Research Article

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A strong convergence theorem for a zero of the sum of a finite family of maximally monotone mappings

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Abstract: The purpose of this article is to study the method of approximation for zeros of the sum of a finite family of maximally monotone mappings and prove strong convergence of the proposed approximation method under suitable conditions. The method of proof is of independent interest. In addition, we give some applications to the minimization problems and provide a numerical example which supports our main result. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

Keywords: firmly nonexpansive, Hilbert spaces, maximally monotone mapping, strong convergence, zero points

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1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|\cdot\|$. Recall that for a mapping $A : H \to 2^H$, the domain of A, $\mathrm{Dom}(A)$, is given by $\mathrm{Dom}(A) = \{x \in H : Ax \neq \emptyset\}$, the graph of A, $\mathrm{Gph}(A)$, is given by $\mathrm{Gph}(A) = \{(x, y) \in H \times H : y \in Ax\}$ and the range of A, $\mathrm{ran}(A)$, is given by $\mathrm{ran}(A) = \{Ax : x \in \mathrm{Dom}(A)\}$. The mapping A is called monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathrm{Gph}(A), \tag{1}$$

and it is called *maximally monotone* if it is monotone and the graph of A is not properly contained in the graph of any other monotone mapping. The resolvent of A with parameter $\lambda > 0$ is $J_{\lambda A} = (I + \lambda A)^{-1}$, where I is the identity mapping on H, and it enjoys firmly nonexpansive property, that is, for any $x, y \in \mathrm{ran}(I + \lambda A)$, we have

$$\|J_{\lambda A}x - J_{\lambda A}y\|^2 \leq \langle x - y, J_{\lambda A}x - J_{\lambda A}y \rangle. \tag{2}$$

A monotone mapping $A : H \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that for any $x, y \in H$ we have

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2. \tag{3}$$

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Let $A_k : H \to 2^H, k = 1, 2, \ldots, m$, be maximally monotone mappings. Consider the inclusion problem of finding $z \in H$ such that

$$0 \in A_1 z + A_2 z + \cdots + A_m z, \tag{4}$$

where $m \geq 2$. We denote the solution set of (4) by $zero(A_1 + A_2 + \cdots + A_m) = (A_1 + A_2 + \cdots + A_m)^{-1}(0)$. This problem, which includes variational inequality problems, equilibrium problems, complementary problems, minimization problems, nonlinear evolution equations, fixed point problems as special cases, is quite general. In fact, a number of problems arising in applied areas such as image recovery, machine learning and signal processing can be mathematically modeled as (4), see [1,2] and references therein. To be more precise, a stationary solution to the initial value problem of the evolution equation

$$0 \in F(t) + \frac{\partial x}{\partial t}, x(0) = x_0 \tag{5}$$

can be formulated as (4) when the governing maximal monotone F is of the form $F := A_1 + A_2 + \cdots + A_m$ (see, e.g., [3]). Furthermore, optimization problems often need (see, e.g., [4]) to solve a minimization problem of the form

$$\min_{x \in H}\{g_1(x) + g_2(x) + \cdots + g_m(x)\}, \tag{6}$$

where $g_i, i = 1, 2, \ldots, m$ are proper lower semicontinuous convex functions from H to the extended real line $\bar{R} := (-\infty, \infty]$. If in (6), we assume that $A_i := \partial g_i$, for $i = 1, 2, \ldots, m$, where ∂ of g_i is the subdifferential operator of g_i in the sense of convex analysis, then (6) is equivalent to (4). Consequently, considerable research efforts have been devoted to methods of finding approximate solutions (when they exist) of inclusions of the form (4) for a sum of a finite number of monotone mappings (see, e.g., [3,5]).

For the case where $m = 2$, the inclusion problem (4) reduces to the problem of finding $z \in H$ such that

$$0 \in Az + Bz, \tag{7}$$

where A and B are monotone mappings. For solving problem (7), several authors have studied different iterative schemes (see, e.g., [6–16] and references therein). The most attractive methods for solving the inclusion problem (7) are the Peaceman-Rachford and Douglas-Rachford iterative methods.

The nonlinear Peaceman-Rachford and Douglas-Rachford, splitting iterative methods, introduced by Lions and Mercier [3], are given by

$$x_{n+1} = (2J_{\lambda A} - I)(2J_{\lambda B} - I)x_n, \quad n \geq 1, \tag{8}$$

and

$$x_{n+1} = J_{\lambda A}(2J_{\lambda B} - I)x_n + (I - J_{\lambda B})x_n, \quad n \geq 1, \tag{9}$$

respectively, where $\lambda > 0$ is a fixed scalar. The nonlinear Peaceman-Rachford algorithm (8) fails, in general, to converge (even in the *weak topology* in the infinite-dimensional setting). This is due to the fact that the generating mapping $(2J_{\lambda A} - I)(2J_{\lambda B} - I)$ is merely nonexpansive. The nonlinear Douglas-Rachford algorithm (9) was initially proposed in [3] for finding a zero of the sum of two maximally monotone mappings and has been studied by many authors (see, e.g., [1,3,11,17,18] and references therein). This method always converges in the weak topology to a solution of (7), since the generating operator $J_{\lambda A}(2J_{\lambda B} - I) + (I - J_{\lambda B})$ for this algorithm is firmly nonexpansive (see, e.g., [11]).

In 1979, Passty [11] studied the *forward-backward* splitting method which is given by

$$x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \quad n \geq 1, \tag{10}$$

where $\{\lambda_n\}$ is a sequence of positive scalars, A and B are maximal monotone mappings. He proved that the sequence in (10) converges weakly to the solution of problem (7). Different authors have used algorithm (10), for the inclusion problem (7), when A is a single-valued α-inversely strong monotone (or α-strongly monotone) mapping and B is a maximal monotone mapping defined in real Hilbert spaces (see, e.g., [18,19]).

We remark that the aforementioned results provide weak convergence. But we also indicate that several authors have studied different iterative methods (see, e.g., [21–24] and references therein) and proved strong convergent results to approximate zeros of the sum of monotone mappings A and B, where $A : H \to H$ is an α-inverse strongly monotone mapping and $B : H \to 2^H$ is a maximally monotone mapping under certain conditions (see, e.g., [19,25–27]).

In 2012, Takahashi et al. [19] studied the following Halpern-type iteration in a Hilbert space setting: for any $x_0 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J_{r_nB}(x_n - r_nAx_n)), \quad \forall n \geq 0, \tag{11}$$

where $u \in H$ is a fixed vector and A is an α-inverse strongly monotone and single-valued mapping on H, and B is a maximally monotone mapping on H. They proved that the sequence $\{x_n\}$ generated by (11) converges strongly to a point $x \in (A + B)^{-1}(0)$ provided that the control sequences $\{\beta_n\}$, $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions.

