpha_n \frac{1}{|J_n|^{1/p}}\chi_{J_n}(x,y)g(x,y)\right| \le \frac{\alpha}{l^{1/p}}|g(x,y)| \in L^1.$$

By the Dominated Convergence Theorem, we have

$$\begin{aligned}
\iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y &= \iint_J \left(\sum_{n=0}^{\infty} \alpha_n A_n(x,y)\right) g(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint_J \sum_{n=0}^{\infty} \alpha_n \frac{1}{|J_n|^{1/p}} \chi_{J_n}(x,y) g(x,y)\mathrm{d}x\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \alpha_n \frac{1}{|J_n|^{1/p}} \iint_J \chi_{J_n}(x,y) g(x,y)\mathrm{d}x\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \alpha_n \frac{1}{|J_n|^{1/p}} \iint_{J_n} g(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Taking the absolute value on both sides of the above equality, we have

$$\left| \iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y \right| \le \sum_{n=0}^{\infty} |\alpha_n| \frac{1}{|J_n|^{1/p}} \left| \iint_{J_n} g(x,y)\mathrm{d}x\mathrm{d}y \right|.$$

Taking the supremum over all subsets J_n of J such that $0 < |J_n| < 1$, we have

$$\left| \iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y \right| \le \left(\sum_{n=0}^{\infty} |\alpha_n|\right) \left(\sup_{\substack{J_n \subseteq J \\ 0<|J_n|<1}} \frac{1}{|J_n|^{1/p}} \left| \iint_{J_n} g(x,y)\mathrm{d}x\mathrm{d}y \right| \right) = \left(\sum_{n=0}^{\infty} |\alpha_n|\right) \|g\|_{D^p}.$$

To conclude, we take the infimum over all representations of f and we obtain

$$\left| \iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y \right| \le \|f\|_{C^p} \|g\|_{D^p}.$$

The proof of the second part of the theorem is similar by noting that in this case for all $n \in \mathbb{N}$ and $(x,y) \in J$, we have

$$|F_n(x,y)| = |\beta_n B_n(x,y) g(x,y)| \le \frac{2\alpha}{l^{1/p}} |g(x,y)| \in L^1,$$

so that the Dominated Convergence Theorem can still be used. □

Remark 5. This proof illustrates the importance of choosing sequences h_n and k_n for which $\liminf_{n\in\mathbb{N}} h_n k_n > 0$. Indeed, suppose $h_n = k_n = \frac{1}{2n}$. Then, $|J_n| = h_n k_n = \frac{1}{n^2}$ and $\liminf_{n\in\mathbb{N}} h_n k_n = 0$. In this case,

$$\iint_J f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y = \iint_J \sum_{n=0}^{\infty} F_n(x,y)\mathrm{d}x\mathrm{d}y,$$

with $F_n(x,y) = \alpha_n n^{2/p} \chi_{J_n}(x,y) g(x,y)$. The sequence F_n is unbounded in general so that the Dominated Convergence Theorem will fail and thus the inequality above cannot be obtained.

4.4 Duality

Theorem 6. *Let* $1 < p < \infty$ *be a real number.*

1. *The dual space* $(C^p)^*$ *of* C^p *is equivalent to* D^p *with equivalent norms, that is,* $\varphi \in (B^p)^*$ *if and only if there is a unique* $g \in D^p$ *such that*

$$\varphi_g(f) = \iint_J f(x,y)g(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Moreover,

$$\|\varphi_g\|_{(C^p)^*} \simeq \|g\|_{D^p}.$$

2. *Likewise, the dual space $(B^p)^*$ of B^p is equivalent to E^p with equivalent norms.*

Proof. Fix $g \in D^p$. Define a functional φ_g on C^p as: $\varphi_g : C^p \mapsto \mathbb{R}$ with

$$\varphi_g(f) = \iint_J f(x,y)g(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The linearity of the integral makes φ_g a linear functional and using the Hölder's-type inequalities above, we have

$$|\varphi_g(f)| = \left|\iint_J f(x,y)g(x,y)\,\mathrm{d}x\mathrm{d}y\right| \le \|f\|_{C^p}\|g\|_{D^p}.$$

It follows that

$$\|\varphi_g\|_{(C^p)^*} = \sup_{\|f\|_{C^p}=1} |\varphi_g(f)| \le \|g\|_{D^p}. \tag{4.1}$$

Now consider $\varphi \in (C^p)^*$. Then, there exists an absolute constant M such that

$$|\varphi(f)| \le M\|f\|_{C^p}, \quad \forall f \in C^p.$$

Since J is a rectangle, we can define a σ-finite measure μ as follows: let $E \subseteq J$ be a rectangle. Put $\mu(E) = \varphi(\chi_E)$. Since $\chi_E = |E|^{1/p}\frac{1}{|E|^{1/p}}\chi_E$, it follows that $\chi_E \in C^p$ with $\|\chi_E\|_{C^p} = |E|^{1/p}$. Moreover, $|\mu(E)| = |\varphi(\chi_E)| \le M|E|^{1/p}$. The latter implies that the measure μ is absolute continuous with respect to the Lebesgue measure $|\cdot|$, which is also σ-finite. Therefore, by the Radon-Nykodym theorem, there exists a measurable function $g \in L^1$ such that

$$\mu(E) = \iint_E g(x,y)\,\mathrm{d}x\mathrm{d}y.$$

It remains to show that $g \in D^p$ and that there exists a constant $K > 0$ such that $\|\varphi\|_{(C^p)^*} \ge K\|g\|_{D^p}$.

