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Research Article

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Generalized parabolic Marcinkiewicz integrals associated with polynomial compound curves with rough kernels

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Abstract: In this article, we study the generalized parabolic parametric Marcinkiewicz integral operators $\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}$ related to polynomial compound curves. Under some weak conditions on the kernels, we establish appropriate estimates of these operators. By the virtue of the obtained estimates along with an extrapolation argument, we give the boundedness of the aforementioned operators from Triebel-Lizorkin spaces to L^p spaces under weaker conditions on Ω and h. Our results represent significant improvements and natural extensions of what was known previously.

Keywords: parametric Marcinkiewicz integrals, rough kernels, Triebel-Lizorkin spaces, extrapolation

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1 Introduction

Throughout this article, let $\mathbf{R}^n$ ($n \geq 2$) be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Let $\alpha_1, \alpha_2,\ldots,\alpha_n$ be fixed real numbers in the interval $[1,\infty)$. Define the function H: $\mathbf{R}^n \times \mathbf{R}^+ \to \mathbf{R}$ by $H(x,\rho) = \sum_{i=1}^n \frac{x_i^2}{\rho^{2\alpha_i}}$ with $x = (x_1, x_2,\ldots,x_n) \in \mathbf{R}^n$. Then, for each fixed $x \in \mathbf{R}^n$, the function $H(x,\rho)$ is a strictly decreasing function in $\rho > 0$. We denote the unique solution of the equation $H(x,\rho) = 1$ by $\rho = \rho(x)$. Fabes and Riviére showed in ref. [1] that $(\mathbf{R}^n,\rho)$ is a metric space, which is known by the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$.

For $\rho > 0$, let A_ρ be the diagonal $n \times n$ matrix:

$$A_\rho = \begin{bmatrix} \rho^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \rho^{\alpha_n} \end{bmatrix}.$$

The change of variables related to the space $(\mathbf{R}^n,\rho)$ is given by the transformation

$$\begin{aligned} x_1 &= \rho^{\alpha_1} \cos \vartheta_1 \cdots \cos \vartheta_{n-2} \cos \vartheta_{n-1}, \\ x_2 &= \rho^{\alpha_2} \cos \vartheta_1 \cdots \cos \vartheta_{n-2} \sin \vartheta_{n-1}, \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}} \cos \vartheta_1 \sin \vartheta_2, \\ x_n &= \rho^{\alpha_n} \sin \vartheta_1. \end{aligned}$$

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Hence, $\mathrm{d}x = \rho^{\alpha-1}J(x')\mathrm{d}\rho\mathrm{d}\sigma(x')$, where $\rho^{\alpha-1}J(x')$ is the Jacobian of the above transforms,

$$x' \in \mathbf{S}^{n-1}, \quad \alpha = \sum_{i=1}^{n}\alpha_i, \quad \text{and} \quad J(x') = \sum_{i=1}^{n}\alpha_i(x'_i)^2.$$

The authors of ref. [1] showed that $J(x')$ is a $C^\infty(\mathbf{S}^{n-1})$ function, and there exists a constant L, such that $1 \le J(x') \le L$.

For a suitable mapping Φ: $\mathbf{R}^n \to \mathbf{R}^n$, we define the generalized parabolic parametric Marcinkiewicz integral operators $\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}$, initially for C_0^∞ functions on $\mathbf{R}^n$, by

$$\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)(x) = \left(\int_0^\infty \left|\frac{1}{t^\lambda}\int_{\rho(u)\le t} f(x-\Phi(u))\frac{\Omega(x)h(\rho(x))}{\rho(x)^{\alpha-\lambda}}\mathrm{d}u\right|^r \frac{\mathrm{d}t}{t}\right)^{1/r},$$

where $r > 1$; $\lambda = \tau + \sigma i$ ($\tau, \sigma \in \mathbf{R}$ with $\tau > 0$); h: $\mathbf{R}^+ \to \mathbf{C}$ is a measurable function; and Ω is a real valued function on $\mathbf{R}^n$, integrable on $\mathbf{S}^{n-1}$ and satisfies the conditions.

$$\Omega(A_\rho x) = \Omega(x), \quad \forall \rho > 0, \tag{1.1}$$

$$\int_{\mathbf{S}^{n-1}} \Omega(x')J(x')\mathrm{d}\sigma(x') = 0. \tag{1.2}$$

We point out if $\alpha_1 = \cdots = \alpha_n = 1$, then we have $\alpha = n$, $\rho(x) = |x|$ and $(\mathbf{R}^n,\rho) = (\mathbf{R}^n,|\cdot|)$. In this case, $\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}$ is denoted by $\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}$. Also, when $\Phi(u) = u$, $h = 1$, and $r = 2$, then the operator $\mathcal{M}^{(r,c)}_{\Omega,h,\Phi,\lambda}$, denote by $\mathcal{M}_{\Omega,\lambda}$, reduces to the classical parametric Marcinkiewicz integral operator. Historically, the operator $\mathcal{M}_{\Omega,\lambda}$ was introduced by Stein [2] and proved the L^p $(1 < p \le 2)$ boundedness of $\mathcal{M}_{\Omega,1}$ provided that $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $0 < \alpha \le 1$. Subsequently, this result was investigated and improved by many researchers (see, for example, refs. [3–6]). The study of the boundedness of the operator $\mathcal{M}_{\Omega,\lambda}$ was performed by Hörmander [7]. As a matter of fact, he showed that if $\lambda > 0$ and $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $\alpha > 0$, then $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ boundedness of $\mathcal{M}_{\Omega,\lambda}$ is satisfied. Later on, the study of the operator $\mathcal{M}^{(2,c)}_{\Omega,h,\Phi,\lambda}$ under very various conditions on the kernels has been considered by many authors. For more information about the importance and the recent advances on the study of such operators, we refer the readers to refs. [8–15], as well as ref. [16], and the references therein.

Conversely, there has been a considerable amount of mathematicians with respect to the study of the boundedness of the generalized parametric Marcinkiewicz integrals $\mathcal{M}^{(r,c)}_{\Omega,h,\Phi,\lambda}$. This operator was first introduced by Chen et al. [17] and showed that whenever $\Phi(u) = u$, $h \equiv 1$, and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$, then a positive constant C exists such that

$$\|\mathcal{M}^{(r,c)}_{\Omega,h,\Phi,\lambda}f\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{1.3}$$

holds for all $1 < p$, $r < \infty$, where f belongs to the homogeneous Triebel-Lizorkin space $\dot{F}^\alpha_{p,r}(\mathbf{R}^n)$. Afterward, Le [18] improved the aforementioned result. Precisely, he established the inequality (1.3) for all $p, r \in (1,\infty)$ under the conditions that $\Phi(u) = u$, $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ and $h \in \Delta_{\max\{r',2\}}(\mathbf{R}^+)$. For the significance and recent advances on the study of such operators, readers may refer to [16,19–22,23].