Recently, Bot et al. [25] studied the following classical Douglas-Rachford method:

$$\begin{cases} y_n = J_{yB}(\beta_n x_n); \\ z_n = J_{yA}(2y_n - \beta_n x_n); \\ x_{n+1} = \beta_n x_n + \lambda_n(z_n - y_n), \quad n \geq 1, \end{cases} \tag{12}$$

where $A : H \to 2^H$ and $B : H \to 2^H$ are maximally monotone mappings and $\{\lambda_n\}$ and $\{\beta_n\}$ are real sequences satisfying certain conditions and showed that $\{x_n\}$ converges strongly to $\bar{x} = P_{F(R_{yA}R_{yB})}(0)$, as $n \to \infty$, where $R_{yA} = 2J_{yA} - I$ while $\{y_n\}$ and $\{z_n\}$ converge strongly to $J_{yB}(\bar{x}) \in \text{zer}(A + B)$, as $n \to \infty$.

More recently, Wega and Zegeye [15] constructed an algorithm that converges strongly to a solution of the sum of two maximally monotone mappings using a different technique.

Question 1. A natural question arises whether we can obtain a strong convergence result for approximating zeros of the sum of a finite family of maximally monotone mappings via *the extended solution set* of the sum of maximally monotone mappings?

In 2009, Svaiter [28] constructed a new approach for splitting algorithms, which starts by reformulating (4) as the problem of locating a point in a certain *extended solution set* $S_e(A_1, A_2, \ldots, A_m)$ subset of $H \times H^m$, which is defined by:

$$S_e(A_1, \ldots, A_m) = \{(z, w_1, \ldots, w_m) \in V : z \in H, w_k \in A_k(z), k = 1, 2, \ldots, m\},$$

where

$$V = \{(z, w_1, \ldots, w_m) \in H \times H^m : w_1 + \cdots + w_m = 0\}.$$

He proved weak convergence results provided that H has a finite dimension or $A_1 + A_2 + \cdots + A_m$ is maximal monotone.

We remark that the extended solution set is associated with the common fixed points of a countable family of nonexpansive mappings and so the methods of approximating fixed points are used to approximate the solution of problem (4).

Motivated and inspired by the above results, our purpose in this article is to construct a viscosity-type algorithm for finding zeros of the sum of a finite family of maximally monotone mappings via the extended solution set $S_e(A_1, A_2, \ldots, A_m)$ and discuss its strong convergence. The viscosity method introduced by Moudafi [30] involves a contraction mapping f in the procedure and it can be regarded as a regularization process for the solution of problem (4), which is supposed to induce the convergence in norm of the iterates. Another advantage of this method is that it allows one to select a particular solution point of (4), which satisfies some variational inequality. The assumption that one of the mappings is α-inverse strongly monotone is dispensed with. Our results provide an affirmative answer to our question. Our method of proof is of independent interest. Our results improve and generalize several results in the literature.

2 Preliminaries

In this section, we recall some definitions and known results that will be used in the sequel.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be a *contraction* if there exists $\alpha \in [0, 1)$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$, $\forall x, y \in C$ and it is said to be *nonexpansive* if $\alpha = 1$. The set of fixed points of T is defined by $F(T) := \{x \in C : Tx = x\}$.

Lemma 2.1. [29] *Let C be a closed and convex subset of a real Hilbert space H, and $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\} \subseteq C$ and $x \in C$ such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then, $x \in F(T)$.*

The following lemmas shall be used in the later section.

Lemma 2.2. [28] *Finding a point in $S_e(A_1, \ldots, A_m)$ is equivalent to solving* (4) *in the sense that*

$$0 \in A_1(z) + \cdots + A_m(z) \Leftrightarrow \exists w_1, \ldots, w_m \in H : (z, w_1, \ldots, w_m) \in S_e(A_1, \ldots, A_m).$$

Lemma 2.3. [28] *If the monotone operators $A_1, \ldots, A_m$ are maximally monotone, the corresponding extended solution set $S_e(A_1, \ldots, A_m)$ is closed and convex in H^{m+1}.*

Lemma 2.4. [28] *Given $(x_k, y_k) \in Gph(A_k)$, $k = 1, \ldots, m$, define $\varphi : V \to \mathbb{R}$ via*

$$\varphi(z, w_1, \ldots, w_m) = \sum_{k=1}^{m} \langle z - x_k, y_k - w_k \rangle. \tag{13}$$

Then, for any $(z, w_1, \ldots, w_m) \in S_e(A_1, \ldots, A_m)$, one has $\varphi(z, w_1, \ldots, w_m) \leq 0$, that is,

$$S_e(A_1, \ldots, A_m) \subseteq \{(z, w_1, \ldots, w_m) \in V : \varphi(z, w_1, \ldots, w_m) \leq 0\}.$$

In addition, φ is affine on V, with

$$\nabla\varphi = \left(\sum_{k=1}^{m} y_k, x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_m - \bar{x}\right), \quad \text{where } \bar{x} = \frac{1}{m}\sum_{k=1}^{m} x_k \tag{14}$$

and

$$\begin{aligned} \nabla\varphi = 0 \quad &\Leftrightarrow (x_1, y_1, \ldots, y_m) \in S_e(A_1, \ldots, A_m), \quad x_1 = x_2 = \cdots = x_m \\ &\Leftrightarrow \varphi(z, w_1, \ldots, w_m) = 0, \quad \forall (z, w_1, \ldots, w_m) \in V. \end{aligned} \tag{15}$$

The function φ in Lemma 2.4 is called *decomposable separators*.

Lemma 2.5. [31] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, \, n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n\delta_n| < \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [32] *Let $x, y \in H$. If H is a real Hilbert space, then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.7. [33] *Let C be a closed and convex subset of a real Hilbert space H and $x \in H$ be given. The metric projection of H onto C, P_C, is characterized by the following*:

(i) $P_C x \in C$, $\langle x - P_C x, P_C x - z\rangle \geq 0$, *for all* $z \in C$;
(ii) P_C *is firmly nonexpansive and hence nonexpansive.*

3 Main results

In this section, we introduce an algorithm for finding a point in $S_e(A_1,\dots,A_m)$, which will lead us to a solution of the sum of a finite family of maximally monotone mappings in a Hilbert H space and discuss its strong convergence.

In what follows, let H be a real Hilbert space and $A_k : H \to 2^H$, $k = 1,\dots,m$, where $m \geq 2$, be a family of maximal monotone mappings satisfying $S_e(A_1,\dots,A_m) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$, and let $\{\beta_n\}$ be a decreasing sequence in $(0,1]$ with $\beta_0 = 1$.

We now propose the following algorithm which basically uses Algorithm 3 of [28].