We note that

$$|\mu(E)| = |\varphi(\chi_E)| = \left|\iint_E g(x,y)\,\mathrm{d}x\mathrm{d}y\right| \le M|E|^{1/p}.$$

That is,

$$\frac{1}{|E|^{1/p}}\left|\iint_E g(x,y)\,\mathrm{d}x\mathrm{d}y\right| \le M.$$

Taking the supremum over all rectangles $E \subseteq J$ such that $0 < |E| < 1$, we have

$$\sup_{\substack{E\subseteq J\\ 0<|E|<1}} \frac{1}{|E|^{1/p}}\left|\iint_E g(x,y)\,\mathrm{d}x\mathrm{d}y\right| \le M < \infty.$$

This means simply by definition that $g \in D^p$. Now let us note that

$$\varphi(\chi_E) = \iint_E g(x,y)\,\mathrm{d}x\mathrm{d}y.$$

By linearity of φ, we have that

$$\varphi\left(\frac{1}{|E|^{1/p}}\chi_E\right) = \iint_J \frac{1}{|E|^{1/p}}\chi_E(x,y)g(x,y)\,dx\,dy.$$

The continuity of φ allows us to write it in the form

$$\varphi(f) = \varphi_g(f) = \iint_J f(x,y)g(x,y)\,dx\,dy.$$

Now suppose that for all constants $K > 0$, $\|\varphi_g\|_{(C^p)^*} < K\|g\|_{D^p}$. In particular, for all $n \in \mathbb{N}$, we would have $\|\varphi_g\|_{(C^p)^*} < \frac{1}{n}\|g\|_{D^p}$. Now consider $f_0 = \frac{1}{|E|^{1/p}}\chi_E$. We observe that $f_0 \in C^p$ with $\|f_0\|_{C^p} = 1$. So, in particular, we have that

$$|E|^{1/p}|\varphi_g(f_0)| = |\varphi_g(\chi_E)| \leq \sup_{\|f\|_{C^p}=1} |\varphi_g(f)| = \|\varphi_g\|_{(C^p)^*} < \frac{1}{n}\|g\|_{D^p}.$$

In other words,

$$n|\varphi_g(\chi_E)| = n\left|\iint_E g(x,y)\,dx\,dy\right| < \|g\|_{D^p}.$$

So dividing both sides by $|E|^{1/p}$ and taking the supremum over all $E \subseteq J$ such that $0 < |E| < 1$, we have

$$n\|g\|_{D^p} < |E|^{1/p}\|g\|_{D^p}.$$

The latter inequality fails to be true once we chose $n = [|E|^{1/p}] + 1$. Therefore, there must exist $K > 0$ such that $\|\varphi_g\|_{(C^p)^*} \geq K\|g\|_{D^p}$. This and the inequality in (4.1) prove that $\|\varphi_g\|_{(C^p)^*} \simeq \|g\|_{D^p}$. A similar approach will yield the second part of the theorem. ☐

Remark 6. A more direct proof that $\|\varphi\| \geq K\|g\|_{D^p}$ is as follows:

$$\|\varphi_g\|_{(C^p)^*} = \sup_{\|f\|_{C^p}=1} |\varphi_g(f)| \geq |\varphi_g(f_0)| = \left|\frac{1}{|E|^{1/p}}\iint_E g(x,y)\,dx\,dy\right| \geq \|g\|_{D^p}.$$

The result follows with $K = 1$.

5 Extension to the space

Consider the cube $J = I^3$ and let a sub-cube J_n of J be defined, for a given integer n, real numbers a_n, b_n, c_n, h_n, k_n, and m_n with $h_n, k_n, m_n > 0$ as

$$J_n = [a_n - h_n, a_n + h_n] \times [b_n - k_n, b_n + k_n] \times [c_n - m_n, c_n + m_n].$$

Definition 10. Let

$$\begin{aligned}
L_{n,1,1} &= [a_n - h_n, a_n] \times [b_n - k_n, b_n] \times [c_n - m_n, c_n),\\
L_{n,1,2} &= [a_n - h_n, a_n] \times [b_n - k_n, b_n] \times [c_n, c_n + m_n],\\
L_{n,2,1} &= [a_n - h_n, a_n] \times [b_n, b_n + k_n] \times [c_n - m_n, c_n),\\
L_{n,2,2} &= [a_n - h_n, a_n] \times [b_n, b_n + k_n] \times [c_n, c_n + m_n],\\
R_{n,1,1} &= [a_n, a_n + h_n] \times [b_n - k_n, b_n) \times [c_n - m_n, c_n),\\
R_{n,1,2} &= [a_n, a_n + h_n] \times [b_n - k_n, b_n) \times [c_n, c_n + m_n],\\
R_{n,2,1} &= [a_n, a_n + h_n] \times (b_n, b_n + k_n] \times [c_n - m_n, c_n),\\
R_{n,2,2} &= [a_n, a_n + h_n] \times (b_n, b_n + k_n] \times [c_n, c_n + m_n].
\end{aligned}$$

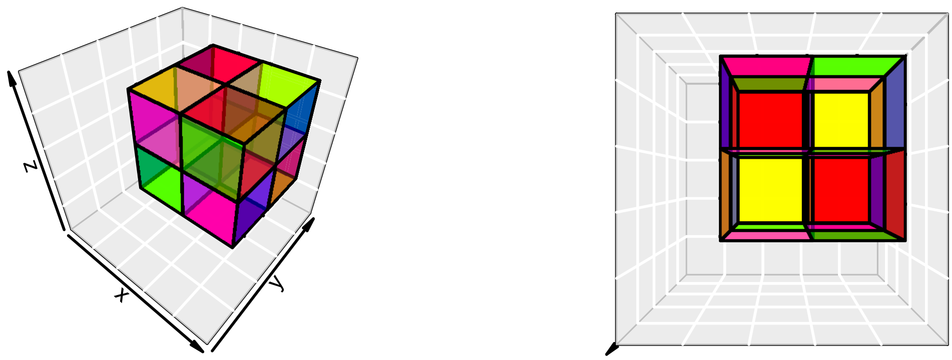


Figure 3: To understand the picture, there are six colors: red and yellow on the *xy*-plane, green and magenta on the *xz*-plane, and brown and blue on the *yz*-plane. Each cube has two faces with the same color, by projection onto that plane.