Although many problems concerning the boundedness of the operator $\mathcal{M}^{(r,c)}_{\Omega,h,\Phi,\lambda}$ remain open, the investigation to verify the boundedness of the parametric Marcinkiewicz operators with mixed homogeneity has been started.

Again when $\Phi(u) = u$, $\lambda = h = 1$, and $r = 2$, then the operator $\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}$ recovers the classical parabolic Marcinkiewicz integral operator, denoted by μ_Ω, which was introduced by Ding et al. [24]. In particular, Ding et al. [24] proved that the parabolic Littlewood-Paley operator μ_Ω is of type (p,p) for all $p \in (1,\infty)$ provided that

$\Omega \in L^q(\mathbf{S}^{n-1})$ for $q > 1$. Subsequently, the study of the L^p boundedness of $\mathcal{M}^{(2)}_{\Omega,h,\Phi,\lambda}$ under various conditions on the kernel functions has been carried out by many researchers (see, for example, refs. [25–30]).

Let us recall the definition of the Triebel-Lizorkin spaces. For $1 < p, r < \infty$ and $\alpha \in \mathbf{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n)$ is defined by

$$\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) \colon \|f\|_{\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n)} = \left\| \left(\sum_{k\in\mathbf{Z}} 2^{k\alpha r} |\Lambda_k \times f|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} < \infty \right\},$$

where $\mathcal{S}'$ denotes the tempered distributions class on $\mathbf{R}^n$, $\widehat{\Lambda_k}(\xi) = \Gamma(2^{-k}\xi)$ for $k \in \mathbf{Z}$, and $\Gamma \in C_0^{\infty}(\mathbf{R}^n)$ is a radial function satisfying the following conditions:
(i) $0 \le \Gamma \le 1$;
(ii) $\mathrm{supp}\Gamma \subset \left\{\xi\colon \frac{1}{2} \le |\xi| \le 2\right\}$;
(iii) $\Gamma(\xi) \ge c > 0 \quad \text{if } \frac{3}{5} \le |\xi| \le \frac{5}{3}$;
(iv) $\sum_{j\in\mathbf{Z}} \Gamma(2^{-j}\xi) = 1 \ (\xi \ne 0)$.

The following properties of the Triebel-Lizorkin space are well known (for more details, see ref. [31]).
(a) $\mathcal{S}(\mathbf{R}^n)$ is dense in $\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n)$;
(b) $\dot{F}^{0}_{p,2}(\mathbf{R}^n) = L^p(\mathbf{R}^n) \quad \text{for } 1 < p < \infty$;
(c) $\dot{F}^{\alpha}_{p,r_1}(\mathbf{R}^n) \subset \dot{F}^{\alpha}_{p,r_2}(\mathbf{R}^n) \quad \text{if } r_1 < r_2$;
(d) $(\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n))^{*} = \dot{F}^{-\alpha}_{p',r'}(\mathbf{R}^n)$.

Let $\Delta_{\gamma}(\mathbf{R}^+)$ (for $\gamma \ge 1$) denote the collection of all measurable functions $h{:}[0,\infty) \to \mathbf{C}$, satisfying

$$\|h\|_{\Delta_\gamma(\mathbf{R}^+)} = \sup_{R>0} \left(\frac{1}{R} \int_0^R |h(\rho)|^{\gamma} \mathrm{d}\rho \right)^{1/\gamma} < \infty.$$

Also, let $\mathcal{N}^{\gamma}(\mathbf{R}^+)$ denote the set of all measurable functions $h\colon \mathbf{R}^+ \to \mathbf{C}$ that satisfy the condition

$$N_{\gamma}(h) = \sum_{k=1}^{\infty} 2^k k^{\gamma} d_k(h) < \infty,$$

where $d_k(h) = \sup_{j\in\mathbf{Z}} 2^{-j}|E(j,k)|$ with $E(j,1) = \{\rho \in (2^j,2^{j+1}]\colon|h(\rho)| \le 2\}$ and $E(j,k) = \{\rho \in (2^j,2^{j+1}]\colon 2^{k-1} < |h(\rho)| \le 2^k\}$ for $k \ge 2$.

It is clear that $\Delta_{\gamma}(\mathbf{R}^+) = L^{\infty}(\mathbf{R}^+) \subset \Delta_{\gamma_1}(\mathbf{R}^+) \subset \Delta_{\gamma_2}(\mathbf{R}^+)$ for $1 < \gamma_2 < \gamma_1 < \infty$ and $\Delta_{\gamma}(\mathbf{R}^+) \subset \mathcal{N}^{\beta}(\mathbf{R}^+)$ for any $\beta > 0$ and $1 < \gamma < \infty$.

In this study, the class $\mathfrak{F}$ denoted the set of all positive, increasing C^1 functions $\phi{:}(0,\infty) \to \mathbf{R}^+$ satisfying the following conditions:
(i) $t\phi'(t) \ge C_{\phi}\phi(t)$ for all $t > 0$; and
(ii) $\phi(2t) \le c_{\phi}\phi(t)$ for all $t > 0$,where C_{ϕ}, c_{ϕ} are independent of t. There are many model examples for the class $\mathfrak{F}$ such as t^d with $d > 0$, $t^{\iota}(\ln(1+t)^{\kappa})$ with ι, $\kappa > 0$, real-valued polynomials P on $\mathbf{R}$ with positive coefficients and $P(0) = 0$, and so on.

Let us recall some useful spaces related to our work. For $\kappa > 0$, the space $L(\log L)^{\kappa}(\mathbf{S}^{n-1})$ is denoted to the set of all measurable functions Ω that satisfies

$$\|\Omega\|_{L(\log L)^{\kappa}(\mathbf{S}^{n-1})} = \int_{(\mathbf{S}^{n-1})} |\Omega(u)| \log(2 + |\Omega(u)|) \mathrm{d}\sigma(u) < \infty.$$

The block space that was introduced in ref. [32] is denoted by $B_q^{(0,\nu)}(\mathbf{S}^{n-1})$ (for $\nu > -1$ and $q > 1$).

