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Two new forms of ordered soft separation axioms

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Abstract: The goal of this work is to introduce and study two new types of ordered soft separation axioms, namely soft T_i -ordered and strong soft T_i -ordered spaces ($i = 0, 1, 2, 3, 4$). These two types are formulated with respect to the ordinary points and the distinction between them is attributed to the nature of the monotone neighborhoods. We provide several examples to elucidate the relationships among these concepts and to show the relationships associate them with their parametric topological ordered spaces and p-soft T_i -ordered spaces. Some open problems on the relationships between strong soft T_i -ordered and soft T_i -ordered spaces ($i = 2, 3, 4$) are posed. Also, we prove some significant results which associate both types of the introduced ordered axioms with some notions such as finite product soft spaces, soft topological and soft hereditary properties. Furthermore, we describe the shape of increasing (decreasing) soft closed and open subsets of soft regularly ordered spaces; and demonstrate that a condition of strong soft regularly ordered is sufficient for the equivalence between p-soft T_1 -ordered and strong soft T_1 -ordered spaces. Finally, we establish a number of findings that associate soft compactness with some ordered soft separation axioms initiated in this work.

Keywords: monotone soft open set, monotone soft neighborhood, soft T_i -ordered and strong soft T_i -ordered spaces ($i = 0, 1, 2, 3, 4$)

MSC: 54D10, 54D15, 54D30, 54F05

1 Introduction

The study of the concept of topological ordered spaces was presented for the first time by Nachbin [1]. He has constructed this concept by adding a partial order relation to the structure of a topological space. With regard to Nachbin's definition of topological ordered spaces, two points can be considered, the first one is that the topology and the partial order relation operate independently of one another, and the second one is that the topological ordered spaces are one of the generalizations of topological spaces. After Nachbin's work, many researchers carried out various studies on ordered spaces (see, for example, [2–5]).

Zadeh [6] introduced the notion of fuzzy sets in 1965 as mathematical instruments for dealing with uncertainties. To put a topological structure to fuzzy set theory, Chang [7] has defined fuzzy topological spaces. Then Katsaras [8] combined a partial order relation and a fuzzy topology to define a fuzzy topological ordered space.

In 1999, the notion of soft sets was proposed by Molodtsov [9] to overcome problems associated with uncertainties, vagueness, impreciseness and incomplete data. This notion includes enough parameters which make it a suitable alternative for the previous mathematical approaches such as fuzzy and rough sets. The useful applications of soft sets to several directions contribute to progress work on it rapidly (see, for example, [10, 11]). The concept of soft topological spaces was introduced by Shabir and Naz in their pioneer work

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[12]. Then many studies on soft topological spaces have been done (see, for example, [13–18]). El-Shafei et al. [19] introduced partial belong and total non-belong relations which are more functional and flexible for theoretical and application studies via the soft set theory and soft topologies. Then they employed these two new notions to present new soft separation axioms, namely p -soft T_i -spaces ($i = 0, 1, 2, 3, 4$). The authors of [20–25] have done some amendments for some alleged results on soft axioms. Al-shami and Kočinac [26] explored the equivalence between the extended and enriched soft topologies and has obtained some interesting results related to the parametric topologies. The authors of [27, 28] introduced different types of soft axioms on supra soft topological spaces.

In [29], the authors formulated the concepts of monotone soft sets and soft topological ordered spaces as a new soft structure. They also have utilized the natural belong and total non-belong relations to introduce the notions of p -soft T_i -ordered spaces ($i = 0, 1, 2, 3, 4$). In [30] we studied and investigated these notions on supra soft topological ordered spaces.

The topic of soft separation axioms is one of the most significant and interesting in soft topology. In general, soft separation axioms are utilized to obtain more restricted families of soft topological spaces. It turns out, from the previous studies, that there are many points of view to study soft separation axioms. The diversity of these perspectives is attributed to the relations of belong and non-belong that are used in the definitions; and the objects of study, ordinary points or soft points (see, for example, [12, 19, 31–34]). The variety of ordered soft separation axioms will be more extended, because the soft neighborhoods and soft open sets is distinguished according to the partially ordered soft set.

As a contribution of study ordered soft separation axioms, the authors devote this work to defining and investigating two types of ordered soft separation axioms, namely soft T_i -ordered and strong soft T_i -ordered spaces ($i = 0, 1, 2, 3, 4$). With the help of examples, we illustrate the relationships among them. Also, we derive their fundamental features such as the finite product of soft T_i -ordered (resp. strong soft T_i -ordered) spaces is soft T_i -ordered (resp. strong soft T_i -ordered) for $i = 0, 1, 2$; and the property of being a soft T_i -ordered (strong soft T_i -ordered) space is a soft topological ordered property for $i = 0, 1, 2, 3, 4$. Moreover, we investigate certain properties of them that associated with some notions of soft ordered topology such as soft ordered topological invariant and soft compatibly ordered subspaces. In the end of both Section (3) and Section (4), we discuss some results about the relationships between soft compact spaces and some of the initiated ordered soft separation axioms.

2 Preliminaries

This section is allocated to recall some definitions and well known results which we shall utilize them in the next parts of this work.

2.1 Soft set

Definition 2.1. [9] A pair (G, E) is said to be a soft set over X provided that G is a mapping of a parameters set E into 2^X .

For short, we use the notation G_E instead of (G, E) and we express a soft set G_E as follows: $G_E = \{(e, G(e)) : e \in E \text{ and } G(e) \in 2^X\}$. Also, we use the notation $S(X_E)$ to denote the collection of all soft sets defined over X under a set of parameters E .

Definition 2.2. [12, 19] For a soft set G_E over X and $x \in X$, we say that:

- (i) $x \in G_E$ if $x \in G(e)$ for each $e \in E$; and we say that $x \notin G_E$ if $x \notin G(e)$ for some $e \in E$;
- (ii) $x \in G_E$ if $x \in G(e)$ for some $e \in E$; and we say that $x \notin G_E$ if $x \notin G(e)$ for each $e \in E$.