Let

$$L_n = L_{n,1,1} \cup L_{n,1,2} \cup R_{n,2,1} \cup R_{n,2,2} \text{ and } R_n = L_{n,2,1} \cup L_{n,2,2} \cup R_{n,1,1} \cup R_{n,1,2}.$$

We define the function $B_n(x, y, z)$ in the space as (see Figure 3 for an illustration)

$$B_n(x, y, z) = \frac{1}{|J_n|^{1/p}} \{\chi_{R_n}(x, y, z) - \chi_{L_n}(x, y, z)\}. \tag{5.1}$$

The eight cubes represent, respectively, $L_{n,i,j}$ for $i, j = 1, 2$ defined above in the following way:

$$\begin{aligned}
L_{n,1,1} &\mapsto xy\text{-yellow}, \quad xz\text{-green}, \quad yz\text{-blue},\\
L_{n,1,2} &\mapsto xy\text{-yellow}, \quad xz\text{-magenta}, \quad yz\text{-brown},\\
L_{n,2,1} &\mapsto xy\text{-red}, \quad xz\text{-green}, \quad yz\text{-brown},\\
L_{n,2,2} &\mapsto xy\text{-red}, \quad xz\text{-magenta}, \quad yz\text{-blue},\\
R_{n,1,1} &\mapsto xy\text{-red}, \quad xz\text{-magenta}, \quad yz\text{-blue},\\
R_{n,1,2} &\mapsto xy\text{-red}, \quad xz\text{-green}, \quad yz\text{-brown},\\
R_{n,2,1} &\mapsto xy\text{-yellow}, \quad xz\text{-magenta}, \quad yz\text{-brown},\\
R_{n,2,2} &\mapsto xy\text{-yellow}, \quad xz\text{-green}, \quad yz\text{-blue}.
\end{aligned}$$

The goal of the colors in the picture is to illustrate the fact that by restricting the definition to either plane, we will recover the definition of $B_n(\cdot,\cdot)$ in the bisphere as above, or by restricting it to either coordinate line, we will recover the original definition of the special atom. With this definition in hand, we see that the results of the previous section naturally extend to the trisphere space (more generally to the space, see Section 6). It can even be extended to the polysphere $\mathbb{T}^k$ for $k \geq 3$ by observing that there will be 2^{k-1} intervals $L_{n,\cdot}$ and 2^{k-1} intervals $R_{n,\cdot}$ and by combining them adequately.

6 Discussion

The special atom space may have been understudied in the literature because of its relative simplicity. That simplicity seemingly hides deep connections to well-known spaces.

6.1 Relationship with the weighted Bergman spaces

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{T} = \{z \in C : |z| = 1\}$ be the unit circle. Given $k > 1$, the polydisk is defined as $\mathbb{D}^k = \{(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k : |z_i| < 1, \forall 1 \leq i \leq k\}$ and the polysphere is given as $\mathbb{T}^k = \{(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k : |z_i| = 1, \forall 1 \leq i \leq k\}$. Also, in this section, $I = [0, 1]$.

In their inception, functions in B^p are given in their atomic decomposition forms. The space B^p, however, has an analytic form using the following result.

Theorem 7. [5] *Let $f \in B^p$ and $w(t) = t^{1/p}$. Define an analytic function $F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}-z}{e^{it}+z} f(t)\,dt$. Let $A(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. Let*

$$S_w = \left\{ F \in A(\mathbb{D}) : \|F\|_{S_w} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| \, \frac{w(1-r)}{1-r} d\theta dr < \infty \right\}.$$

Then, B^p is continuously contained in S_w with

$$\|f\|_{B^p} \simeq \|F\|_{S^w}. \tag{6.1}$$

This result means that B^p is the real characterization (or the boundary value space) of S_w in that

- If $F \in S_w$, then $f \in B^p$ where $f(\theta) = \lim_{r\to 1} \operatorname{Re} F(re^{i\theta})$.
- If $f \in B^p$, then $F \in S_w$ where $F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}-z}{e^{it}+z} f(t)\,dt$.

Now considering a weight w satisfying certain conditions, we can replace $|J_n|$ in the definition of B^p by $w(J_n)$ to obtain a weighted special atoms space B_w. It was shown in [5] that the aforementioned theorem can be extended to the weighted case.