The main results of this paper are formulated as follows:

Theorem 1.1. *Let $\phi \in \mathfrak{F}$, and let $\Phi(u) = (P_1(\phi(\rho(u)))u_1', P_2(\phi(\rho(u)))u_2', \cdots, P_n(\phi(\rho(u)))u_n')$ with P_j being real valued polynomials on* $\mathbf{R}$ *satisfying $P_j(0) = 0$ for $j = 1, 2, \ldots, n$. Suppose that Ω satisfies the conditions* (1.1) and (1.2), *$\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma \le 2$. Then, for any $f \in \dot{F}^0_{p,r}(\mathbf{R}^n)$, there exists a constant $C > 0$, such that*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{1.4}$$

for $1 < p < r$; and

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1/r}(\gamma-1)^{-1/r}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{1.5}$$

for $r \le p < \infty$.

Theorem 1.2. *Φ and Ω be given as in Theorem 1.1. Assume that $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 2$. Then, there is a positive constant C, such that*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1/r}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{1.6}$$

for $1 < p < r$ if $r \le \gamma'$ and $2 < \gamma < \infty$; and

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1/r}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{1.7}$$

for $\gamma' < p < \infty$ if $2 < \gamma \le \infty$ and $\gamma' < r$.

By the conclusions from Theorems 1.1 and 1.2 and following the same extrapolation arguments used in refs. [9,20,29,33,34], we have the following:

Theorem 1.3. *Suppose that Ω satisfies* (1.1) and (1.2), *Φ is given as in Theorem 1.1 and $h \in \mathcal{N}^{1/r}(\mathbf{R}^+)$.*

(i) *If $\Omega \in B_q^{(0,\frac{1}{r}-1)}(\mathbf{S}^{n-1})$* for some $q > 1$, *then*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C\left(1 + \|\Omega\|_{B_q^{(0,\frac{1}{r}-1)}(\mathbf{S}^{n-1})}\right)(1 + N_{1/r}(h))\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $r \le p < \infty$; and

(ii) *If $\Omega \in L(\log L)^{1/r}(\mathbf{S}^{n-1})$, then*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C\left(1 + \|\Omega\|_{L(\log L)^{1/r}(\mathbf{S}^{n-1})}\right)(1 + N_{1/r}(h))\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $r \le p < \infty$.

Theorem 1.4. *Let Ω, Φ be given as in Theorem 1.3, and let $h \in \mathcal{N}^1(\mathbf{R}^+)$.*

(i) *If $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$* for some $q > 1$, *then*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C\left(1 + \|\Omega\|_{B_q^{(0,0)}(\mathbf{S}^{n-1})}\right)(1 + N_1(h))\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$; and

(ii) *If $\Omega \in L(\log L)(\mathbf{S}^{n-1})$, then*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C\left(1 + \|\Omega\|_{L(\log L)(\mathbf{S}^{n-1})}\right)(1 + N_1(h))\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$.

Theorem 1.5. *Let Ω satisfies* (1.1) and (1.2), $h \in \Delta_\gamma(\mathbf{R}^+)$ *for some* $\gamma > 2$ *and* Φ *be given as in Theorem 1.1.*
(i) *If* $\Omega \in B_q^{\left(0,\frac{1}{r}-1\right)}(\mathbf{S}^{n-1})$ for some $q > 1$, *then*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C\left(1 + \|\Omega\|_{B_q^{\left(0,\frac{1}{r}-1\right)}(\mathbf{S}^{n-1})}\right)\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$ *if* $r \le \gamma'$ *and* $2 < \gamma < \infty$; *and for* $\gamma' < p < \infty$ *if* $\gamma' < r$ *and* $2 < \gamma \le \infty$.
(ii) *If* $\Omega \in L(\log L)^{1/r}(\mathbf{S}^{n-1})$, *then*

$$\|\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)\|_{L^p(\mathbf{R}^n)} \le C\left(1 + \|\Omega\|_{L(\log L)^{1/r}(\mathbf{S}^{n-1})}\right)\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$ *if* $r \le \gamma'$ *and* $2 < \gamma < \infty$; *and for* $\gamma' < p < \infty$ *if* $\gamma' < r$ *and* $2 < \gamma \le \infty$.

The constant $C = C_{n,\lambda,p,\phi,\max_{1\le j\le n}\deg(P_j)}$ *in Theorems 1.1–1.5 is independent of* Ω, h, γ, q, *and the coefficients of* P_j *for* $1 \le j \le n$.

It is worth mentioning to the following remark related to our results and their optimality.

Remark 1.6. (1) Al-Qassem and Al-Salman [6] found that $\mathcal{M}_{\Omega,1}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ under the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ with $q > 1$. Moreover, they established the optimality of the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ in the sense that the exponent $-1/2$ in $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ cannot be replaced by any smaller number $-1 < \varepsilon < -1/2$ for the L^2 boundedness of $\mathcal{M}_{\Omega,1}$ to hold.

(2) Walsh [4] proved that $\mathcal{M}_{\Omega,1}$ is bounded on $L^2(\mathbf{R}^n)$ whenever $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$. Furthermore, he showed that the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is optimal in the sense that the operator $\mathcal{M}_{\Omega,1}$ may lose the L^2 boundedness if Ω is assumed to be in the space $\Omega \in L(\log L)^{\varepsilon}(\mathbf{S}^{n-1})$ for some $0 < \varepsilon < 1/2$.

(3) If $\Phi(u) = u$, then, Al-Qassem et al. [20] established the boundedness of the parametric Marcinkiewicz integral operator $\mathcal{M}^{(r,c)}_{\Omega,h,\Phi,\lambda}$ under the same our conditions on Ω, h, and r.

(4) The L^p boundedness of the parametric Marcinkiewicz operators with mixed homogeneity $\mathcal{M}^{(2)}_{\Omega,h,\Phi,\lambda}$ was satisfied [29] only when $h \in \mathcal{N}^{1/2}(\mathbf{R}^+)$, $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, and Φ is given as in Theorem 1.1.

Here and henceforth, the letter C denotes a positive constant that may be different at different occurrences and independent of the essential variables.

2 Some notations and lemmas

In this section, we give some lemmas, which we shall need in the proof of the main results. Let $N = \max_{1\le j\le n}\deg(P_j)$.