Algorithm 3.1.
Step 0: Select initial guess $u_0 = (z_0, w_{1,0},\dots,w_{m,0}) \in V$.
Step 1: Given nth iterates $(n \geq 0)$, compute $x_{k,n}$ and $y_{k,n}$ satisfying:

$$\begin{aligned} x_{k,n} &= (I + \lambda_{k,n}A_k)^{-1}(z_n + \lambda_{k,n}w_{k,n}), \quad k = 1,2,\dots,m,\\ y_{k,n} &= w_{k,n} + \frac{z_n - x_{k,n}}{\lambda_{k,n}}, \quad k = 1,2,\dots,m, \end{aligned}$$

where the real sequence $\lambda_{k,n} > 0$, for each $k = 1,2,\dots,m$ and $n \geq 0$.
Step 2: Define $\varphi_i : V \to \mathbb{R}$ by

$$\varphi_i(z, w_1,\dots,w_m) = \sum_{k=1}^{m}\langle z - x_{k,i}, y_{k,i} - w_k\rangle, \tag{16}$$

and compute

$$T_iu_n = (z_n, w_{1,n},\dots,w_{m,n}) - \max\left\{0, \frac{\varphi_i(z_n, w_{1,n},\dots,w_{m,n})}{\|\nabla\varphi_i\|^2}\right\}\nabla\varphi_i, \tag{17}$$

where $\nabla\varphi_i = (\sum_{k=1}^{m}y_{k,i}, x_{1,i} - \overline{x_i}, x_{2,i} - \overline{x_i},\dots,x_{m,i} - \overline{x_i})$ and $\|\nabla\varphi_i\|^2 = \sum_{k=1}^{m}\|x_{k,i} - \overline{x_i}\|^2 + \|\sum_{k=1}^{m}y_{k,i}\|^2$ with $\overline{x_i} = \frac{1}{m}\sum_{k=1}^{m}x_{k,i}$, which is the projection of $u_n = (z_n, w_{1,n},\dots,w_{m,n})$ onto the half space

$$H_i = \{(z, w_1,\dots,w_m) \in V : \varphi_i(z, w_1,\dots,w_m) \leq 0\}.$$

Step 3: Compute

$$\begin{cases} v_n = \beta_nu_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_iu_n,\\ u_{n+1} = \alpha_nf(u_n) + (1 - \alpha_n)v_n, \quad \forall n \geq 0, \end{cases} \tag{18}$$

where $f : V \to V$ is a contraction mapping with constant α. Set $n := n + 1$ and go to step 1.

Remark 3.1. The following points indicate that Algorithm 3.1 is well defined.

(i) By maximality of A_k the resolvent mapping $(I + \lambda_{k,n}A_k)^{-1}$ is single-valued and everywhere defined for each $k = 1,2,\dots,m$ (see, e.g., [34]). By rearranging the equation in step 1, one has the following equation:

$$x_{k,n} + \lambda_{k,n}y_{k,n} = z_n + \lambda_{k,n}w_{k,n}, \quad \forall n \geq 0, \tag{19}$$

where for each $k = 1,2,\dots,m$ and $n \geq 0$, $\lambda_{k,n} > 0$ and $y_{k,n} \in A_k(x_{k,n})$. Hence, for each $k = 1,2,\dots,m$, $(x_{k,n}, y_{k,n}) \in \mathrm{Gph}(A_k)$ exists and is unique. Thus, the decomposable separator function φ in Lemma 2.4 is well defined.

(ii) Since $\varphi_i : V \to \mathbb{R}$ is affine on V (see Lemma 2.4) and the half space H_i is closed and convex subspace of V for all $i \geq 1$, the projection of u_n onto H_i given by (17) exists and is firmly nonexpansive (see, e.g., [20,28]).

Lemma 3.2. *The sequence $\{u_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $u^* = (z^*, w_1^*, w_2^*, \ldots, w_m^*) \in S_e(A_1, \ldots, A_m)$, then by Lemma 2.4 we have $u^* \in H_i$ for all $i \geq 0$, which implies that $S_e(A_1, \ldots, A_m) \subset \bigcap_{i=1}^{\infty} F(T_i) = \mathcal{F} \neq \emptyset$. Now, the fact that T_i is nonexpansive and (18) yields

$$\begin{aligned} \|v_n - u^*\| &= \left\| \beta_n u_n + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i u_n - u^* \right\| \\ &= \left\| \beta_n u_n + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i u_n - \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) u^* + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) u^* - u^* \right\| \\ &= \left\| \beta_n (u_n - u^*) + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)(T_i u_n - u^*) \right\| \\ &\leq \beta_n \|u_n - u^*\| + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \|u_n - u^*\| = \|u_n - u^*\|. \end{aligned} \tag{20}$$

Thus, it follows that

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|\alpha_n f(u_n) + (1 - \alpha_n) v_n - u^*\| \\ &\leq \alpha_n \|f(u_n) - u^*\| + (1 - \alpha_n) \|v_n - u^*\| \\ &\leq \alpha_n \|f(u_n) - f(u^*)\| + \alpha_n \|f(u^*) - u^*\| + (1 - \alpha_n) \|v_n - u^*\| \\ &\leq (1 - \alpha_n (1 - \alpha)) \|u_n - u^*\| + \alpha_n \|f(u^*) - u^*\| \\ &\leq \max \left\{ \|u_n - u^*\|, \frac{\|f(u^*) - u^*\|}{1 - \alpha} \right\}. \end{aligned}$$

Then, we have

$$\|u_n - u^*\| \leq \max \left\{ \|u_0 - u^*\|, \frac{\|f(u^*) - u^*\|}{1 - \alpha} \right\},$$

which implies that $\{u_n\}$ is bounded. Hence, we can obtain that $\{T_i u_n\}$, $\{v_n\}$ and $\{f(u_n)\}$ are bounded. □

Lemma 3.3. *The sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to $u^* = (z^*, w_1^*, \ldots, w_m^*) \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. We proceed with the following steps.
Step 1. First we show that $\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0$.
From (18), we have

$$\begin{aligned} \|v_n - v_{n-1}\| &= \left\| \beta_n u_n + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i u_n - \beta_{n-1} u_{n-1} - \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) T_i u_{n-1} \right\| \\ &\leq \beta_n \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|u_{n-1}\| + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \|T_i u_n - T_i u_{n-1}\| + |\beta_n - \beta_{n-1}| \|T_i u_{n-1}\| \\ &\leq \beta_n \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|u_{n-1}\| + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| \|T_i u_{n-1}\| \\ &\leq \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| M, \end{aligned} \tag{21}$$

for some $M > 0$ as the sequences $\{u_n\}$ and $\{T_i u_{n-1}\}$ are bounded. Thus, the inequalities in (18) and (21) imply that