Definition 2.3. [11] G_E over X is called a null soft set (resp. an absolute soft set) if $G(e) = \emptyset$ (resp. $G(e) = X$) for each $e \in E$; and it is denoted by $\tilde{\Phi}$ (resp. \tilde{X}).

Definition 2.4. [35] G_E over X is called a soft point if there are $e \in E$ and $x \in X$ such that $G(e) = \{x\}$ and $G(b) = \emptyset$ for each $b \in E \setminus \{e\}$.

A soft point will be shortly denoted by P_e^x and we say that $P_e^x \in G_E$ provided that $x \in G(e)$.

Definition 2.5. [19] G_E over X is said to be stable if there is $S \subseteq X$ such that $G(e) = S$ for each $e \in E$.

Definition 2.6. [10] The relative complement of G_E , denoted by G_E^c , is a mapping $G^c : E \rightarrow 2^X$ defined by $G^c(e) = X \setminus G(e)$ for each $e \in E$.

Definition 2.7. [36] G_A is a soft subset of G_B if $A \subseteq B$ and $G(a) \subseteq F(a)$ for all $a \in A$.

Definition 2.8. [11] The union of soft sets G_A and F_B over X , denoted by $G_A \tilde{\cup} F_B$, is the soft set V_D , where $D = A \cup B$ and a mapping $V : D \rightarrow 2^X$ is defined as follows:

$$V(d) = \begin{cases} G(d) & : d \in A - B \\ F(d) & : d \in B - A \\ G(d) \cup F(d) & : d \in A \cap B \end{cases}$$

Definition 2.9. [10] The intersection of soft sets G_A and F_B over X , denoted by $G_A \tilde{\cap} F_B$, is the soft set V_D , where $D = A \cap B \neq \emptyset$, and a mapping $V : D \rightarrow 2^X$ is defined by $V(d) = G(d) \cap F(d)$ for all $d \in D$.

Definition 2.10. [37] Let G_A and H_B be two soft sets over X and Y , respectively. Then the cartesian product of G_A and H_B is denoted by $(G \times H)_{A \times B}$ and is defined as $(G \times H)(a, b) = G(a) \times H(b)$ for each $a \in A$ and $b \in B$.

Definition 2.11. [35] A soft mapping f_ϕ of $S(X_A)$ into $S(Y_B)$ is a pair of mappings $f : X \rightarrow Y$ and $\phi : A \rightarrow B$ such that for soft subsets G_K and H_L of $S(X_A)$ and $S(Y_B)$, respectively, we have:

(i) $f_\phi(G_K) = (f_\phi(G))_B$ is a soft subset of $S(Y_B)$ such that

$$f_\phi(G)(b) = \begin{cases} \bigcup_{a \in \phi^{-1}(b) \cap K} f(G(a)) & : \phi^{-1}(b) \cap K \neq \emptyset \\ \emptyset & : \phi^{-1}(b) \cap K = \emptyset \end{cases}$$

for each $b \in B$;

(ii) $f_\phi^{-1}(H_L) = (f_\phi^{-1}(H))_A$ is a soft subset of $S(X_A)$ such that

$$f_\phi^{-1}(H)(a) = \begin{cases} f^{-1}(H(\phi(a))) & : \phi(a) \in L \\ \emptyset & : \phi(a) \notin L \end{cases}$$

for each $a \in A$.

Definition 2.12. [35] A soft mapping $f_\phi : S(X_A) \rightarrow S(Y_B)$ is said to be injective (resp. surjective, bijective) if the two mappings f and ϕ are injective (resp. surjective, bijective).

Proposition 2.13. [35] For a soft mapping $f_\phi : S(X_A) \rightarrow S(Y_B)$, we have the following results:

- (i) $G_A \tilde{\subseteq} f_\phi^{-1}f_\phi(G_A)$ for each $G_A \in S(X_A)$; and $f_\phi f_\phi^{-1}(H_B) \tilde{\subseteq} H_B$ for each $H_B \in S(Y_B)$;
- (ii) If f_ϕ is injective (resp. surjective), then $G_A = f_\phi^{-1}f_\phi(G_A)$ (resp. $f_\phi f_\phi^{-1}(H_B) = H_B$).

Proposition 2.14. [29] Let $f_\phi : S(X_A) \rightarrow S(Y_B)$ be a soft mapping. Then

- (i) The image of any soft point is a soft point;

(ii) If f is bijective, then the inverse image of any soft point is a soft point.

Definition 2.15. [38] A binary relation \preceq on $X \neq \emptyset$ is said to be a partial order relation if it is reflexive, anti-symmetric and transitive. An element $x \in X$ is said to be the smallest (resp. largest) element of X if $x \preceq y$ (resp. $y \preceq x$) for all $y \in X$.

Henceforth, a diagonal relation $\{(x, x) : x \in X\}$ on X is denoted by Δ .

Definition 2.16. [29] (X, E, \preceq) is said to be a partially ordered soft set on $X \neq \emptyset$ if (X, \preceq) is a partially ordered set. For two soft points P_α^x and P_α^y in (X, E, \preceq) , we say that $P_\alpha^x \preceq P_\alpha^y$ if $x \preceq y$.

Definition 2.17. [29] An increasing operator i and a decreasing operator d are two soft maps of $(S(X_E), \preceq)$ into $(S(X_E), \preceq)$ defined as follows: for each soft subset G_E of $S(X_E)$

- (i) $i(G_E) = iG_E$, where iG is a mapping of E into $P(X)$ given by $iG(e) = i(G(e)) = \{x \in X : b \preceq x \text{ for some } b \in G(e)\}$;
- (ii) $d(G_E) = dG_E$, where dG is a mapping of E into $P(X)$ given by $dG(e) = d(G(e)) = \{x \in X : x \preceq b \text{ for some } b \in G(e)\}$.