For $1 \le s \le N$ and $1 \le l \le n$, let $P_l^{(s)}(t) = \sum_{i=1}^{s} c_{i,l}t^i$ and $P^{(s)}(t) = (P_1^{(s)}(t),\dots,P_n^{(s)}(t))$. Set $P^{(0)}(t) = 0$, $P_l(t) = \sum_{i=1}^{N} c_{i,l}t^i$ with $1 \le l \le n$ and $\Phi_s(u) = (P_1^{(s)}(\phi(\rho(u)))u_1', P_2^{(s)}(\phi(\rho(u)))u_2',\dots,P_n^{(s)}(\phi(\rho(u)))u_n')$.

Let $\theta \ge 2$. For a suitable measurable function h: $\mathbf{R}^+ \to \mathbf{C}$, a suitable function ϕ: $\mathbf{R}^+ \to \mathbf{R}$, and Ω: $\mathbf{S}^{n-1} \to \mathbf{R}$, we define the family of measures $\{\sigma^s_{\Omega,\phi,h,t} := \sigma^s_{h,t}: t \in \mathbf{R}^+, 1 \le s \le N\}$ and the corresponding maximal operators $\sigma^\star_{h,s}$ and $M_{h,\theta,s}$ on $\mathbf{R}^n$ by

$$\widehat{\sigma^s_{h,t}}(\xi) = t^{-\lambda} \int_{t/2\le\rho(u)\le t} e^{-i\xi\cdot\Phi_s(u)}\frac{\Omega(u)h(\rho(u))}{\rho(u)^{\alpha-\lambda}})du;$$

$$\sigma^\star_{h,s}(f)(x) = \sup_{t\in\mathbf{R}^+}||\sigma^s_{h,t}| \times f(x)|;$$

and

$$M_{h,\theta,s}(f)(x) = \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} ||\sigma^s_{h,t}| \times |f(x)|\frac{\mathrm{d}t}{t},$$

where

$$\xi\cdot\Phi_s(u) = \sum_{l=1}^{n} \xi_l u'_l P^{(s)}(\phi(\rho(u))) = \sum_{i=1}^{s} (L_i(\xi)\cdot u'_l)\phi(\rho(u))^i,$$

L_i: $\mathbf{R}^n \to \mathbf{R}^n$ is given by $L_i(\xi) = (c_{i,1}\xi_1,\dots,c_{i,n}\xi_n)$, and $|\sigma^s_{h,t}|$ is defined in the same way as $\sigma^s_{h,t}$, but with replacing h by $|h|$ and Ω by $|\Omega|$. We write $\|\sigma^s_{h,t}\|$ for the total variation of $\sigma^s_{h,t}$.

We shall need the following lemma from ref. [29].

Lemma 2.1. *Let $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ satisfying (1.1) and (1.2), $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma \le 2$, and $\theta = 2^{q'\gamma'}$. Suppose that $\phi \in \mathfrak{F}$. Then, for $0 \le s \le N$ and any $1 < p < \infty$, the following inequalities*

$$\|M_{h,\theta,s}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{L^p(\mathbf{R}^n)}, \tag{2.1}$$

$$\|\sigma^\star_{h,s}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{L^p(\mathbf{R}^n)}, \tag{2.2}$$

hold, where the positive constant $C = C_{n,p,\phi}$ is independent of h, Ω, γ, q, and the coefficients of P_j for $1 \le j \le n$.

By using Lemma 2.2 from [29], we directly obtain the following lemma.

Lemma 2.2. *Let Ω, ϕ be given as in Lemma 2.1, and let $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 1$. Then, for any $1 \le s \le N$, $t > 0$ and $\xi \in \mathbf{R}^n$, there exists a constant $C > 0$, such that*

$$\max\{\|\sigma^s_{h,t}\|, |\widehat{\sigma^s_{h,t}}(\xi)|, |\widehat{|\sigma^s_{h,t}|}(\xi)|\} \le C\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}, \tag{2.3}$$

$$\max\{|\widehat{\sigma^s_{h,t}}(\xi) - \widehat{\sigma^{s-1}_{h,t}}(\xi)|, |\widehat{|\sigma^s_{h,t}|}(\xi)| - \widehat{|\sigma^{s-1}_{h,t}|}(\xi)|\} \le C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}(\phi(t)^s|L_s(\xi)|)^{\frac{1}{2sq'A(\gamma)}}, \tag{2.4}$$

$$\max\{|\widehat{\sigma^s_{h,t}}(\xi)|, |\widehat{|\sigma^s_{h,t}|}(\xi)|\} \le C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}(\phi(t)^s|L_s(\xi)|)^{\frac{1}{2sq'A(\gamma)}}, \tag{2.5}$$

where $A(\gamma) = \begin{cases} \gamma' & \text{if } 1 < \gamma \le 2, \\ 1 & \text{if } \gamma > 2. \end{cases}$ *The constant C is independent of Ω, h, γ,* and *q, but depends on ϕ.*

To prove Theorem 1.1, we employ the next lemmas with arguments similar to those in refs. [20] and [29].

Lemma 2.3. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma \le 2$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $\theta = 2^{q'\gamma'}$. Assume that ϕ is given as in Lemma 2.1, and r is a real number with $r > 1$. Then, for $0 \le s \le N$, there exists a constant $C > 0$, such that*

$$\begin{aligned} \left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma^s_{h,t} \times g_k|^r \frac{\mathrm{d}t}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} &\le C(q-1)^{-\frac{1}{r}}(\gamma-1)^{-\frac{1}{r}} \\ &\quad \times \|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)}, \quad r \le p < \infty \end{aligned} \tag{2.6}$$

and

$$\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma^s_{h,t}\times g_k|^r\frac{\mathrm{d}t}{t}\Bigg)^{1/r}\Bigg\|_{L^p(\mathbf{R}^n)}\le C(q-1)^{-1}(\gamma-1)^{-1}\times\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}}|g_k|^r\Bigg)^{1/r}\Bigg\|_{L^p(\mathbf{R}^n)},\quad 1<p<r \tag{2.7}$$

hold for arbitrary functions $\{g_k(\cdot),\ k\in\mathbf{Z}\}$ *on* $\mathbf{R}^n$. *The constant* $C=C_{n,p,\phi}$ *is independent of* Ω, h, γ, q, *and the coefficients of* $\{P_j\}$ *for all* $1\le j\le n$.