Now, recall that weighted Bergman-Besov-Lipschitz spaces BBL_w are defined for a weight function w (defined above) as:

$$\mathrm{BBL}_w = \left\{ f : I = [0,1] \to \mathbb{R} : \int_0^{2\pi} \int_0^{2\pi} \frac{|f(x) - f(y)|}{|x-y|} w(x-y)\,dx dy < \infty \right\}.$$

We note that this space is just the weighted version of the generalized Hölder spaces $\Lambda\left(1 - \frac{1}{p}, 1, 1\right)$ with weight $w(t) = t^{1/p}$ defined as:

$$\Lambda\left(1 - \frac{1}{p}, 1, 1\right) = \left\{ f : I \to \mathbb{R} : \int_0^{2\pi} \int_0^{2\pi} \frac{|f(x) - f(y)|}{|x-y|^{2-1/p}} dx dy < \infty \right\}.$$

It was shown in [6] that $B_w \simeq \mathrm{BLL}_w$ with equivalent norms, meaning that B_w is the atomic decomposition of BLL_w.

From now on, we will write $B^p(I)$ as B^p. Let q such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $z = x + iy = re^{i\theta} \in \mathbb{D}$. Then, $d\theta dr = 2r dx dy = 2|z| dx dy = 2\pi |z| dA(z)$, where $dA(z) = \frac{dxdy}{\pi}$. Now define the function $\Phi(z) = |z|(1-|z|)^{\frac{1}{p-1}} = |z|\frac{w(1-|z|)}{1-|z|}$ on $\mathbb{D}$.

Then, (6.1) can be written as

$$\|f\|_{B^p(I)} \simeq \int_{\mathbb{D}} |F'(z)| \Phi(z)\, dA(z). \tag{6.2}$$

Noting that $\Phi \in L^1(\mathbb{D})$, the right-hand side of (6.2) means that $F' \in A^1_\Phi(\mathbb{D}) = L^1_\Phi(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$, the weighted Bergman space with weight Φ, where $\mathcal{H}(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$. We note that the standard Bergman weights are given as $\Phi_0(z) = (\alpha + 1)(1 - |z|)^\alpha$ for $\alpha > -1$. With the choice of $p \geq 1$, $\Phi(z)$ can be transformed into a standard weight since in that case $\alpha = -\frac{1}{q} > -1$. When $k \geq 2$, we note that $z = (z_1, z_2, \ldots, z_k)$ is a vector in $\mathbb{C}^k$, with $z_j = r_j e^{ji\theta_j}$ for $1 \leq j \leq k$, which can also be written as $z_j = x_j + iy_j$. Then, for a differentiable function F on $\mathbb{C}^k$, we put $F'(z) = (f_1(z), f_2(z), \dots, f_k(z))$, where $f_j(z) = \frac{\partial F(z)}{\partial z_j}$, $1 \le j \le k$. Note that when $k = 1$, this becomes $f_1(z) = F'(z)$. For $k \ge 2$, we have $|F'(z)| := \|F\|_2 = \sqrt{\sum_{j=1}^k |f_j(z)|^2}$.

With the definition of $B^p(I^k)$ at hand and for an integer $k \ge 2$, we have the following.

Conjecture 1. *There exist weights $\Phi(z)$ and an holomorphic function F on $\mathbb{D}^k$ such that*

$$\|f\|_{B^p(I^k)} \simeq \int_{\mathbb{D}^k} |F'(z)|\Phi(z)\,\mathrm{d}A(z), \tag{6.3}$$

for $z = (z_1, z_2, \dots, z_k) \in \mathbb{D}^k$ with $z_j = x_j + iy_j$ and $\mathrm{d}A(z) = \frac{1}{(\pi)^k}\prod_{j=1}^k \mathrm{d}x_j\mathrm{d}y_j$.

6.2 Lacunary functions

In Section 6.1, we saw that the weighted special atom space has an analytic characterization as the weighted Bergman-Besov-Lipschitz space. Above, we also mentioned that B^p can be extended to the bisphere or polysphere. However, to be able to define analytic functions on the bidisk and polydisk, we need to make sure that lacunary functions are properly characterized and removed. First, recall that

Definition 11. A lacunary function F is an analytic function possessing the so-called Hadamard gaps, that is,

$$F(z) = \sum_{k=1}^{n_k} a_k z^{n_k} \quad \text{such that } \lambda = \inf_k \frac{n_{k+1}}{n_k} > 1.$$

Let us mention the Ostrowski-Hadamard gap theorem, see for example [7].

Theorem 8. (Ostrowski-Hadamard) *Suppose F is a lacunary function with radius of convergence 1. Then, f cannot be analytically continued from the open disc $\mathbb{D}$ to any larger open set, including even a single point of the boundary $\mathbb{T}$ of $\mathbb{D}$.*

This result essentially says that if we hope to extend the result of [5] to the space or to a higher dimension space, we need to discard lacunary functions. Better, we need to characterize the sub-space of lacunary functions. On the unit sphere, this was done in [8], where the space $b^p(I)$ of lacunary functions on I was characterized as:

$$b^p(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{Z} : f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} 2^n K(n,p)\left(\sum_{k\in I_n} |a_k|^2\right)^{\frac{1}{2}} < \infty, K(n,p) > 0 \right\},$$

where $K(n, p)$ is a positive weight function satisfying certain conditions. Basically, it was proved that if F is lacunary on $\mathbb{D}$, then

$$\|F\|_{B^p(I)} \simeq \|F\|_{b^p(\mathbb{D})}. \tag{6.4}$$

The result in (6.4) relies heavily on the following theorem by A. Zygmund.

Theorem 9. [9] *Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be a lacunary function defined on the unit disk $\mathbb{D}$. Then, there is an absolute constant c independent of f such that*

$$\left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} \le c\|f\|_{L^1(\mathbb{D})}.$$

Paley extended the result of A. Zygmund in [9] to the polydisk of $H^1(\mathbb{D}^m)$.