Definition 2.18. [29] A soft subset G_E of (X, E, \preceq) is said to be increasing (resp. decreasing) provided that $G_E = i(G_E)$ (resp. $G_E = d(G_E)$).

Theorem 2.19. [29] The finite product of increasing (resp. decreasing) soft sets is increasing (resp. decreasing).

Definition 2.20. [29] A soft map $f_\phi : (S(X_A), \preceq_1) \rightarrow (S(Y_B), \preceq_2)$ is said to be:

- (i) increasing (resp. decreasing) provided that $P_\alpha^x \preceq_1 P_\alpha^y$ implies $f_\phi(P_\alpha^x) \preceq_2 f_\phi(P_\alpha^y)$ (resp. $f_\phi(P_\alpha^y) \preceq_2 f_\phi(P_\alpha^x)$);
- (iii) an ordered embedding provided that $P_\alpha^x \preceq_1 P_\alpha^y$ if and only if $f_\phi(P_\alpha^x) \preceq_2 f_\phi(P_\alpha^y)$.

Theorem 2.21. [29] Let $f_\phi : (S(X_A), \preceq_1) \rightarrow (S(Y_B), \preceq_2)$ be a bijective ordered embedding soft mapping. Then the image of each increasing (resp. decreasing) soft set is increasing (resp. decreasing).

2.2 Soft topology

Definition 2.22. [12] A family τ of soft sets over X under a fixed parameters set E which contains \tilde{X} and $\tilde{\Phi}$ and is closed under finite soft intersection and arbitrary soft union is said to be a soft topology on X .

The triple (X, τ, E) is said to be a soft topological space (briefly, STS). Every member of τ is called soft open and its relative complement is called soft closed.

Proposition 2.23. [12] If (X, τ, E) is an STS, then a family $\tau_e = \{G(e) : G_E \in \tau\}$ forms a topology on X for each $e \in E$.

The notation τ_e , which is given in the proposition above, is said to be a parametric topology and (X, τ_e) is said to be a parametric topological space.

Definition 2.24. [35] A soft subset W_E of (X, τ, E) is called a soft neighborhood of $x \in X$ if there exists a soft open set G_E such that $x \in G_E \tilde{\subseteq} W_E$.

Definition 2.25. [35, 39] A soft mapping $f_\phi : (X, \tau, A) \rightarrow (Y, \theta, B)$ is said to be:

- (i) soft continuous if the inverse image of each soft open set is soft open;

- (ii) soft open (resp. soft closed) if the image of each soft open (resp. soft closed) set is soft open (resp. soft closed);
- (iii) a soft homeomorphism if it is bijective, soft continuous and soft open.

Definition 2.26. [13] A collection $\{G_{i_E} : i \in I\}$ of soft open sets is called a soft open cover of (X, τ, E) if $\tilde{X} = \bigcup_{i \in I} G_{i_E}$. And (X, τ, E) is called soft compact if every soft open cover of \tilde{X} has a finite subcover.

Proposition 2.27. [13] Every soft closed subset H_E of a soft compact space is soft compact.

Theorem 2.28. [13] Let (X, τ, A) and (Y, θ, B) be two STSs. Let $\Omega = \{G_A \times F_B : G_A \in \tau \text{ and } F_B \in \theta\}$. Then the family of all arbitrary unions of elements of Ω is a soft topology on $X \times Y$.

Definition 2.29. [1] A triple (X, τ, \preceq) is said to be a topological ordered space, where \preceq and τ are respectively a partial order relation and a topology on $X \neq \emptyset$.

Definition 2.30. [29] A quadrable system (X, τ, E, \preceq) is said to be a soft topological ordered space, where (X, τ, E) is a soft topological space and (X, \preceq) is a partially ordered set.

We will write from now on STOS instead of a soft topological ordered space.

Definition 2.31. [29] A soft subset W_E of (X, τ, E, \preceq) is said to be an increasing (resp. a decreasing) soft neighborhood of $x \in X$ if W_E is increasing (resp. decreasing) and a soft neighborhood of $x \in X$.

Proposition 2.32. [29] In (X, τ, E, \preceq) we find that for each $e \in E$, the family $\tau_e = \{G(e) : G_E \in \tau\}$ with a partial order relation \preceq form an ordered topology on X .

Definition 2.33. [29] Let $Y \subseteq X$. Then $(Y, \tau_Y, \preceq_Y, E)$ is called a soft ordered subspace of (X, τ, \preceq, E) provided that (Y, τ_Y, E) is soft subspace of (X, τ, E) and $\preceq_Y = \preceq \cap Y \times Y$.

Lemma 2.34. [29] If U_E is an increasing (resp. a decreasing) soft subset of (X, τ, \preceq, E) , then $U_E \tilde{\cap} \tilde{Y}$ is an increasing (resp. a decreasing) soft subset of a soft ordered subspace $(Y, \tau_Y, \preceq_Y, E)$.

Definition 2.35. [29] The product of a finite family of soft topological ordered spaces $\{(X_i, \tau_i, \preceq_i, E_i) : i \in \{1, 2, \dots, n\}\}$ is an STOS (X, τ, \preceq, E) , where $X = \prod_{i=1}^n X_i$, τ is the product soft topology on X , $E = \prod_{i=1}^n E_i$ and $\preceq = \{(x, y) : x, y \in X\}$ such that $(x_i, y_i) \in \preceq_i$ for every i .

Lemma 2.36. [29] If $H_{E_1 \times E_2}$ is a decreasing (resp. an increasing) soft closed subset of a soft ordered product space $(X \times Y, \tau_1 \times \tau_2, E_1 \times E_2, \preceq)$, then $H_{E_1 \times E_2} = [G_{E_1}^c \times \tilde{Y}] \tilde{\cup} [\tilde{X} \times F_{E_2}^c]$, for some increasing (resp. decreasing) soft open sets $G_{E_1} \in \tau_1$ and $F_{E_2} \in \tau_2$.