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|\alpha_n f(u_n) + (1 - \alpha_n) v_n - \alpha_{n-1} f(u_{n-1}) - (1 - \alpha_{n-1}) v_{n-1}\| \\ &\leq \alpha_n \|f(u_n) - f(u_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(u_{n-1})\| + (1 - \alpha_n) \|v_n - v_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|v_{n-1}\| \\ &\leq \alpha \alpha_n \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| (\|f(u_{n-1})\| + \|v_{n-1}\|) + (1 - \alpha_n)(\|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}| M) \\ &= (1 - \alpha_n (1 - \alpha)) \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| (\|f(u_{n-1})\| + \|v_{n-1}\|) + (1 - \alpha_n)(\beta_{n-1} - \beta_n) M. \end{aligned} \tag{22}$$

Since $\{\beta_n\}$ is strictly decreasing, it implies that $\sum_{n=1}^{\infty}(\beta_{n-1}-\beta_n)<\infty$. Hence, from (22), conditions of $\{\alpha_n\}$ and Lemma 2.5, we immediately obtain that

$$||u_{n+1}-u_n||\to 0 \quad \text{as } n\to\infty. \tag{23}$$

Step 2. We show that $||T_iu_n-u_n||\to 0$ as $n\to\infty$. Take $u^*=P_{\mathcal{F}}(f(u^*))$. Note that

$$\begin{aligned}\|u_n-u^*\|^2 &\geq \|T_iu_n-T_iu^*\|^2\\ &= \|T_iu_n-u_n+u_n-T_iu^*\|^2\\ &= \|T_iu_n-u_n\|^2+\|u_n-u^*\|^2+2\langle T_iu_n-u_n,u_n-u^*\rangle,\end{aligned} \tag{24}$$

which yields

$$\frac{1}{2}\|T_iu_n-u_n\|^2\leq\langle u_n-T_iu_n,u_n-u^*\rangle. \tag{25}$$

Furthermore, from (18) and (25), we immediately obtain

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_iu_n-u_n\|^2 &\leq \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\langle u_n-T_iu_n,u_n-u^*\rangle\\ &= \langle(1-\beta_n)u_n-\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_iu_n,u_n-u^*\rangle\\ &= \langle(1-\beta_n)u_n-v_n+\beta_nu_n,u_n-u^*\rangle\\ &= \langle u_n-v_n,u_n-u^*\rangle\\ &= \langle u_n-u_{n+1},u_n-u^*\rangle+\langle u_{n+1}-v_n,u_n-u^*\rangle,\end{aligned} \tag{26}$$

and from (18) and (26), we have

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)||T_iu_n-u_n||^2 &\leq \langle u_n-u_{n+1},u_n-u^*\rangle+\langle\alpha_nf(u_n)+(1-\alpha_n)v_n-v_n,u_n-u^*\rangle\\ &\leq \|u_n-u_{n+1}\|\|u_n-u^*\|+\langle\alpha_nf(u_n)-\alpha_nv_n,u_n-u^*\rangle\\ &\leq \|u_n-u_{n+1}\|\|u_n-u^*\|+\alpha_n\|f(u_n)-v_n\|\|u_n-u^*\|.\end{aligned} \tag{27}$$

Thus, from Lemma 3.2, (23), (27) and the condition on $\{\alpha_n\}$, we conclude that

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_iu_n-u_n\|^2=0. \tag{28}$$

Since $\{\beta_n\}$ is strictly decreasing, for every $i\in\mathbb{N}$, equality (28) yields

$$\lim_{n\to\infty}\|T_iu_n-u_n\|=0 \text{ and } \lim_{n\to\infty}\|T_nu_n-u_n\|=0. \tag{29}$$

Step 3. We show that $\limsup_{n\to\infty}\langle f(u^*)-u^*,u_n-u^*\rangle\leq 0$.

Since $\{u_n\}$ is bounded, there exist $u\in H\times H^m$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle f(u^*)-u^*,u_n-u\rangle^*=\lim_{k\to\infty}\langle f(u^*)-u^*,u_{n_k}-u^*\rangle$$

and $u_{n_k}\rightharpoonup u$. From (29) and Lemma 2.1, we obtain that $u\in F(T_i)$ for each $i\geq 1$ and hence $u\in\mathcal{F}$. Since $u^*=P_{\mathcal{F}}(f(u^*))$, by Lemma 2.7, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(u^*)-u^*,u_n-u^*\rangle &= \lim_{k\to\infty}\langle f(u^*)-u^*,u_{n_k}-u^*\rangle\\ &= \langle f(u^*)-u^*,u-u^*\rangle\leq 0, \quad \text{as required.}\end{aligned} \tag{30}$$

Step 4. Finally, we show that $u_n\to u^*$ as $n\to\infty$. From Lemma 2.6, (18) and (20), we have

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &= \|\alpha_n f(u_n)+(1-\alpha_n)v_n-u^*\|^2\\ &\le (\alpha\alpha_n\|u_n-u^*\|+(1-\alpha_n)\|u_n-u^*\|)^2+2\alpha_n\langle f(u^*)-u^*,u_{n+1}-u^*\rangle\\ &\le (1-(1-\alpha)\alpha_n)\|u_n-u^*\|^2+2\alpha_n\langle f(u^*)-u^*,u_{n+1}-u^*\rangle\\ &\le (1-(1-\alpha)\alpha_n)\|u_n-u^*\|^2+2\alpha_n\langle f(u^*)-u^*,u_n-u^*\rangle+2\alpha_n\|f(u^*)-u^*\|\|u_{n+1}-u_n\|.\end{aligned}\tag{31}$$

Therefore, from (30), (31) and Lemma 2.5, we conclude that $u_n \to u^*$ as $n \to \infty$. □

Lemma 3.4. *If in Algorithm 3.1, there exist $\overline{\lambda} > \underline{\lambda} > 0$ such that the sequence $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \overline{\lambda}]$, for each $k = 1, 2, \dots, m$ and $n \ge 0$, then there exists $\eta > 0$ such that*

$$\varphi_n(u_n) \ge \eta\|\nabla\varphi_n(u_n)\|^2 \quad \forall\, n \ge 0, \tag{32}$$

where $\nabla\varphi_n(u_n) = (\sum_{k=1}^m y_{k,n}, x_{1,n}-\overline{x_n}, x_{2,n}-\overline{x_n}, \dots, x_{m,n}-\overline{x_n})$ and $\|\nabla\varphi_n(u_n)\|^2 = \sum_{k=1}^m \|x_{k,n}-\overline{x_n}\|^2 + \|\sum_{k=1}^m y_{k,n}\|^2$ with $\overline{x_n} = \frac{1}{m}\sum_{k=1}^m x_{k,n}$.