Definition 12. A function $f(z) = \sum_{k\in\mathbb{N}^m} a_{k_1} \cdots a_{k_m} z_1^{n_{k_1}} \cdots z_m^{n_{k_m}}$ defined on the polydisk $\mathbb{D}^m$ will be called lacunary if

$$\lambda = \min\{\lambda_1, \lambda_2, \ldots, \lambda_m\} > 1, \quad \text{where } \lambda_j = \inf_{k_j} \frac{n_{k_j+1}}{n_{k_j}} > 1, \ 1 \le j \le m.$$

Theorem 10. (Paley 1960) *Let $f(z) = \sum_{k\in\mathbb{N}^m} a_{k_1} \cdots a_{k_m} z_1^{n_{k_1}} \cdots z_m^{n_{k_m}}$ be a lacunary function defined on the polydisk $\mathbb{D}^m$. Then, there is an absolute constant c independent of f such that*

$$\left(\sum_{k\in\mathbb{N}^m} |a_{k_1} \cdots a_{k_m}|^2\right)^{\frac{1}{2}} \le c\|f\|_{H^1(\mathbb{D}^m)}.$$

Now, with the definition of the special atom space in high dimension, we claim the following.

Conjecture 2. *There exists a weight function $K(n,p)$ characterizing $b^p(\mathbb{D}^k)$ such that if F is lacunary on the polydisk $\mathbb{D}^k$,*

$$\|F\|_{B^p(I^k)} \simeq \|F\|_{b^p(\mathbb{D}^k)}.$$

The result in (6.4) also means that if $B^p(I)$ is the space of functions defined on $\mathbb{D}$ having an analytic continuation on $\mathbb{D}$, then $B^p(I) = H^1(\mathbb{D})\backslash b^p(\mathbb{D})$, where $H^1(\mathbb{D})$ is the Hardy's space consisting of functions such that

$$\|f\|_{H^1(\mathbb{D})} = \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|d\theta < \infty.$$

Now that we have an extension of the special atom space to higher dimensions, then the same endeavor could be carried out in higher dimensions:

Conjecture 3. *If $B^p(I^k)$ is the space of functions defined on $\mathbb{D}^k$ having an analytic continuation on $\mathbb{D}^k$, then $B^p(I^k) \cup b^p(\mathbb{D}^k) = H^1(\mathbb{D}^k)$.*

6.3 Relationship with Haar wavelets

We will define a Haar wavelet based on the special atom space in high dimensions given above and we show that it is an extension of the classical Haar wavelet in $L^2(I)$.

Definition 13. Let

$$\psi(x,y) = \chi_R(x,y) - \chi_L(x,y), \quad \text{and} \quad \phi(x,y) = \chi_J(x,y), \tag{6.5}$$

where

$$\begin{aligned} R &= \left[0, \frac{1}{2}\right) \times \left[0, \frac{1}{2}\right) \cup \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right], \\ L &= \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right], \\ J &= [0,1] \times [0,1]. \end{aligned}$$

In Figure 4, we show a representation of $\psi(x,y)$ and $\phi(x,y)$ in the space.

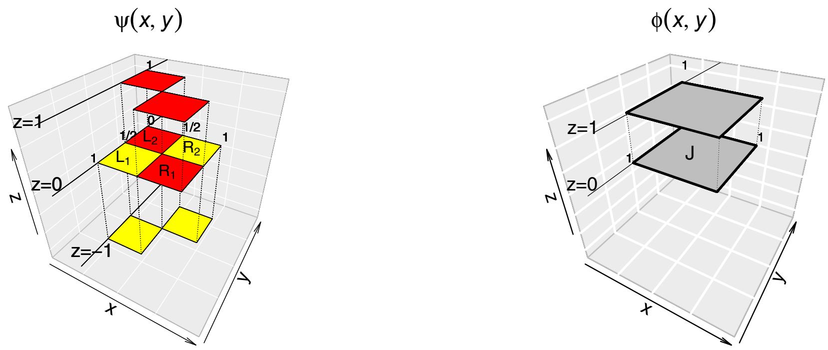


Figure 4: An illustration of $\psi(x,y)$ and $\phi(x,y)$.

6.3.1 Haar functions and Haar systems

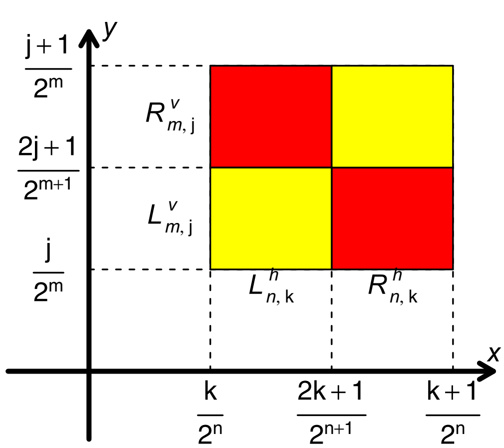
For $k = 0, 1, \ldots, 2^n - 1$ and $n \in \mathbb{N}$ and for $j = 0, 1, \ldots, 2^m - 1$ and $m \in \mathbb{N}$, let

$$J_{n,k}^{m,j} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right], \tag{6.6}$$

$$\begin{aligned} L_{n,k}^h &= \left[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}\right), \quad R_{n,k}^h = \left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right], \\ L_{m,j}^v &= \left[\frac{j}{2^m}, \frac{2j+1}{2^{m+1}}\right), \quad R_{m,j}^v = \left[\frac{2j+1}{2^{m+1}}, \frac{j+1}{2^m}\right]. \end{aligned} \tag{6.7}$$