Proof. First, we prove (2.6). For fixed p with $r\le p<\infty$, by duality, there is a nonnegative function $\psi\in L^{(p/r)'}(\mathbf{R}^n)$ with $\|\psi\|_{L^{(p/r)'}(\mathbf{R}^n)}\le 1$, such that

$$\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma^s_{h,t}\times g_k|^r\frac{\mathrm{d}t}{t}\Bigg)^{1/r}\Bigg\|^r_{L^p(\mathbf{R}^n)}=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma^s_{h,t}\times g_k(x)|^r\frac{\mathrm{d}t}{t}\psi(x)\mathrm{d}x. \tag{2.8}$$

A simple change of variable and Hölder's inequality lead to

$$|\sigma^s_{h,t}\times g_k(x)|^r\le C\|h\|^{(r/r')}_{\Delta_1(\mathbf{R}^+)}\|\Omega\|^{(r/r')}_{L^1(\mathbf{S}^{n-1})}\times\int_{t/2}^{t}\int_{\mathbf{S}^{n-1}}|g_k(x-\Phi_s(A_\rho u)|^r|\Omega(u)|J(u)\mathrm{d}\sigma(u)|h(\rho)|\frac{\mathrm{d}\rho}{\rho}. \tag{2.9}$$

Hence, by (2.8) and (2.9) and Hölder's inequality, we have that

$$\begin{aligned}\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{h,t}\times g_k|^r\frac{\mathrm{d}t}{t}\Bigg)^{1/r}\Bigg\|^r_{L^p(\mathbf{R}^n)}&\le C\|h\|^{(r/r')}_{\Delta_1(\mathbf{R}^+)}\|\Omega\|^{(r/r')}_{L^1(\mathbf{S}^{n-1})}\int_{\mathbf{R}^n}\Bigg(\sum_{k\in\mathbf{Z}}|g_k(x)|^r\Bigg)M_{|h|,\theta,s}\tilde{\psi}(-x)\mathrm{d}x\\&\le C\|h\|^{(r/r')}_{\Delta_1(\mathbf{R}^+)}\|\Omega\|^{(r/r')}_{L^1(\mathbf{S}^{n-1})}\Bigg\|\sum_{k\in\mathbf{Z}}|g_k|^r\Bigg\|_{L^{(p,r)}(\mathbf{R}^n)}\|M_{|h|,\theta,s}(\tilde{\psi})\|_{L^{(p/r)'}(\mathbf{R}^n)},\end{aligned}$$

where $\tilde{\psi}(-x)=\psi(x)$. Therefore, by Lemma 2.1 and the assumption on ψ, we obtain

$$\Bigg\|\Bigg(\sum_{k\in Z}\int_{\theta^k}^{\theta^{k+1}}|\sigma^s_{h,t}\times g_k|^r\frac{\mathrm{d}t}{t}\Bigg)^{1/r}\Bigg\|_{L^p(\mathbf{R}^n)}\le C(q-1)^{-\frac{1}{r}}(\gamma-1)^{-\frac{1}{r}}\times\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}}|g_k|^r\Bigg)^{1/r}\Bigg\|_{L^p(\mathbf{R}^n)} \tag{2.10}$$

for $r<p<\infty$. Now if $p=r$, then by Hölder's inequality (2.9) and Lemma 2.1, we obtain

$$\begin{aligned}\Bigg\|\Bigg(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma^s_{h,t}\times g_k|^r\frac{\mathrm{d}t}{t}\Bigg)^{1/r}\Bigg\|^r_{L^p(\mathbf{R}^n)}&\le C\|h\|^{(r/r')}_{\Delta_1(\mathbf{R}^+)}\|\Omega\|^{(r/r')}_{L^1(\mathbf{S}^{n-1})}\\&\quad\times\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\theta^k}^{\theta^{k+1}}\int_{t/2}^{t}\int_{\mathbf{S}^{n-1}}|g_k(x-\Phi_s(A_\rho u))|^r|\Omega(u)|J(u)|h(\rho)|\mathrm{d}\sigma(u)\frac{\mathrm{d}\rho}{\rho}\frac{\mathrm{d}t}{t}\mathrm{d}x\\&\le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|^{(r/r')+1}_{\Delta_1(\mathbf{R}^+)}\|\Omega\|^{(r/r')+1}_{L^1(\mathbf{S}^{n-1})}\int_{\mathbf{R}^n}\Bigg(\sum_{k\in\mathbf{Z}}|g_k(x)|^r\Bigg)^{p/r}\mathrm{d}x\end{aligned} \tag{2.11}$$

which shows that (2.6) is satisfied for the case $p=r$.

Next, we prove (2.7). Let $1 < p < r$. By the duality, there exist functions $\{\varphi_k(x,t)\}$ defined on $\mathbf{R}^n \times \mathbf{R}^+$ with $\left\| \left\| \|\varphi_k\|_{L^{r'}\left([\theta^k,\theta^{k+1}], \frac{dt}{t}\right)} \right\|_{l^{r'}} \right\|_{L^{p'}(\mathbf{R}^n)} \le 1$, such that

$$\begin{aligned}\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} |\sigma^s_{h,t} \times g_k|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} (\sigma^s_{h,t} \times g_k(x)) \varphi_k(x,t) \frac{dt}{t} dx \\ &\le C(q-1)^{-1/r} (\gamma-1)^{-1/r} \|(H(\varphi))^{1/r'}\|_{L^{p'}(\mathbf{R}^n)} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)},\end{aligned} \tag{2.12}$$

where

$$H(\varphi)(x) = \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} |\sigma^s_{h,t} \times \tilde{\varphi}_k(x,t)|^{r'} \frac{dt}{t} \quad \text{and} \quad \tilde{\varphi}_k(x,t) = \varphi_k(-x,t).$$

Since $p' > r'$, there is a nonnegative function $b \in L^{(p'/r')}(\mathbf{R}^n)$, such that

$$\|H(\varphi)\|_{L^{(p'/r')}(\mathbf{R}^n)} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{k+1}} |\sigma^s_{h,t} \times \tilde{\varphi}_k(x,t)|^{r'} \frac{dt}{t} b(x) dx. \tag{2.13}$$

Following the same above argument, we obtain

$$\begin{aligned}\|H(\varphi)\|_{L^{(p'/r')}(\mathbf{R}^n)} &\le C\|h\|^{(r'/r)}_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(r'/r)}_{L^1(\mathbf{S}^{n-1})} \times \|\sigma^\star_{|h|,s}(\tilde{b})\|_{L^{(p'/r')'}(\mathbf{R}^n)} \left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\varphi_k(\cdot,t)|^{r'} \frac{dt}{t} \right) \right\|_{L^{(p'/r')}(\mathbf{R}^n)} \\ &\le C(q-1)^{-1}(\gamma-1)^{-1} \|h\|^{(r'/r)+1}_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(r'/r)+1}_{L^q(\mathbf{S}^{n-1})} \|\tilde{b}\|_{L^{(p'/r')'}(\mathbf{R}^n)},\end{aligned} \tag{2.14}$$

where $\tilde{b}(x) = b(-x)$. Therefore, the inequality (2.7) follows from (2.12) and (2.14). This completes the proof of Lemma 2.3. □