Proof. From Eq. (19), we have

$$\frac{z_n - x_{k,n}}{\lambda_{k,n}} = y_{k,n} - w_{k,n}. \tag{33}$$

From (16), (33) and condition of $\lambda_{k,n}$, we get

$$\begin{aligned}\varphi_n(u_n) &= \sum_{k=1}^m \langle z_n - x_{k,n}, y_{k,n} - w_{k,n}\rangle = \sum_{k=1}^m \left\langle z_n - x_{k,n}, \frac{z_n - x_{k,n}}{\lambda_{k,n}}\right\rangle\\ &= \sum_{k=1}^m \frac{1}{\lambda_{k,n}}\|z_n - x_{k,n}\|^2 \ge \frac{1}{\overline{\lambda}}\sum_{k=1}^m \|z_n - x_{k,n}\|^2.\end{aligned}\tag{34}$$

By rearranging equation (33), we can also get

$$y_{k,n} = \frac{z_n - x_{k,n}}{\lambda_{k,n}} + w_{k,n}, \tag{35}$$

which implies that

$$\sum_{k=1}^m y_{k,n} = \sum_{k=1}^m \frac{z_n - x_{k,n}}{\lambda_{k,n}} + \sum_{k=1}^m w_{k,n} = \sum_{k=1}^m \frac{z_n - x_{k,n}}{\lambda_{k,n}}, \tag{36}$$

and hence

$$\begin{aligned}\left\|\sum_{k=1}^m y_{k,n}\right\|^2 &= \left\|\sum_{k=1}^m \frac{z_n - x_{k,n}}{\lambda_{k,n}}\right\|^2 \le \frac{1}{\underline{\lambda}^2}\left\|\sum_{k=1}^m z_n - x_{k,n}\right\|^2\\ &= \frac{1}{\underline{\lambda}^2}\|m(z_n - \overline{x_n})\|^2 = \frac{m^2}{\underline{\lambda}^2}\|z_n - \overline{x_n}\|^2,\end{aligned}\tag{37}$$

which is equivalent to

$$\frac{\underline{\lambda}^2}{m}\left\|\sum_{k=1}^m y_{k,n}\right\|^2 \le m\,\|z_n - \overline{x_n}\|^2. \tag{38}$$

In addition, from the properties of inner product, we have

$$\begin{aligned}
\sum_{k=1}^{m}\|x_{k,n}-\overline{x_n}\|^2 &= \sum_{k=1}^{m}\|x_{k,n}-z_n+z_n-\overline{x_n}\|^2\\
&= \sum_{k=1}^{m}\|(x_{k,n}-z_n)-(\overline{x_n}-z_n)\|^2\\
&= \sum_{k=1}^{m}\|x_{k,n}-z_n\|^2-2\left\langle \sum_{k=1}^{m}(x_{k,n}-z_n),\overline{x_n}-z_n\right\rangle+\sum_{k=1}^{m}\|\overline{x_n}-z_n\|^2\\
&= \sum_{k=1}^{m}\|x_{k,n}-z_n\|^2-2m\langle\overline{x_n}-z_n,\overline{x_n}-z_n\rangle+m\,\|\overline{x_n}-z_n\|^2\\
&= \sum_{k=1}^{m}\|x_{k,n}-z_n\|^2-2m\,\|\overline{x_n}-z_n\|^2+m\,\|\overline{x_n}-z_n\|^2\\
&= \sum_{k=1}^{m}\|x_{k,n}-z_n\|^2-m\|\overline{x_n}-z_n\|^2,
\end{aligned} \tag{39}$$

which implies that

$$\sum_{k=1}^{m}\|x_{k,n}-\overline{x_n}\|^2+m\,\|\overline{x_n}-z_n\|^2=\sum_{k=1}^{m}\|x_{k,n}-z_n\|^2. \tag{40}$$

From (38) and (40), we get

$$\sum_{k=1}^{m}\|x_{k,n}-\overline{x_n}\|^2+\frac{\lambda^2}{m}\left\|\sum_{k=1}^{m}y_{k,n}\right\|^2\le\sum_{k=1}^{m}\|x_{k,n}-z_n\|^2 \tag{41}$$

and hence

$$\frac{1}{\overline{\lambda}}\left(\sum_{k=1}^{m}\|x_{k,n}-\overline{x_n}\|^2+\frac{\lambda^2}{m}\left\|\sum_{k=1}^{m}y_{k,n}\right\|^2\right)\le\frac{1}{\overline{\lambda}}\sum_{k=1}^{m}\|x_{k,n}-z_n\|^2. \tag{42}$$

Thus, from (42), (34) and setting $\eta=\min\left\{\frac{1}{\overline{\lambda}},\frac{\lambda^2}{m\overline{\lambda}}\right\}$, we get

$$\eta\left(\sum_{k=1}^{m}\|x_{k,n}-\overline{x_n}\|^2+\left\|\sum_{k=1}^{m}y_{k,n}\right\|^2\right)=\eta\|\nabla\varphi_n(u_n)\|^2\le\varphi_n(u_n). \tag{43}$$

□

Next, we state and prove our main theorem.

Theorem 3.5. *Let H be a real Hilbert space and let $A_k : H\to 2^H$, for $k\in\{1,2,\dots,m\}$ and $m\ge 2$, be maximally monotone mappings satisfying $\Omega := \{z\in H : 0\in A_1z+A_2z+\cdots+A_mz\}\ne\emptyset$. Let $f : V\to V$ be a contraction mapping with constant α. Let the real sequence $\{\lambda_{k,n}\}\subset[\underline{\lambda},\overline{\lambda}]$ for some $\overline{\lambda}>\underline{\lambda}>0$ and for each $k=1,2,\dots,m$ and $n\ge 0$. Then, the sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to an element $u^*=(z^*,w_1^*,\dots,w_m^*)\in S_e(A_1,A_2,\dots,A_m)$, satisfying the variational inequality*

$$\langle(f-I)u^*,u^*-x\rangle\ge 0,\quad\forall x\in S_e(A_1,A_2,\dots,A_m), \tag{44}$$

where $z^*\in\Omega$.