Now define

$$\begin{aligned} L_{n,m,j,k} &= (L_{n,k}^h \times R_{m,j}^v) \cup (L_{m,j}^v \times R_{n,k}^h), \\ R_{n,m,j,k} &= (L_{m,j}^v \times L_{n,k}^h) \cup (R_{m,j}^v \times R_{n,k}^h). \end{aligned} \tag{6.8}$$



Now put

$$h_{n,k}^{m,j}(x,y) = \chi_{L_{n,k}^h \times R_{m,j}^v}(x,y) + \chi_{R_{n,k}^h \times L_{m,j}^v}(x,y) - \chi_{L_{n,k}^h \times L_{m,j}^v}(x,y) - \chi_{R_{n,k}^h \times R_{m,j}^v}(x,y).$$

We normalize the later function on $L^2(J)$ to obtain the Haar System

$$H_{n,k}^{m,j}(x,y) = 2^{\frac{n+m}{2}} h_{n,k}^{m,j}(x,y). \tag{6.9}$$

This system is illustrated in three dimension in Figure 5.

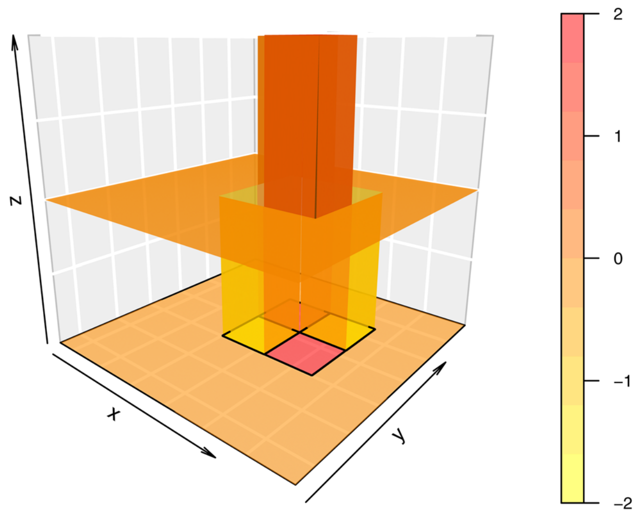


Figure 5: Representation of $H_{n,k}^{m,j}$ for $n = m = 1$ and $k = j = 0$ generated using a grid of 300 × 300 points over the rectangle $\left[-\frac{1}{2}, \frac{4}{5}\right] \times \left[-\frac{1}{2}, \frac{4}{5}\right]$. We observe that by projecting onto the xy-space, we obtain the sets L_n and R_n in Definition 5.

Theorem 11. *The Haar system defined in (6.9) can be generated by a single function ψ defined in (6.5) as*

$$H_{n,k}^{m,j}(x, y) = 2^{\frac{n+m}{2}} \psi(2^n x - k, 2^m y - j).$$

Moreover, the family $\{H_{n,k}^{m,j}\}$ is an orthonormal system in $L^2(J)$.

Proof. The proof can easily be obtained by noticing that by construction, if we project $H_{n,k}^{m,j}$ to the x-axis or to the y-axis, we obtain an orthonormal basis of $L^2[0, 1]$. □

Definition 14. Let $J = [0, 1] \times [0, 1]$. A wavelet on J is a function $\psi \in L^2(J)$ such that for integers m, n, k, j, the family

$$\{H_{n,k}^{m,j}\} = \left\{2^{\frac{n+m}{2}} \psi(2^n x - k, 2^m y - j)\right\}$$

is an orthonormal basis in $L^2(J)$. A similar definition applies to $L^2(J)$.

Remark 7.

- We observe that by restricting (or projecting) $H_{n,k}^{m,j}(x, y)$ to the real line, we will obtain the Haar function or the special atom.
- For $J = I \times I \times I$ or $\mathbb{R}^3$, we have the Haar system defined similarly and generated by a single function ψ as

$$H_{n,k}^{m,j,l,q}(x, y, z) = 2^{\frac{n+m+l}{2}} \psi(2^n x - k, 2^m y - j, 2^l z - q).$$

- In general, we can extend it to $\mathbb{R}^d$ for $d \geq 2$.
- The relationship between the Haar functions and the special atoms in higher dimension is very similar to the one in one-dimension.

  Indeed, the special atom defined on the dyadic interval can be written in terms of the Haar function using Definition 5.1 with the notation

$$B_{n,k}^{m,j}(x,y) = \frac{1}{|J_{n,k}^{m,j}|}[\chi_{R_{n,m,k,j}}(x,y) - \chi_{L_{n,m,k,j}}(x,y)],$$

where $J_{n,k}^{m,j}$ is the dyadic interval defined in (6.6) and $L_{n,m,k,j}$, $R_{n,m,k,j}$ are defined in (6.8). Then,

$$|J_{n,k}^{m,j}| = \frac{1}{2^n}\cdot\frac{1}{2^m} = \frac{1}{2^{n+m}} \quad \text{so that} \quad \frac{1}{|J_{n,k}^{m,j}|} = 2^{n+m}.$$

Therefore,

$$B_{n,k}^{m,j}(x,y) = 2^{\frac{n+m}{2}}\cdot 2^{\frac{n+m}{2}}[\chi_{R_{n,m,k,j}}(x,y) - \chi_{L_{n,m,k,j}}(x,y)] = 2^{\frac{n+m}{2}}h_{n,k}^{m,j}(x,y).$$

Ultimately, the point of this discussion is to show what could be investigated using the extension of the special atom space proposed in this article.