Definition 2.37. [29] A soft ordered subspace $(Y, \tau_Y, \preceq_Y, E)$ of (X, τ, \preceq, E) is called a soft compatibly ordered provided that for each increasing (resp. decreasing) soft closed subset H_E of $(Y, \tau_Y, \preceq_Y, E)$, there exists an increasing (resp. a decreasing) soft closed subset H_E^* of (X, τ, \preceq, E) such that $H_E = \tilde{Y} \tilde{\cap} H_E^*$.

Definition 2.38. [29] A soft topological ordered property or soft ordered topological invariant is a property of a soft topological ordered space which is invariant under ordered embedding soft homeomorphism mappings.

Theorem 2.39. [29] (X, τ, E, \preceq) is upper (resp. lower) p -soft T_1 -ordered if and only if $(i(x))_E$ (resp. $(d(x))_E$) is soft closed, for all $x \in X$.

3 Ordered soft separation axioms

In this section, we formulate the concepts of soft T_i -ordered spaces ($i = 0, 1, 2, 3, 4$) by using monotone soft neighborhoods and establish some of their properties. With the help of illustrative examples, we elucidate the relationship between them; and the interrelations between them and their parametric topological ordered spaces.

Definition 3.1. A soft subset W_E of (X, τ, E, \preceq) is said to be:

- (i) partially containing x provided that $x \in W_E$;
- (ii) partially soft neighborhood of x , provided that there exists a soft open set G_E such that $x \in G_E \tilde{\subseteq} W_E$;
- (iii) an increasing (resp. a decreasing) partially soft neighborhood of $x \in X$ provided that W_E is an increasing (resp. a decreasing) and partially soft neighborhood of x .

The following example illustrates the above definition.

Example 3.2. Let $E = \{e_1, e_2\}$ and $\preceq = \triangle \cup \{(x, z)\}$ be a partial order relation on $X = \{x, y, z\}$. Then $\tau = \{\tilde{\Phi}, \tilde{X}, G_E\}$ is a soft topology on X , where $G_E = \{(e_1, \emptyset), (e_2, \{x\})\}$. Now, it can be noted that:

- (i) A soft set $W_E = \{(e_1, \{y\}), (e_2, \{x\})\}$ partially contains x because $x \in W_E$. But W_E does not partially contain z because $z \notin W_E$;
- (ii) W_E is a partially soft neighborhood of x because G_E is a soft open set such that $x \in G_E \tilde{\subseteq} W_E$;
- (iii) W_E is a decreasing partially soft neighborhood of x because W_E is decreasing and partially soft neighborhood of x . On the other hand, $i(W_E) = \{(e_1, \{y\}), (e_2, \{x, z\})\} \neq W_E$. Then W_E is not increasing. Hence, W_E is not increasing partially soft neighborhood of x .

Definition 3.3. (X, τ, E, \preceq) is said to be:

- (i) upper (resp. lower) soft T_1 -ordered if for every $x \not\preceq y$ in X , there exists a decreasing (resp. an increasing) soft neighborhood W_E of y (resp. x) such that $x \notin W_E$ (resp. $y \notin W_E$);
- (ii) soft T_0 -ordered if it is upper soft T_1 -ordered or lower soft T_1 -ordered;
- (iii) soft T_1 -ordered if it is upper soft T_1 -ordered and lower soft T_1 -ordered;
- (iv) soft T_2 -ordered if for every $x \not\preceq y$ in X , there exist disjoint an increasing soft neighborhood W_E of x and a decreasing soft neighborhood V_E of y .

Remark 3.4. The definition of a p -soft T_2 -ordered space in [29] reports that for every $x \not\preceq y$ in X , there exist two disjoint soft neighborhoods W_E and V_E containing x and y , respectively. This means that $y \notin W_E$ and $x \notin V_E$. Since W_E and V_E are disjoint then $y \notin W_E$ if and only if $y \notin W_E$ and $x \notin V_E$ if and only if $x \notin V_E$. So the definitions of soft T_2 -ordered and p -soft T_2 -ordered spaces are equivalent. Hence, all results concerning p -soft T_2 -ordered spaces in [29] are still valid for soft T_2 -ordered spaces.

Proposition 3.5. Every soft T_i -ordered space (X, τ, E, \preceq) is soft T_{i-1} -ordered for $i = 1, 2$.

Proof. The proof follows immediately from Definition (3.3). \square

To show that the converse of the above proposition is not always true, we give the following two examples.

Example 3.6. Let $E = \{e_1, e_2\}$ be a set of parameters and $\preceq = \triangle \cup \{(x, y), (x, z)\}$ be a partial order relation on $X = \{x, y, z\}$. We define the soft sets $\{G_{i_E} : i = 1, 2, 3, 4\}$ as follows:

$$\begin{aligned} G_{1_E} &= \{(e_1, \{y\}), (e_2, \{x, y\})\}; \\ G_{2_E} &= \{(e_1, \{z\}), (e_2, \{x, z\})\}; \end{aligned}$$

$$G_{3_E} = \{(e_1, \{y, z\}), (e_2, X)\} \text{ and} \\ G_{4_E} = \{(e_1, \emptyset), (e_2, \{x\})\}.$$

Then $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, 3, 4\}$ forms a soft topology on X . Now, for $y \not\preceq x$ and $y \not\preceq z$, we find that $W_E = \{(e_1, \{y\}), (e_2, X)\}$ is an increasing soft neighborhood of y such that $x \notin W_E$ and $z \notin W_E$. Also, for $z \not\preceq x$ and $z \not\preceq y$, we find that $W_E = \{(e_1, \{z\}), (e_2, X)\}$ is an increasing soft neighborhood of z such that $x \notin W_E$ and $y \notin W_E$. Therefore (X, τ, E, \preceq) is a lower soft T_1 -ordered space. Hence, it is soft T_0 -ordered. On the other hand, there does not exist a soft neighborhood W_E of x such that $y \notin W_E$ or $z \notin W_E$. This means that it is not an upper soft T_1 -ordered space. Hence, (X, τ, E, \preceq) is not soft T_1 -ordered.