In the same manner, we establish the following:

Lemma 2.4. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $2 \le \gamma < \infty$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $\theta = 2^{q'}$. Suppose that ϕ is given as in Lemma 2.1, and r is a real number with $r \le \gamma'$. Then, for $0 \le s \le N$ and $1 < p < r$, a positive constant C exists such that the inequality*

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma^s_{h,t} \times g_k|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1/r} \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)}$$

holds for arbitrary functions $\{g_k(\cdot), k \in \mathbf{Z}\}$ on $\mathbf{R}^n$. The constant $C = C_{n,p,\phi}$ is independent of Ω, h, γ, q, and the coefficients of $\{P_j\}$ for all $1 \le j \le n$.

Proof. Let $1 < p < r$ with $r \leq \gamma'$, by the duality, there are functions $\{\varphi_k(x,t)\}$ defined on $\mathbf{R}^n \times \mathbf{R}^+$ with $\left\| \left\| \|\varphi_k\|_{L^{r'}\left([\theta^k,\theta^{k+1}],\frac{dt}{t}\right)} \right\|_{l^{r'}} \right\|_{L^{p'}(\mathbf{R}^n)} \leq 1$, such that

$$\begin{aligned}\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} |\sigma_{h,t}^s \times g_k|^r \frac{\mathrm{d}t}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} (\sigma_{h,t}^s \times g_k(x))\varphi_k(x,t) \frac{\mathrm{d}t}{t} \mathrm{d}x \\ &\leq C(q-1)^{-1/r} \|(H(\varphi))^{1/r'}\|_{L^{p'}(\mathbf{R}^n)} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)},\end{aligned} \tag{2.15}$$

where

$$H(\varphi)(x) = \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} |\sigma_{h,t}^s \times \tilde{\varphi}_k(x,t)|^{r'} \frac{\mathrm{d}t}{t} \quad \text{and} \quad \tilde{\varphi}_k(x,t) = \varphi_k(-x,t).$$

Since $\gamma \geq 2$ and $\gamma \leq r'$, we get that $r \leq \gamma' \leq 2 \leq \gamma$. So by Hölder's inequality, we obtain

$$|\sigma_{h,t}^s \times \tilde{\varphi}_k(x,t)|^{r'} \leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+)}^{r'} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)} \int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \times |\varphi_k(x - \Phi_s(A_\rho u), t)|^{r'} \mathrm{d}\sigma(u) \frac{\mathrm{d}\rho}{\rho}. \tag{2.16}$$

Notice that for any $b \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$, we have

$$|\sigma_{1,t}^s| \times |b|(x) \leq \int_{\mathbf{S}^{n-1}} |\Omega(u)| \int_{t/2}^{t} |b(x - \Phi_s(A_\rho u))| \frac{\mathrm{d}\rho}{\rho} \mathrm{d}\sigma(u) \leq C \int_{\mathbf{S}^{n-1}} |\Omega(u)| \mathcal{M}_{P(\phi)} b(x) \mathrm{d}\sigma(u),$$

where

$$\mathcal{M}_{P(\phi)} b(x) = \sup_{t>0} \frac{1}{t} \int_0^t |b(x - \Phi_s(A_\rho u))| \mathrm{d}\rho.$$

So, by using Lemma 2.2 from [35], we obtain

$$\|\sigma_{1,s}^\star(b)\|_{L^p(\mathbf{R}^n)} \leq C \int_{\mathbf{S}^{n-1}} |\Omega(u)| \|\mathcal{M}_{P(\phi)}(b)\|_{L^p(\mathbf{R}^n)} \mathrm{d}\sigma(u) \leq C_\phi \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|b\|_{L^p(\mathbf{R}^n)}. \tag{2.17}$$

Since $p' > r'$, there is a nonnegative function $b \in L^{(p'/r')}(\mathbf{R}^n)$, such that

$$\|H(\varphi)\|_{L^{(p'/r')}(\mathbf{R}^n)} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{h,t}^s \times \tilde{\varphi}_k(x,t)|^{r'} \frac{\mathrm{d}t}{t} b(x) \mathrm{d}x. \tag{2.18}$$

Hence, by simple change of variables, Hölder's inequality, and (2.16)–(2.18), we obtain

$$\begin{aligned}\|H(\varphi)\|_{L^{(p'/r')}(\mathbf{R}^n)} &\leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+)}^{r'} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)} \|\sigma_{1,s}^\star(b)\|_{L^{(p'/r')'}(\mathbf{R}^n)} \times \left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\varphi_k(\cdot,t)|^{r'} \frac{\mathrm{d}t}{t} \right) \right\|_{L^{(p'/r')}(\mathbf{R}^n)} \\ &\leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+)}^{r'} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)+1} \|b\|_{L^{(p'/r')'}(\mathbf{R}^n)}.\end{aligned} \tag{2.19}$$

Therefore, when we combine (2.19) by (2.15), we complete the proof of Lemma 2.4 □

Lemma 2.5. *Let Ω, h, ϕ, and θ be given as in Lemma 2.4, and let r be a real number with $r > \gamma'$. Then, for $0 \le s \le N$ and $\gamma' < p < \infty$, there exists a constant $C > 0$, such that*

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma^s_{h,t} \times g_k|^r \frac{\mathrm{d}t}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1/r} \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)}$$

for arbitrary functions $\{g_k(\cdot),\ k \in \mathbf{Z}\}$ on $\mathbf{R}^n$. The constant $C = C_{n,p,\phi}$ is independent of Ω, h, γ, q, and the coefficients of $\{P_j\}$ for all $1 \le j \le n$.