7 Applications

In this section, we show how to use the special atom above to estimate functions in the plane and space (Figures 6–9).

7.1 Applications in the plane

Let $f \in L^2(J)$. Then,

$$f(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} 2^{\frac{n}{2}}\alpha_{n,k}h_{n,k}(x).$$

We know that $\alpha_{n,k} = \langle f, h_{n,k}\rangle = \int_J f(x)h_{n,k}(x)\mathrm{d}x$. Therefore, a consistent estimator of $\alpha_{n,k}$, for a fixed integer P is given as:

$$\alpha_{n,k}(P) = \frac{1}{P}\sum_{i=1}^{P} f(x_i)h_{m,k}(x_i).$$

This also means that $\alpha_{n,k}(P)$ is a Riemann sum of $\alpha_{n,k}$. In addition, fix an integer N (resolution level). An estimator of f is the sequence of functions

$$f_N^P(x) = \sum_{n=0}^{N}\sum_{k=0}^{2^n-1} 2^{\frac{n}{2}}\alpha_{n,k}(P)h_{n,k}(x).$$

By construction, we have $f_N^P \to f$ uniformly as $N, P \to \infty$.

7.2 Applications in the space

Let $f \in L^2(J)$. Then, since $\{h_{n,k}^{m,j}\}$ is an orthonormal basis in $L^2(J)$, we have

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1}\sum_{m=0}^{\infty}\sum_{j=0}^{2^m-1} 2^{\frac{n+m}{2}}\alpha_{n,k}^{m,j}h_{n,k}^{m,j}(x,y).$$

We know that

$$\alpha_{n,k}^{m,j} = \langle f, h_{n,k}^{m,j}\rangle = \iint_J f(x,y) h_{n,k}^{m,j}(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Therefore, a consistent estimator of $\alpha_{n,k}^{m,j}$, for fixed integers P and Q is given as:

$$\alpha_{n,k}^{m,j}(P,Q) = \frac{1}{PQ}\sum_{i=1}^{P}\sum_{l=1}^{Q} f(x_i, y_l) h_{n,k}^{m,j}(x_i, y_l).$$

In addition, fix two integers N, M (resolution levels). An estimator of f is the sequence of functions

$$f_{NM}^{PQ}(x,y) = \sum_{n=0}^{N}\sum_{k=0}^{2^n-1}\sum_{m=0}^{M}\sum_{j=0}^{2^m-1} 2^{\frac{n+m}{2}}\alpha_{n,k}^{m,j}(P,Q) h_{n,k}^{m,j}(x,y).$$

By construction, we have

$$f_{NM}^{PQ} \to f \quad \text{uniformly as } N, M, Q, P \to \infty.$$

In the example below, we show that even for small values of N, M, P, Q the estimation of f using f_{NM}^{PQ} is quite good.

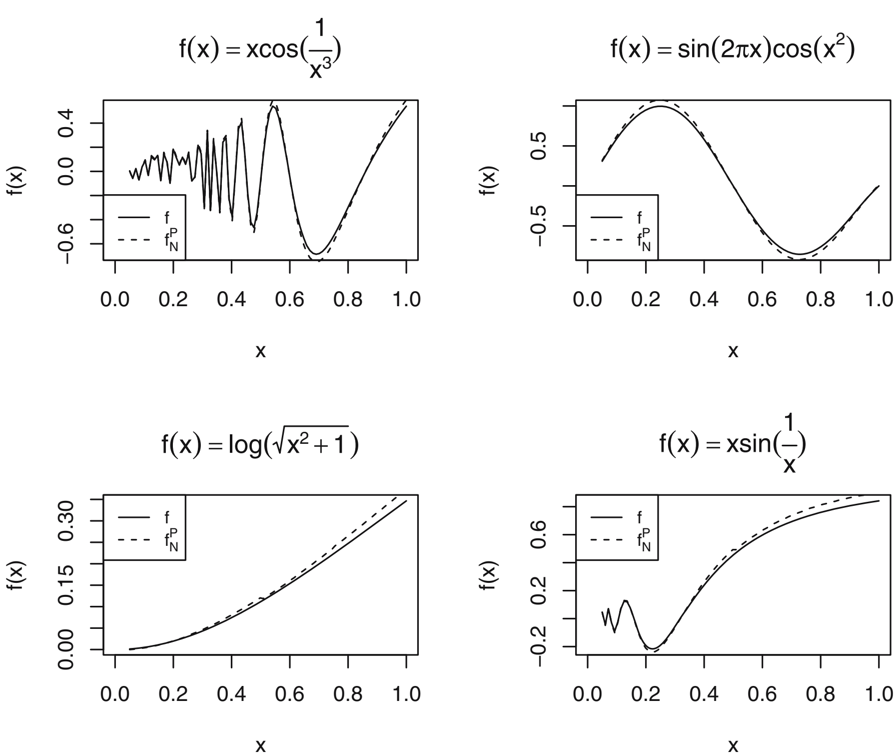


Figure 6: In the figures, we used $N = 17$; $P = 90$ to construct f_N^P. We observe that even for low resolution level and relatively small number of points, we obtain a good approximation.