Example 3.7. Let $E = \{e_1, e_2\}$ be a set of parameters and $\preceq = \Delta \cup \{(1, 2), (2, 3), (1, 3)\}$ be a partial order relation on $X = \{1, 2, 3\}$. The soft sets $\{G_{i_E} : i = 1, 2, \dots, 9\}$ are defined as follows:

$$G_{1_E} = \{(e_1, \{1\}), (e_2, \{2\})\}; \\ G_{2_E} = \{(e_1, \{1\}), (e_2, \{1, 2\})\}; \\ G_{3_E} = \{(e_1, \{1, 2\}), (e_2, \{2\})\}; \\ G_{4_E} = \{(e_1, \{1, 2\}), (e_2, \{1, 2\})\}; \\ G_{5_E} = \{(e_1, \{3\}), (e_2, X)\}; \\ G_{6_E} = \{(e_1, \emptyset), (e_2, \{1, 2\})\}; \\ G_{7_E} = \{(e_1, \emptyset), (e_2, \{2\})\}; \\ G_{8_E} = \{(e_1, \emptyset), (e_2, \{1\})\} \text{ and} \\ G_{9_E} = \{(e_1, \{1, 3\}), (e_2, X)\}.$$

Then $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 9\}$ forms a soft topology on X . Now, for $3 \not\preceq 2$ and $3 \not\preceq 1$, we find that G_{5_E} is an increasing soft neighborhood of 3 such that $2 \notin G_{5_E}$ and $1 \notin G_{5_E}$; and G_{4_E} is a decreasing soft neighborhood of 2 and 1 such that $3 \notin G_{4_E}$. Also, for $2 \not\preceq 1$, we find that $W_E = \{(e_1, X), (e_2, \{2, 3\})\}$ is an increasing soft neighborhood of 2 and G_{2_E} is a decreasing soft neighborhood of 1 such that $1 \notin W_E$ and $2 \notin G_{2_E}$. Therefore, (X, τ, E, \preceq) is soft T_1 -ordered. In contrast, any soft open set containing 1 intersects any soft open set containing 2 . Hence (X, τ, E, \preceq) is not soft T_2 -ordered.

Proposition 3.8. If a is the smallest element of a finite upper soft T_1 -ordered space (X, τ, E, \preceq) , then there is a decreasing soft neighborhood W_E of a such that $y \notin W_E$ for each $y \in X \setminus \{a\}$.

Proof. Let a be the smallest element in (X, \preceq) . Then $a \preceq x$ for all $x \in X$. Since \preceq is anti-symmetric, then $x \not\preceq a$ for all $x \in X$. Therefore, there exists a decreasing soft neighborhood W_E of a such that $x \notin W_E$. Since X is finite, then $\bigcap W_E$ is a decreasing soft neighborhood of a such that $y \notin W_E$ for each $y \in X \setminus \{a\}$. \square

Proposition 3.9. If a is the largest element of a finite lower soft T_1 -ordered space (X, τ, E, \preceq) , then there is an increasing soft neighborhood W_E of a such that $y \notin W_E$ for each $y \in X \setminus \{a\}$.

Proof. The proof is similar to that of Proposition (3.8). \square

Proposition 3.10. If (X, τ, E, \preceq) is a finite soft T_1 -ordered space, then for each $x \in X$ there is a soft open set G_E containing x such that $y \notin G_E$ for each $y \in X \setminus \{x\}$.

Proof. Let $x \in X$. Since \preceq is anti-symmetric, then $x \not\preceq y$ or $y \not\preceq x$ for all $y \in X$. By hypothesis, there exists an increasing soft neighborhood W_E of x or a decreasing soft neighborhood W_E of x such that $y \notin W_E$. Since X is finite, then $\bigcap W_E$ is a soft neighborhood of x such that $y \notin W_E$ for each $y \in X \setminus \{a\}$. Hence, there is a soft open set G_E such that $x \in G_E \subseteq \bigcap W_E$. \square

Theorem 3.11. (X, τ, E, \preceq) is soft T_2 -ordered if and only if for all $(x, y) \notin \preceq$ there are soft open sets U_E and V_E containing x and y , respectively, such that $(a, b) \notin \preceq$ for every $a \in U(e)$ and $b \in V(e)$.

Proof. From Remark (3.4) and Theorem 4.13 in [29], the proof follows. \square

Proposition 3.12. *If (X, τ, E, \preceq) is soft T_2 -ordered, then all parametric topological ordered spaces (X, τ_e, \preceq) are T_2 -ordered.*

Proof. The proof follows from Remark (3.4) and Proposition 4.15 in [29]. \square

Corollary 3.13. *The minimum number of soft open subsets of a finite soft T_2 -ordered space (X, τ, E, \preceq) is $2^{|X|}$ soft open sets.*

The next example clarifies that the converse of Proposition (3.12) fails.

Example 3.14. *Consider a partial order relation $\preceq = \triangle \cup \{(x, y)\}$ on $X = \{x, y\}$ and let $E = \{e_1, e_2\}$ be a parameters set. The collection $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2\}$ is a soft topology on X , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, \{y\})\} \text{ and} \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \{x\})\}. \end{aligned}$$

Obviously, (X, τ_{e_1}, \preceq) and (X, τ_{e_2}, \preceq) are T_2 -ordered spaces; notwithstanding, (X, τ, E, \preceq) is not a soft T_0 -ordered space.

Also, the example below combined with the above example illustrate that the soft T_i -ordered spaces ($i = 0, 1$) and their parametric topological ordered spaces are independent of each other.