Proof. By Lemma 3.4, there exists $\eta>0$ such that

$$\varphi_n(u_n)\ge\eta\,\|\nabla\varphi_n\|^2\quad\forall n\ge 0. \tag{45}$$

This implies that $\varphi_n(u_n)$ is always nonnegative, and from (17), we obtain

$$T_nu_n-u_n=-\left(\frac{\varphi_n(u_n)}{\|\nabla\varphi_n\|^2}\right)\nabla\varphi_n, \tag{46}$$

which implies

$$\|T_n u_n - u_n\| = \left\| -\left(\frac{\varphi_n(u_n)}{\|\nabla\varphi_n\|^2}\right)\nabla\varphi_n \right\| = \frac{\varphi_n(u_n)}{\|\nabla\varphi_n\|}, \tag{47}$$

for all n such that $\nabla\varphi_n \neq 0$. Thus, dividing both sides of inequality (45) by $\|\nabla\varphi_n\|$, we obtain

$$\|T_n u_n - u_n\| \geq \eta\|\nabla\varphi_n\|, \tag{48}$$

which is also true for n having $\nabla\varphi_n = 0$. From (29), we have $\|T_n u_n - u_n\| \to 0$ as $n \to \infty$, so (48) implies $\nabla\varphi_n \to 0$ as $n \to \infty$. Thus, it follows from (47) that

$$\lim_{n\to\infty} \varphi_n(u_n) = 0. \tag{49}$$

From the expression for $\nabla\varphi_n$, we have $\sum_{k=1}^{m} y_{k,n} \to 0$ and $x_{k,n} - \overline{x_n} \to 0$ for $k = 1, 2, \ldots, m$ as $n \to \infty$. Moreover, subtracting $x_{k,n} + \lambda_{k,n} w_{k,n}$ from both sides of Eq. (19), we obtain

$$z_n - x_{k,n} = \lambda_{k,n}(y_{k,n} - w_{k,n}). \tag{50}$$

This and the definition of φ_n imply that

$$\varphi_n(u_n) = \sum_{k=1}^{m} \langle z_n - x_{k,n}, y_{k,n} - w_{k,n}\rangle = \sum_{k=1}^{m} \langle \lambda_{k,n}(y_{k,n} - w_{k,n}), y_{k,n} - w_{k,n}\rangle = \sum_{k=1}^{m} \lambda_{k,n}\|y_{k,n} - w_{k,n}\|^2. \tag{51}$$

Hence, from (49), (51) and the fact that $\lambda_{k,n} > \underline{\lambda}$, we have

$$\lim_{n\to\infty} \|y_{k,n} - w_{k,n}\| = 0, \quad \text{for all } k = 1, 2, \ldots, m, \tag{52}$$

and from (50) and (52), we obtain

$$\lim_{n\to\infty} \|z_n - x_{k,n}\| = 0, \quad \text{for all } k = 1, 2, \ldots, m. \tag{53}$$

Moreover, from Lemma 3.3 the sequence $\{u_n\} = \{(z_n, w_{1,n}, \ldots, w_{m,n})\}$ converges strongly to a point $u^* = (z^*, w_1^*, \ldots, w_m^*) \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$. In addition, Eqs. (52) and (53) imply that $y_{k,n} \to w_k^*$ and $x_{k,n} \to z^*$, for each $k = 1, 2, \ldots, m$. Since $\mathrm{Gph}(A_k)$ is closed and $(x_{k,n}, y_{k,n}) \in \mathrm{Gph}(A_k)$ for all $k = 1, 2, \ldots, m$ and $n \geq 0$, we get $w_k^* \in A_k(z^*)$. Furthermore, since $\{u_n\} \subset V$ and V is a closed subspace, we also have $u^* \in V$ and hence $u^* = (z^*, w_1^*, \ldots, w_m^*) \in S_e(A_1, A_2, \ldots, A_m)$ and by Lemma 2.2 we obtain $z^* \in (A_1 + A_2 + \cdots + A_m)^{-1}(0)$. Moreover, since $u^* = P_{\mathcal{F}}(f(u^*))$ by Lemma 2.7, we obtain the variational inequality

$$\langle (f - I)u^*, u^* - x\rangle \geq 0, \quad \forall x \in S_e(A_1, A_2, \ldots, A_m), \tag{54}$$

as $S_e(A_1, A_2, \ldots, A_m) \subset \mathcal{F}$. The proof is complete. □

Remark 3.6. We observe that Algorithm 3.1 is equivalent to the following scheme:

$$\begin{cases} u_0 = (z_0, w_{1,0}, \ldots, w_{m,0}) \in V \text{ chosen arbitrarily}, \\ x_{k,n} = (I + \lambda_{k,n}A_k)^{-1}(z_n + \lambda_{k,n}w_{k,n}), & k = 1, 2, \ldots, m, \\ y_{k,n} = w_{k,n} + \dfrac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \ldots, m, \\ u_{n+1} = \alpha_n f(z_n, w_{1,n}, \ldots, w_{m,n}) + (1 - \alpha_n)(c_n, d_{1,n}, \ldots, d_{m,n}), & n \geq 0, \end{cases} \tag{55}$$

where $f : V \to V$ is a contraction mapping with constant α, $\lambda_{k,n} > 0$ for $k = 1, 2, \ldots, m$, and $n \geq 0$, and

$$c_n = \beta_n z_n + \sum_{i=1}^{m} (\beta_{i-1} - \beta_i)\hat{c}_i,$$

$$d_{k,n} = \beta_n w_{k,n} + \sum_{i=1}^{m} (\beta_{i-1} - \beta_i)\hat{d}_{k,i},$$

$$\hat{c}_i = \begin{cases} z_n, & \text{if } \varphi_i(u_n) \le 0, \\ z_n - \delta_i\left(\sum_{k=1}^{m} y_{k,i}\right), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

$$\hat{d}_{k,i} = \begin{cases} w_{k,n}, & \text{if } \varphi_i(u_n) \le 0, \\ w_{k,n} - \delta_i(x_{k,i} - \overline{x_i}), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

with

$$\delta_i = \frac{\sum_{k=1}^{m} \langle z_n - x_{k,i}, y_{k,i} - w_{k,n} \rangle}{\|\sum_{k=1}^{m} y_{k,i}\|^2 + \sum_{k=1}^{m} \|x_{k,i} - \overline{x_i}\|^2},$$

for $\overline{x_i} = \frac{1}{m}\sum_{k=1}^{m} x_{k,i}$.

Remark 3.7. At this point, we know that $S_e(A_1, A_2, \ldots, A_m) \subset \mathcal{F}$ and one can show that $\mathcal{F} \subset S_e(A_1, A_2, \ldots, A_m)$ and hence $S_e(A_1, A_2, \ldots, A_m) = \mathcal{F}$.

If in (55) we assume $\lambda_{k,n} = \lambda > 0$, for $k = 1, 2, \ldots, m$ and $n \ge 0$, then we get the following corollary for the sum of a finite family of maximally monotone mappings in Hilbert spaces.