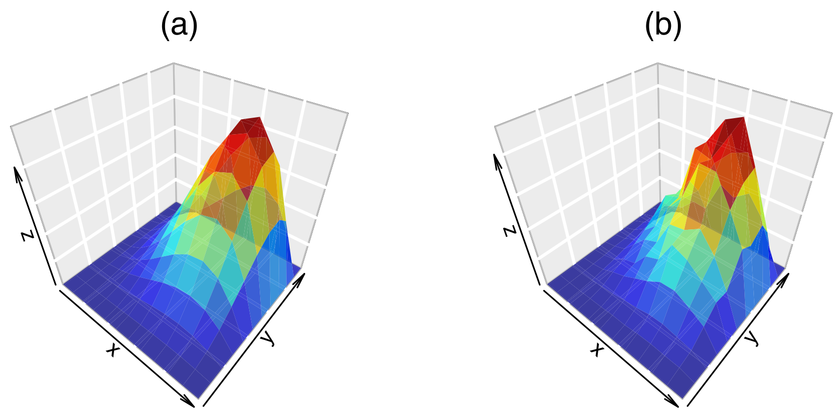


Figure 7: (a) is a representation of $f(x, y) = \sin(\pi x^2)\sin(\pi y^2)$, using a 12×12 grid of points. (b) is an estimate of f, using f_{NM}^{PQ}, for $N, M = 4$; $P = Q = 12$.

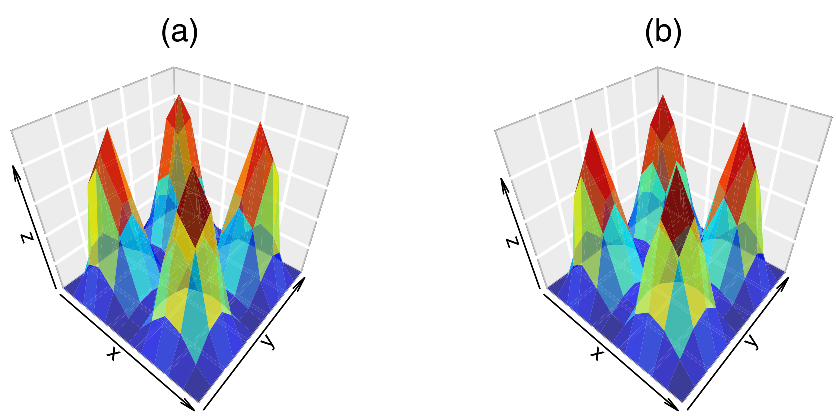


Figure 8: (a) is a representation of $f(x, y) = \sin(\pi x^2)^2 \sin(\pi y^2)^2$, using a 12×12 grid of points. (b) is an estimate of f, using f_{NM}^{PQ}, for $N, M = 4$; $P = Q = 12$.

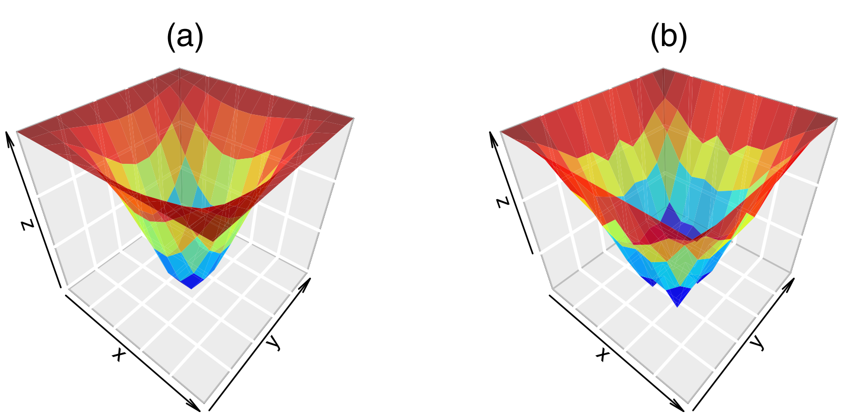


Figure 9: (a) is a representation of $f(x, y) = \frac{-1}{2\pi} \sin(\pi x)^2 \sin(\pi y)^2$, using a 12×12 grid of points. (b) is an estimate of f, using f_{NM}^{PQ}, for $N, M = 4$; $P = Q = 12$.

References

[1] Ronald Coifman, *A real variable characterization of h^p*, Studia Math. **51** (1974), 269–274.
[2] Geraldo De Souza, *Spaces formed by special atoms*, PhD thesis, SUNY at Albany, 1980.
[3] Geraldo De Souza and Gary Sampson, *Function in the Dirichlet space such that its Fourier series diverges almost everywhere*, Proc. Am. Math. Soc. **120** (1994) no. 3, 723–726.
[4] Eddy Kwessi, Paul Alfonso, Geraldo De Souza, and Asheber Abebe, *A note on multiplication and composition operators in Lorentz spaces*, J. Funct. Spaces Appl. **2012** (2012), 1–10, DOI: 10.1155/2012/293613.
[5] Stephen Bloom and Geraldo De Souza, *Atomic decomposition of generalized Lipschitz spaces*, Illinois J. Math. **33** (1989), no. 2, 181–209.
[6] Geraldo De Souza, *The atomic decomposition of Bergman-Besov-Lipschitz spaces*, Proc. Am. Math. Soc. **14** (1985), 682–686.
[7] Steven Krantz, *Handbook of Complex Variables*, Birkhäuser Boston Inc., 1999.
[8] Eddy Kwessi, Geraldo De Souza, Asheber Abebe, and Rauno Aulaskari, *Characterization of Lacunary functions in Bergma-Besov-Lipschitz spaces*, Complex Var. Elliptic Equ. **58** (2013), no. 2, 157–162.
[9] Antoni Zygmund, *Trigonometric Series*, vol. I and II, Cambridge Mathematical Library, 2002.