Example 3.15. *Consider a partial order relation $\preceq = \triangle \cup \{(x, y)\}$ on $X = \{x, y, z\}$ and let $E = \{e_1, e_2, e_3\}$ be a parameters set. The collection $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 7\}$ is a soft topology on X , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, X), (e_3, X)\}; \\ G_{2_E} &= \{(e_1, X), (e_2, \{y\}), (e_3, X)\}; \\ G_{3_E} &= \{(e_1, X), (e_2, X), (e_3, \{z\})\}; \\ G_{4_E} &= \{(e_1, \{x\}), (e_2, \{y\}), (e_3, X)\}; \\ G_{5_E} &= \{(e_1, \{x\}), (e_2, X), (e_3, \{z\})\}; \\ G_{6_E} &= \{(e_1, X), (e_2, \{y\}), (e_3, \{z\})\} \text{ and} \\ G_{7_E} &= \{(e_1, \{x\}), (e_2, \{y\}), (e_3, \{z\})\}. \end{aligned}$$

Obviously, (X, τ_{e_1}, \preceq) , (X, τ_{e_2}, \preceq) and (X, τ_{e_3}, \preceq) are not T_0 -ordered spaces; notwithstanding, (X, τ, E, \preceq) is soft T_1 -ordered.

Proposition 3.16. *The property of being a soft T_i -ordered space is a soft hereditary property for $i = 0, 1, 2$.*

Proof. From Remark (3.4) and Theorem 4.19 in [29], we obtain the proof in the case of $i = 2$.

To prove the proposition in the case of $i = 1$, let $(Y, \tau_Y, E, \preceq_Y)$ be a soft ordered subspace of a soft T_1 -ordered space (X, τ, E, \preceq) . For every $a \preceq_Y b \in Y$, we have $a \preceq b$. Therefore, there is an increasing soft neighborhood W_E of a and a decreasing soft neighborhood V_E of b . By setting $U_E = \tilde{Y} \cap W_E$ and $G_E = \tilde{Y} \cap V_E$, we find from Lemma (2.34) that U_E is an increasing soft neighborhood of a and G_E is a decreasing soft neighborhood of b such that $b \notin U_E$ and $a \notin G_E$. Thus, $(Y, \tau_Y, \preceq_Y, E)$ is soft T_1 -ordered.

The proof in the case of $i = 0$ can be done similarly. \square

Proposition 3.17. *Every p -soft T_i -ordered space is soft T_i -ordered for $i = 0, 1$.*

Proof. The proof is complete by observing that $x \notin G_E$ implies that $x \notin G_E$ for every $G_E \subseteq \tilde{X}$. \square

Corollary 3.18. *(X, τ, E, \preceq) is upper (resp. lower) soft T_1 -ordered if $(i(x))_E$ (resp. $(d(x))_E$) is a soft closed set for each $x \in X$.*

Proof. If $(i(x))_E$ is a soft closed set for each $x \in X$, then it follows from Theorem (2.39) that (X, τ, E, \preceq) is upper p -soft T_1 -ordered. Hence, it is upper soft T_1 -ordered. \square

Remark 3.19. It can be seen that the given STOS in Example (3.6) is soft T_0 -ordered, but is not p -soft T_0 -ordered, since there does not exist a soft neighborhood W_E of y such that $x \notin W_E$. Also, it can be noted that the given STOS in Example (3.7) is soft T_1 -ordered, but is not p -soft T_1 -ordered, since there does not exist a soft neighborhood W_E of 2 such that $1 \notin W_E$. Hence, the converse of the above proposition fails.

Definition 3.20. (X, τ, E, \preceq) is said to be:

- (i) *upper (resp. lower) soft regularly ordered* if for each increasing (resp. decreasing) soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint soft neighbourhoods W_E of H_E and V_E of x such that W_E is increasing (resp. decreasing) and V_E is decreasing (resp. increasing);
- (ii) *soft regularly ordered* if it is both upper soft regularly ordered and lower soft regularly ordered;
- (iii) *upper (resp. lower) soft T_3 -ordered* if it is both upper (resp. lower) soft T_1 -ordered and upper (resp. lower) soft regularly ordered;
- (iv) *soft T_3 -ordered* if it is both upper soft T_3 -ordered and lower soft T_3 -ordered;
- (v) *soft normal ordered* if for every two disjoint an increasing soft closed set H_{1_E} and a decreasing soft closed set H_{2_E} there exist a disjoint increasing soft neighbourhood V_E of H_{1_E} and a decreasing soft neighbourhood W_E of H_{2_E} ;
- (vi) *soft T_4 -ordered* if it is soft normally ordered and soft T_1 -ordered.

Example 3.21. Let (X, τ, E, \preceq) be the soft indiscrete topological space, where E is an arbitrary set of parameters E and \preceq is any partial order relation on X . Then (X, τ, E, \preceq) is lower (upper) soft regularly ordered. So it is soft regularly ordered. Also it is soft normal. But it is not lower (upper) soft T_3 -ordered if $|X| \geq 2$. So it is not soft T_3 -ordered. Moreover, it is not soft T_4 -ordered. On the other hand, if (X, τ, E, \preceq) is the soft discrete topological space, then it is soft T_i -ordered for $i = 3, 4$.

Theorem 3.22. (X, τ, E, \preceq) is upper (resp. lower) soft regularly ordered if and only if for all $x \in X$ and every decreasing (resp. increasing) soft open set U_E partially containing x , there is a decreasing (resp. an increasing) partially soft neighbourhood V_E of x satisfying $\overline{V_E} \tilde{\subseteq} U_E$.

Proof. We prove the theorem in the lower soft regularly ordered case. The other case follows similar manner.

Necessity: Suppose that U_E is an increasing soft open set such that $x \in U_E$. Then U_E^c is a decreasing soft closed set and $x \notin U_E^c$. Therefore, there is an increasing soft neighbourhood V_E of x and a decreasing soft neighbourhood W_E of U_E^c such that $V_E \tilde{\cap} W_E = \tilde{\Phi}$. Thus, there exists a soft open set G_E such that $U_E^c \tilde{\subseteq} G_E \tilde{\subseteq} W_E$. Since $V_E \tilde{\subseteq} W_E^c$, then $V_E \tilde{\subseteq} W_E^c \tilde{\subseteq} G_E^c \tilde{\subseteq} U_E$ and since G_E^c is soft closed, then $\overline{V_E} \tilde{\subseteq} G_E^c \tilde{\subseteq} U_E$.