Corollary 3.8. *Let H be a real Hilbert space and let $A_k : H \to 2^H$, for $k \in \{1, 2, \ldots, m\}$ and $m \ge 2$, be maximally monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \cdots + A_m z\} \neq \emptyset$. Let $f : V \to V$ be a contraction mapping with constant α. For arbitrary $u_0 = (z_0, w_{1,0}, \ldots, w_{m,0}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (I + \lambda A_k)^{-1}(z_n + \lambda w_{k,n}), & k = 1, 2, \ldots, m, \\ y_{k,n} = w_{n,k} + \dfrac{z_n - x_{k,n}}{\lambda}, & k = 1, 2, \ldots, m, \\ u_{n+1} = \alpha_n f(z_n, w_{1,n}, \ldots, w_{m,n}) + (1 - \alpha_n)(c_n, d_{1,n}, \ldots, d_{m,n}), & n \ge 0, \end{cases} \tag{56}$$

where $\{c_n\}$ and $\{d_{k,n}\}$ are as in (55) and $\lambda > 0$. Then, $\{u_n\}$ converges strongly to an element $u^* = (z^*, w_1^*, \ldots, w_m^*)$ of $S_e(A_1, A_2, \ldots, A_m)$, where $z^* \in \Omega$.

If in Theorem 3.5 we replace the contraction mapping f by constant $u \in V$, then we get the following corollary for the sum of a finite family of maximally monotone mappings in Hilbert spaces.

Corollary 3.9. *Let H be a real Hilbert space and let $A_k : H \to 2^H$, for $k \in \{1, 2, \ldots, m\}$ and $m \ge 2$, be maximally monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \cdots + A_m z\} \neq \emptyset$. For arbitrary $u_0 = (z_0, w_{1,0}, \ldots, w_{m,0}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n} A_k)^{-1}(z_n + \lambda_{k,n} w_{k,n}), & k = 1, 2, \ldots, m, \\ y_{k,n} = w_{k,n} + \dfrac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \ldots, m, \\ u_{n+1} = \alpha_n u + (1 - \alpha_n)(c_n, d_{1,n}, \ldots, d_{m,n}), & n \ge 0, \end{cases} \tag{57}$$

where $\{c_n\}$ and $\{d_{k,n}\}$ are as in (55), $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \overline{\lambda}]$ for some $\overline{\lambda} > \underline{\lambda} > 0$ and for each $k = 1, 2, \ldots, m$ and $n \ge 0$, and $u = (z, w_1, \ldots, w_m) \in V \subset H^{m+1}$. Then, $\{u_n\}$ converges strongly to an element $u^* = (z^*, w_1^*, \ldots, w_m^*)$ of $S_e(A_1, A_2, \ldots, A_m)$, where $z^* \in \Omega$.

We note that if in Corollary 3.9 we assume that $u = (0, 0, \ldots, 0) \in V$ and we get the following theorem for approximating the minimum-norm point of the extended solution set of the sum of a finite family of maximally monotone mappings in Hilbert spaces.

Theorem 3.10. *Let H be a real Hilbert space and let $A_k : H \to 2^H$, for $k \in \{1, 2, \dots, m\}$ and $m \geq 2$, be maximally monotone mappings satisfying $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \dots + A_m z\} \neq \emptyset$. For arbitrary $u_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n} A_k)^{-1}(z_n + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = w_{k,n} + \dfrac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \dots, m, \\ u_{n+1} = (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), & n \geq 0, \end{cases} \tag{58}$$

where $\{c_n\}$ and $\{d_{k,n}\}$ are as in (55) and $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \overline{\lambda}]$ for some $\overline{\lambda} > \underline{\lambda} > 0$ and for each $k = 1, 2, \dots, m$ and $n \geq 0$. Then, $\{u_n\}$ converges strongly to the minimum-norm point $u^* = (z^*, w_1^*, \dots, w_m^*)$ of $S_e(A_1, A_2, \dots, A_m)$, where $z^* \in \Omega$.

Proof. We note that since $u^* = (z^*, w_1^*, \dots, w_m^*) = P_{\mathcal{F}}(0)$, where $u^* \in S_e(A_1, A_2, \dots, A_m) \subset \mathcal{F}$, we obtain that u^* is the minimum-norm point of $S_e(A_1, A_2, \dots, A_m)$. □

4 Application to convex minimization problem

In this section, we apply Theorem 3.5 to study the convex minimization problem.

Let $g_k : H \to \mathbb{R}$, $k = 1, 2, \dots, m$, where $m \geq 2$, be a finite family of convex and lower semicontinuous functions. We consider the problem of finding $z^* \in H$ such that

$$g_1(z^*) + g_2(z^*) + \dots + g_m(z^*) = \min_{z \in H}\{g_1(z) + g_2(z) + \dots + g_m(z)\}. \tag{59}$$

We note that problem (59) is equivalent, by Fermat's rule, to the problem of finding $z^* \in H$ such that

$$0 \in \partial g_1(z^*) + \partial g_2(z^*) + \dots + \partial g_m(z^*), \tag{60}$$

where ∂g is a subdifferential of g, which is maximally monotone (see, e.g., [35]). So, we obtain the following theorem from Theorem 3.5.

Theorem 4.1. *Let H be a real Hilbert space. Let $g_k : H \to \mathbb{R}$, for each $k = 1, 2, 3, \dots, m$ and $m \geq 2$, be a convex and lower semicontinuous function such that $\Omega = \min_{z \in H}\{g_1(z) + g_2(z) + \dots + g_m(z)\} \neq \emptyset$. Let $f : V \to V$ be a contraction mapping with constant α. For arbitrary $u_0 = (z_0, w_{0,1}, \dots, w_{0,m}) \in V$ define an iterative algorithm by*

$$\begin{cases} x_{k,n} = (I + \lambda_{k,n} \partial g_k)^{-1}(z_n + \lambda_{k,n} w_{k,n}), & k = 1, 2, \dots, m, \\ y_{k,n} = w_{k,n} + \dfrac{z_n - x_{k,n}}{\lambda_{k,n}}, & k = 1, 2, \dots, m, \\ u_{n+1} = \alpha_n f(z_n, w_{1,n}, \dots, w_{m,n}) + (1 - \alpha_n)(c_n, d_{1,n}, \dots, d_{m,n}), & n \geq 0, \end{cases} \tag{61}$$

where for each $k = 1, 2, \dots, m$ and $n \geq 0$, $\{\lambda_{k,n}\} \subset [\underline{\lambda}, \overline{\lambda}]$ for some $\overline{\lambda} > \underline{\lambda} > 0$,

$$c_n = \beta_n z_n + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\hat{c}_i,$$

$$d_{k,n} = \beta_n w_{k,n} + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\hat{d}_{k,i},$$

$$\hat{c}_i = \begin{cases} z_n, & \text{if } \varphi_i(u_n) \leq 0, \\ z_n - \delta_i\left(\displaystyle\sum_{k=1}^{m} y_{k,i}\right), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

$$\hat{d}_{k,i} = \begin{cases} w_{k,n}, & \text{if } \varphi_i(u_n) \le 0, \\ w_{k,n} - \delta_i(x_{k,i} - \overline{x_i}), & \text{if } \varphi_i(u_n) > 0, \end{cases}$$

with

$$\delta_i = \frac{\sum_{k=1}^{m}\langle z_n - x_{k,i}, y_{k,i} - w_{k,n}\rangle}{\|\sum_{k=1}^{m} y_{k,i}\|^2 + \sum_{k=1}^{m}\|x_{k,i} - \overline{x_i}\|^2},$$

for $\overline{x_i} = \frac{1}{m}\sum_{k=1}^{m} x_{k,i}$. *Then,* $\{u_n\}$ *converges strongly to an element* $u^* = (z^*, w_1^*, \ldots, w_m^*)$ *of* $S_e(\partial g_1, \partial g_2, \ldots, \partial g_m)$, *where* $z^* \in \Omega$.