Proof. We follow the same aforementioned procedure as in (2.9); by a change of variable and Hölder's inequality, we obtain

$$\left| \sigma^s_{h,\theta^k t} \times g_k(x) \right|^{\gamma'} \le C\|h\|^{\gamma'}_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(\gamma'/\gamma)}_{L^1(\mathbf{S}^{n-1})} \times \int_{\theta^k t}^{\theta^k t} \int_{\mathbf{S}^{n-1}} |g_k(x - \Phi_s(A_\rho u)|^{\gamma'} |\Omega(u)| J(u) \mathrm{d}\sigma(u) \frac{\mathrm{d}\rho}{\rho}. \qquad (2.20)$$

Since $\gamma' < p < \infty$ with $\gamma' < r$, then by duality, there exists a nonnegative function $\psi \in L^{(p/r')}(\mathbf{R}^n)$ with $\|\psi\|L^{(p/r')'}(\mathbf{R}^n) \le 1$, such that

$$\left\| \sum_{k\in\mathbf{Z}} \int_1^\theta \left| \sigma^s_{h,\theta^k t} \times g_k \right|^{\gamma'} \frac{\mathrm{d}t}{t} \right\|_{L^{(p/\gamma')}(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^\theta \left| \sigma^s_{h,\theta^k t} \times g_k(x) \right|^{\gamma'} \frac{\mathrm{d}t}{t} \psi(x) \mathrm{d}x.$$

Hence, by (2.20), simple change of variable, Hölder's inequality, and (2.17), we obtain

$$\begin{aligned} \left\| \sum_{k\in\mathbf{Z}} \int_1^\theta \left| \sigma^s_{h,\theta^k t} \times g_k \right|^{\gamma'} \frac{\mathrm{d}t}{t} \right\|_{L^{(p/\gamma')}(\mathbf{R}^n)} &\le C\|h\|^{\gamma'}_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(\gamma'/\gamma)}_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |g_k(x)|^{\gamma'} \sigma^\star_{1,s} \tilde{\psi}(-x) \mathrm{d}x \\ &\le C\|h\|^{\gamma'}_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(\gamma'/\gamma)}_{L^1(\mathbf{S}^{n-1})} \left\| \sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right\|_{L^{(p/\gamma')}(\mathbf{R}^n)} \|\sigma^\star_{1,s}(\tilde{\psi})\|_{L^{(p/\gamma')'}(\mathbf{R}^n)} \\ &\le C\|h\|^{\gamma'}_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(\gamma'/\gamma)+1}_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|^{\gamma'}_{L^p(\mathbf{R}^n)}, \end{aligned}$$

where $\tilde{\psi}(x) = \psi(-x)$. Since $1 < q \le 2$, we deduce that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^\theta \left| \sigma^s_{h,\theta^k t} \times g_k \right|^{\gamma'} \frac{\mathrm{d}t}{t} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1/\gamma'} \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)} \qquad (2.21)$$

for any $\gamma' < p < \infty$. Define the linear operator T on $\{g_k(x)\}$ by $T(g_k(x)) = \sigma^s_{h,\theta^k t} \times g_k(x)$. On the one hand, by (2.21), we have

$$\begin{aligned} \left\| \left\| \|T(g_k)\|_{L^{\gamma'}\left([1,\theta], \frac{\mathrm{d}t}{t}\right)} \right\|_{l^{\gamma'}(\mathbf{Z})} \right\|_{L^p(\mathbf{R}^n)} &\le \left\| \left(\sum_{k\in\mathbf{Z}} \int_1^\theta \left| \sigma^s_{h,\theta^k t} \times g_k \right|^{\gamma'} \frac{\mathrm{d}t}{t} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)} \\ &\le C(q-1)^{-1/\gamma'} \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)} \end{aligned} \qquad (2.22)$$

for all $\gamma' < p < \infty$ with $\gamma \geq 2$. On the other hand, by Hölder's inequality and (2.17), one can check that

$$\left\| \sup_{k\in\mathbf{Z}} \sup_{t\in[1,\theta]} \left| \sigma^{s}_{h,\theta^k t} \times g_k \right| \right\|_{L^p(\mathbf{R}^n)} \leq \left\| \sigma^{\star}_{h,s}\left(\sup_{k\in\mathbf{Z}} |g_k| \right) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \sup_{k\in\mathbf{Z}} |g_k| \right\|_{L^p(\mathbf{R}^n)}$$

for all $\gamma' < p < \infty$, which gives

$$\left\| \left\| \|T(g_k)\|_{L^\infty\left([1,\theta],\frac{dt}{t}\right)} \right\|_{l^\infty(\mathbf{Z})} \right\|_{L^p(\mathbf{R}^n)} \leq C_p \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \times \left\| \|g_k\|_{l^\infty(\mathbf{Z})} \right\|_{L^p(\mathbf{R}^n)}. \tag{2.23}$$

Consequently, by interpolation (2.22) with (2.23), and using the fact

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma^{s}_{h,t} \times g_k|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \leq C \left\| \left(\sum_{k\in\mathbf{Z}} \int_{1}^{\theta} \left| \sigma^{s}_{h,\theta^k t} \times g_k \right|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)}.$$

This completes the proof of Lemma 2.5. □

3 Proof of the main results

Proof of Theorem 1.1. We prove Theorem 1.1 by applying similar techniques used in [20] and [29]. Assume that $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma \in (1,2]$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \in (1,2]$ satisfy (1.1) and (1.2). Thanks to Minkowski's inequality, we have that

$$\begin{aligned} \mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)(x) &\leq \sum_{k=0}^{\infty} \left(\int_0^\infty \left| t^{-\lambda} \int_{2^{-k-1}t<\rho(t)\leq 2^{-k}t} f(x-\Phi(u)) \frac{\Omega(x)h(\rho(x))}{\rho(x)^{\alpha-\lambda}} du \right|^r \frac{dt}{t} \right)^{1/r} \\ &= \frac{2^\tau}{2^\tau - 1} \left(\int_0^\infty |\sigma^{N}_{h,t} \star f(x)|^r \frac{dt}{t} \right)^{1/r}. \end{aligned} \tag{3.1}$$