Sufficiency: Let H_E be a decreasing soft closed set and $x \notin H_E$. Then H_E^c is an increasing soft open set such that $x \in H_E^c$. Therefore, there exists an increasing soft neighbourhood V_E of x satisfying $\overline{V_E} \tilde{\subseteq} H_E^c$. Obviously, $H_E \tilde{\subseteq} (\overline{V_E})^c$ and $(\overline{V_E})^c$ is soft open. Now, $d((\overline{V_E})^c)$ is a decreasing soft neighbourhood of H_E . Suppose that $V_E \tilde{\cap} d((\overline{V_E})^c) \neq \tilde{\Phi}$. Then there are $x \in X$ and $e \in E$ such that $x \in V(e)$ and $x \in d((\overline{V_E})^c(e))$. This implies that there is $y \in (\overline{V_E})^c(e)$ such that $x \preceq y$. This means that $y \in V(e)$. But this contradicts the disjointness of V_E and $(\overline{V_E})^c$. Hence, $V_E \tilde{\cap} d((\overline{V_E})^c) = \tilde{\Phi}$. \square

Proposition 3.23. Every increasing (decreasing) soft closed or soft open subset of a soft regularly ordered space (X, τ, E, \preceq) is stable.

Proof. Without loss of generality, suppose that H_E is an increasing soft closed set in a soft regularly ordered space (X, τ, E, \preceq) which is not stable. Then there exists $x \in X$ and $\alpha, \beta \in E$ such that $x \in H(\alpha)$ and $x \notin H(\beta)$. This means that $x \notin H_E$. So for any soft neighborhood W_E of x and any soft neighborhood V_E of H_E , we obtain

that $x \in W(\alpha) \cap V(\alpha)$. Thus, we cannot find disjoint soft neighborhoods of x and H_E . This is a contradiction with soft regularly ordered of (X, τ, E, \preceq) . Hence, H_E must be stable.

The proof of the decreasing case can be done similarly. \square

Corollary 3.24. *If all increasing (decreasing) soft closed or soft open subsets in (X, τ, E, \preceq) are stable, then (X, τ, E, \preceq) is p-soft regularly ordered if and only if it is soft regularly ordered.*

Proposition 3.25. *Every soft regularly ordered space is p-soft regularly ordered.*

Proof. Straightforward. \square

The example below shows that the converse of Proposition (3.25) does not hold in general.

Example 3.26. *We define the soft sets $\{G_{i_E} : i = 1, 2, 3, 4\}$ over $X = \{x, y\}$ with a parameters set $E = \{e_1, e_2\}$ as follows:*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, \{x\})\}; \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \{y\})\}; \\ G_{3_E} &= \{(e_1, \{y\}), (e_2, \emptyset)\} \text{ and} \\ G_{4_E} &= \{(e_1, X), (e_2, \{x\})\}. \end{aligned}$$

Then $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, 3, 4\}$ is a soft topology on X . Let $\preceq = \Delta \cup \{(x, y)\}$ be a partial order relation on $X = \{x, y\}$. It can be verified that (X, τ, E, \preceq) is a p-soft regularly ordered space. But it is not soft regularly ordered because an increasing soft open set G_{3_E} is not stable.

Remark 3.27. *In the following we point out that the concepts of soft T_3 -ordered and soft T_4 -ordered spaces are independent of each other.*

- (i) *The given STOS in Example(3.26) is soft T_2 -ordered and soft T_4 -ordered, but it is not soft T_3 -ordered;*
- (ii) *If we consider (X, τ, E, \preceq) is STOS such that E is a singleton set, then (X, τ, E, \preceq) is a topological ordered space. So Example 7 in [2] shows that a soft T_4 -ordered space is a proper extension of a soft T_3 -ordered space.*

The following two problems are still open.

Problem 3.28. *Is a soft T_3 -ordered space a soft T_2 -ordered space?*

Problem 3.29. *Is a soft T_3 -ordered space a p-soft T_3 -ordered space?*

Proposition 3.30. *Every p-soft T_4 -ordered space (X, τ, E, \preceq) is soft T_4 -ordered.*

Proof. Straightforward. \square

The converse of Proposition 3.30 fails. We show this in the next example.

Example 3.31. *Let $E = \{e_1, e_2, e_3\}$ and $\preceq = \Delta \cup \{(x, y)\}$ be a partial order relation on $X = \{x, y\}$. The following six soft sets defined as follows.*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}X), (e_2, X), (e_3, X)\}; \\ G_{2_E} &= \{(e_1, \{y\}X), (e_2, X), (e_3, X)\}; \\ G_{3_E} &= \{(e_1, \emptyset), (e_2, X), (e_3, X)\}; \\ G_{4_E} &= \{(e_1, \{x\}), (e_2, \emptyset), (e_3, \emptyset)\}; \\ G_{5_E} &= \{(e_1, \{y\}), (e_2, \emptyset), (e_3, \emptyset)\} \text{ and} \\ G_{6_E} &= \{(e_1, X), (e_2, \emptyset), (e_3, \emptyset)\}. \end{aligned}$$

The collection $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 6\}$ is a soft topology on X . It can be easily verified that (X, τ, E, \preceq) is soft T_4 -ordered. In contrast, we cannot find a soft open set containing y such that x does not totally belong to it. Therefore, (X, τ, E, \preceq) fails to satisfy a condition of a p -soft T_1 -ordered space. Thus, (X, τ, E, \preceq) is not p -soft T_4 -ordered.

Theorem 3.32. Every soft compatibly ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of a soft regularly ordered space (X, τ, E, \preceq) is soft regularly ordered.