Proof. Set $A_k = \partial g_k$, for $k = 1, 2, \ldots, m$. Then, we have that A_k, $k = 1, 2, \ldots, m$, is maximally monotone with $\Omega := \{z \in H : 0 \in A_1 z + A_2 z + \ldots + A_m z\} \neq \emptyset$. Hence, the conclusion follows from Theorem 3.5. □

5 Numerical example

In this section, we present some numerical experiment results to explain the conclusion of our result. The following numerical example verifies the conclusion of Corollary 3.8.

Example 5.1. Let $H = l_2$, where l_2 is the space of sequences. Let $A_1, A_2, A_3 : l_2 \to l_2$ be defined by $A_1 x = x + (1, 2, 3, 0, 0, \ldots)$, $A_2 x = 2x + (3, 4, 5, 0, 0, \ldots)$ and $A_3 x = 3x - (2, 2, 2, 0, 0, \ldots)$, where $x = (x_1, x_2, \ldots) \in l_2$. We see that A_1, A_2 and A_3 are maximally monotone with $R(I + \lambda A_1) = R(I + \lambda A_2) = R(I + \lambda A_3) = l_2$ for each $\lambda > 0$. Now, by direct calculation we get that

$$x_{1,n} = \frac{z_n + \lambda w_{1,n} - \lambda(1, 2, 3, 0, 0, \ldots)}{1 + \lambda}, \quad y_{1,n} = \frac{z_n + \lambda w_{1,n} + (1, 2, 3, 0, 0, \ldots)}{1 + \lambda},$$
$$x_{2,n} = \frac{z_n + \lambda w_{2,n} - \lambda(3, 4, 5, 0, 0, \ldots)}{1 + 2\lambda}, \quad y_{2,n} = \frac{2z_n + 2\lambda w_{2,n} + (3, 4, 5, 0, 0, \ldots)}{1 + 2\lambda},$$
$$x_{3,n} = \frac{z_n + \lambda w_{3,n} + \lambda(2, 2, 2, 0, 0, \ldots)}{1 + 3\lambda} \text{ and } y_{3,n} = \frac{3z_n + 3\lambda w_{3,n} - (2, 2, 2, 0, 0, \ldots)}{1 + 3\lambda},$$

for some $\lambda > 0$. Thus, if we assume $\lambda = 1$, $\alpha_n = \frac{1}{100(n + 100)}$, $\beta_n = \frac{1}{n}$, for all $n \ge 1$ and $f(x) = \frac{x}{1000}$ then Algorithm (56) reduces to the following:

$$u_{n+1} = \frac{1}{10^5(n + 10^2)}(z_n, w_{1,n}, w_{2,n}, w_{3,n}) + \left(1 - \frac{1}{10^2(n + 10^2)}\right)(c_n, d_{1,n}, d_{2,n}, d_{3,n}), \tag{62}$$

where $\{c_n\}$ and $\{d_{k,n}\}$, for $k = 1, 2, 3$, are as in (55).

Now, if we take initial point $u_1 = (z_1, w_{1,1}, w_{2,1}, w_{3,1})$, where $z_1 = (2, -1, -2, 0, 0, \ldots)$, $w_{1,1} = (1, 1, 1, 0, 0, \ldots)$, $w_{2,1} = (0, 0, 0, 0, 0, \ldots)$ and $w_{3,1} = (-1, -1, -1, 0, 0, \ldots)$, then the numerical experiment results using MATLAB provide that the first component $\{z_n\}$ of $\{u_n\} = \{(z_n, w_{1,n}, w_{2,n}, w_{3,n})\}$ generated by (62) converges strongly to the solution $z^* = \left(-\frac{1}{3}, -\frac{2}{3}, -1, 0, 0, \ldots\right) \in (A_1 + A_2 + A_3)^{-1}(0)$ (Table 1).

6 Conclusion

In this article, we constructed and studied algorithms which start by reformulating (4) as the problem of locating a point in a certain *extended solution set* $S_e(A_1, A_2, \ldots, A_m) \subset H \times H^m$, which converges strongly to a zero of the sum of a finite family of maximally monotone mappings in Hilbert spaces. The assumption that one of the mappings is single-valued, α-inverse strongly monotone or α-strongly monotone is dispensed

Table 1: Convergence of the first component $\{z_n\}$ of $\{u_n\}$ generated by (62)

N	z_n	$\|z_{n+1} - z_n\|_{l_2}$
1	(2.0000, −1.0000, −2.0000, 0, 0, …)	
2	(2.1662, −1.3413, −2.5015, 0, 0, …)	0.690
3	(1.3286, −1.4059, −2.3823, 0, 0, …)	0.8485
4	(0.7003, −1.3819, −2.2615, 0, 0, …)	0.6403
5	(0.4264, −1.3101, −2.1714, 0, 0, …)	0.2971
10	(0.0554, −1.0218, −1.7395, 0, 0, …)	0.0999
100	(−0.3081, −0.5944, −0.9294, 0, 0, …)	9.4868×10^{-4}
200	(−0.2977, −0.6367, −0.9519, 0, 0, …)	3.6056×10^{-4}
300	(−0.3064, −0.6500, −0.9776, 0, 0, …)	2.4495×10^{-4}
400	(−0.3141, −0.6539, −0.9969, 0, 0, …)	1.4142×10^{-4}
500	(−0.3196, −0.6549, −0.9969, 0, 0, …)	1.4142×10^{-4}
1,000	(−0.3300 − 0.6554, −1.0006, 0, 0, …)	0
2,000	(−0.3324, −0.6597, −0.9987, 0, 0, …)	0
3,000	(−0.3330, −0.6635, −1.0001, 0, 0, …)	0
	⋮	⋮
	↓	↓
	$\left(-\frac{1}{3}, -\frac{2}{3}, -1, 0, 0, \ldots\right)$	0

with. In addition, we applied our main results to study the convex minimization problem. Finally, we provided a numerical example to support our results. Our results extend the results of [28] in the sense that our theorems provide strong convergence in arbitrary Hilbert spaces. In particular, Theorem 3.5 extends Proposition 7 of Svaiter [28] from weak to strong convergence. Moreover, our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

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