Let $\psi \in C_0^\infty$ be supported in $\{|t| \leq 1\}$ and $\psi(t) \equiv 1$ for $|t| \leq 1/2$. For $1 \leq s \leq N$, $t > 0$ and $\xi \in \mathbf{R}^n$, define the family of measures $\{\omega_{t,s}\}$ by

$$\widehat{\omega_{t,s}}(\xi) = \widehat{\sigma^{s}_{h,t}}(\xi) \prod_{s<j\leq N} \psi\left(\phi(t)^j |R_j \pi^n_{\nu_j} Q_j(\xi)|\right) - \widehat{\sigma^{s-1}_{h,t}}(\xi) \prod_{s-1<j\leq N} \psi\left(\phi(t)^j |R_j \pi^n_{\nu_j} Q_j(\xi)|\right), \tag{3.2}$$

where $\nu_j = \mathrm{rank}(L_j)$; $R_j\colon \mathbf{R}^{\nu_j} \to \mathbf{R}^{\nu_j}$ and $Q_j\colon \mathbf{R}^n \to \mathbf{R}^n$ are two nonsingular linear transformations satisfying

$$|R_j \pi^n_{\nu_j} Q_j(\xi)| \leq |L_j(\xi)| \leq C |R_j \pi^n_{\nu_j} Q_j(\xi)| \tag{3.3}$$

and $\pi^n_{\nu_j}$ is a projection operator from $\mathbf{R}^n$ to $\mathbf{R}^{\nu_j}$. It is easy to check that

$$\sigma^{N}_{h,t} = \sum_{s=1}^{N} \omega_{t,s}, \tag{3.4}$$

which leads to

$$\mathcal{M}^{(r)}_{\Omega,h,\Phi,\lambda}(f)(x) \le C\sum_{s=1}^{N}\left(\int_0^\infty |\omega_{t,s}\times f(x)|^r\frac{\mathrm{d}t}{t}\right)^{1/r} := C\sum_{s=1}^{N}\mathfrak{M}_s(f)(x). \tag{3.5}$$

Let $\theta = 2^{q'\gamma'}$, and let $\{\Gamma_k\}_{k\in\mathbf{Z}}$ be a smooth partition of unity in $(0,\infty)$, such that

$$\operatorname{supp}\Gamma_k \subseteq [\phi(\theta^{k+1})^{-s}, \phi(\theta^{k-1})^{-s}], \quad \sum_{k\in\mathbf{Z}}\Gamma_k(t) = 1,$$

$$0 \le \Gamma_k \le 1, \quad \text{and} \quad \left|\frac{\mathrm{d}^j\Gamma_k(t)}{\mathrm{d}t^j}\right| \le \frac{C_j}{t^j} \quad \text{for } j\in N, \quad \text{and} \quad t>0.$$

Let $\widehat{\Lambda_k(f)}(\xi) = \Gamma_k(|R_s\pi^n_{\nu_s}Q_s(\xi)|)\hat{f}(\xi)$. Then, for $f\in\mathcal{S}(\mathbf{R}^n)$, one can deduce

$$\mathfrak{M}_s(f)(x) \le C\sum_{j\in\mathbf{Z}}\mathcal{G}^{(r)}_{s,j}(f)(x), \tag{3.6}$$

where

$$\mathcal{G}^{(r)}_{s,j}(f)(x) = \left(\int_0^\infty |\mathcal{F}_{s,j}(x,t)|^r\frac{\mathrm{d}t}{t}\right)^{1/r},$$

$$\mathcal{F}_{s,j}(x,t) = \sum_{k\in\mathbf{Z}}\Lambda_{k+j}\times\omega_{t,s}\times(f)(x)\chi_{(\theta^k,\theta^{k+1})}(t).$$

By the definition of $\omega_{t,s}$, Lemma 2.3, and Littlewood-Paley theorem, we obtain that

$$\begin{aligned}\|\mathcal{G}^{(r)}_{s,j}(f)\|_{L^p(\mathbf{R}^n)} &\le \left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\omega_{t,s}\times\Lambda_{j+k}f|^r\frac{\mathrm{d}t}{t}\right)^{1/r}\right\|_{L^p(\mathbf{R}^n)}\\ &\le C(q-1)^{-1/r}(\gamma-1)^{-1/r}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}|\Lambda_{j+k}f|^r\right)^{1/r}\right\|_{L^p(\mathbf{R}^n)}\\ &\le C(q-1)^{-1/r}(\gamma-1)^{-1/r}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}\end{aligned} \tag{3.7}$$

for $r\le p<\infty$; and

$$\|\mathcal{G}^{(r)}_{s,j}(f)\|_{L^p(\mathbf{R}^n)} \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{3.8}$$

for $1<p<r$. Howevere, the L^p-norm of $\mathcal{G}^{(r)}_{j,s}$ for the case $p=r=2$ can be estimated as follows: Notice that for this case, we have $\|f\|_{\dot{F}^0_{2,2}(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}$. So, by Lemma 2.2, the definition of $\omega_{t,s}$, and Plancherel's theorem, we obtain

$$\begin{aligned}\|\mathcal{G}^{(2)}_{j,s}(f)\|^2_{L^2(\mathbf{R}^n)} &= \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Lambda_{j+k}\left(|R_s\pi^n_{\nu_s}Q_s(\xi)|\right)|^2|\hat{f}(\xi)|^2\left(\int_{\theta^k}^{\theta^{k+1}}|\widehat{\omega_{t,s}}(\xi)|^2\frac{\mathrm{d}t}{t}\right)\mathrm{d}\xi\\ &\le \sum_{k\in\mathbf{Z}}\int_{\mathcal{B}_{j+k,\theta}}\left(\int_{\theta^k}^{\theta^{k+1}}|\widehat{\omega_{t,s}}(\xi)|^2\frac{\mathrm{d}t}{t}\right)|\hat{f}(\xi)|^2\mathrm{d}\xi\\ &\le C(\gamma-1)^{-1}(q-1)^{-1}\|h\|^2_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}B_j^2\|f\|^2_{L^2(\mathbf{R}^n)},\end{aligned} \tag{3.9}$$

where

$$\mathcal{B}_{k,\theta} = \{\xi \in \mathbf{R}^n : \phi(\theta^{k+1})^{-s} \le |R_s \pi^n_{v_s} Q_s(\xi)| \le \phi(\theta^{k-1})^{-s}\};$$
$$B_j = D_\phi^{(2-j)/2}\chi_{\{j\ge 2\}} + D_\phi^{(j)/2}\chi_{\{j<2\}}$$

and $D_\phi > 1$ is a constant satisfies $\phi(2t) \ge D_\phi \phi(t)$ for all $t > 0$. Therefore,

$$\|\mathcal{G}^{(2)}_{j,s}(f)\|_{L^2(\mathbf{R}^n)} \le CB_j(\gamma - 1)^{-1/2}(q-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^0_{2,2}(\mathbf{R}^n)}. \tag{3.10}$$

Consequently, interpolation among (3.7), (3.8), and (3.10) and then using (3.5) and (3.6), we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. To prove Theorem 1.2, we follow the same above arguments by invoking Lemmas 2.4–2.5 instead of Lemma 2.3 as well as $\theta = 2^{q'}$ instead of $\theta = 2^{q'\gamma'}$. □

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