Proof. Suppose that H_E is an increasing soft closed subset of $(Y, \tau_Y, E, \preceq_Y)$ such that $y \notin H_E$. Because $(Y, \tau_Y, E, \preceq_Y)$ is soft compatibly ordered subspace of (X, τ, E, \preceq) , then there is an increasing soft closed set H_E^* in (X, τ, E, \preceq) such that $H_E = \tilde{Y} \cap H_E^*$. By hypothesis, we have a decreasing soft neighborhood V_E of y and an increasing soft neighborhood W_E of H_E^* such that $V_E \cap W_E = \tilde{\Phi}$. From Lemma (2.34) we obtain $\tilde{Y} \cap V_E$ is a decreasing soft neighborhood of y and $\tilde{Y} \cap W_E$ is an increasing soft neighborhood of H_E in $(Y, \tau_Y, E, \preceq_Y)$. The disjointness of $\tilde{Y} \cap V_E$ and $\tilde{Y} \cap W_E$ completes the proof that $(Y, \tau_Y, E, \preceq_Y)$ is upper soft regularly ordered.

In a similar manner it can be proved that $(Y, \tau_Y, E, \preceq_Y)$ is lower soft regularly ordered. Hence, $(Y, \tau_Y, E, \preceq_Y)$ is soft regularly ordered. \square

Corollary 3.33. Every soft compatibly ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of a soft T_3 -ordered space (X, τ, E, \preceq) is soft T_3 -ordered.

The proof of the next proposition is easy and thus it is omitted.

Proposition 3.34. Every soft closed compatibly ordered subspace of a soft T_4 -ordered space is soft T_4 -ordered.

Theorem 3.35. The finite product of soft T_i -ordered spaces is soft T_i -ordered for $i = 0, 1, 2, 3$.

Proof. We only prove the theorem in the case of $i = 2$, and the other cases can be proved similarly.

Assume that $(X, \tau_1, E_1, \preceq_1)$ and $(Y, \tau_2, E_2, \preceq_2)$ are soft T_2 -ordered spaces and let $(X \times Y, \tau, E, \preceq)$ be the soft ordered product space of them. Let $(x_1, y_1) \preceq (x_2, y_2) \in X \times Y$. Then $x_1 \preceq_1 x_2$ or $y_1 \preceq_2 y_2$. Without loss of generality, say $x_1 \preceq_1 x_2$. Since $(X, \tau_1, E_1, \preceq_1)$ is soft T_2 -ordered, then there is an increasing soft neighborhood W_{E_1} of x_1 and a decreasing soft neighborhood V_{E_1} of x_2 such that $x_2 \notin W_{E_1}$ and $x_1 \notin V_{E_1}$ which are disjoint. Therefore, $W_{E_1} \times \tilde{Y}$ is an increasing soft neighborhood of (x_1, y_1) and $V_{E_1} \times \tilde{Y}$ is a decreasing soft neighborhood of (x_2, y_2) such that $(x_2, y_2) \notin [W_{E_1} \times \tilde{Y}]$ and $(x_1, y_1) \notin [V_{E_1} \times \tilde{Y}]$. The disjointness of $W_{E_1} \times \tilde{Y}$ and $V_{E_1} \times \tilde{Y}$ finishes the proof that $(X \times Y, \tau, E, \preceq)$ is soft T_2 -ordered. \square

Theorem 3.36. The property of being a soft T_i -ordered space is a soft topological ordered property for $i = 0, 1, 2, 3, 4$.

Proof. We only prove the theorem in the cases of $i = 2, 4$, and the other cases can be proved similarly.

(i) Let $f_\phi : (X, \tau, A, \preceq_1) \rightarrow (Y, \theta, B, \preceq_2)$ be an ordered embedding soft homeomorphism map such that (X, τ, A, \preceq_1) is soft T_2 -ordered. Suppose that $x \preceq_2 y \in Y$. Then $P_\beta^x \preceq_2 P_\beta^y$ for each $\beta \in B$. Since f_ϕ is bijective, then there are P_α^a and P_α^b in \tilde{X} such that $f_\phi(P_\alpha^a) = P_\beta^x$ and $f_\phi(P_\alpha^b) = P_\beta^y$ and since f_ϕ is an ordered embedding, then $P_\alpha^a \preceq_1 P_\alpha^b$. So $a \preceq_1 b$. By hypothesis, we have an increasing soft neighborhood V_E of a and a decreasing soft neighborhood W_E of b such that $V_E \cap W_E = \tilde{\Phi}$. Since f_ϕ is bijective soft open, then $f_\phi(V_E)$ and $f_\phi(W_E)$ are disjoint soft neighborhoods of x and y , respectively. From Theorem (2.21), we obtain $f_\phi(V_E)$ and $f_\phi(W_E)$ are increasing and decreasing, respectively. Hence, the proof is complete.

(ii) Let $f_\phi : (X, \tau, A, \preceq_1) \rightarrow (Y, \theta, B, \preceq_2)$ be an ordered embedding soft homeomorphism map such that (X, τ, A, \preceq_1) is soft normally ordered. Suppose that the two disjoint soft closed sets H_E and F_E are increasing and decreasing, respectively. Since f_ϕ is bijective soft continuous, then $f_\phi^{-1}(H_E)$ and $f_\phi^{-1}(F_E)$ are disjoint soft closed sets and since f_ϕ is ordered embedding, then $f_\phi^{-1}(H_E)$ is increasing and $f_\phi^{-1}(F_E)$ is decreasing. By hy-

pothesis, there are disjoint soft neighborhoods V_E and W_E of $f_\phi^{-1}(H_E)$ and $f_\phi^{-1}(F_E)$, respectively, such that V_E is increasing and W_E is decreasing. So $H_E \tilde{\subseteq} f_\phi(V_E)$ and $F_E \tilde{\subseteq} f_\phi(W_E)$. It follows by Theorem (2.21) that $f_\phi(V_E)$ is increasing and $f_\phi(W_E)$ is decreasing. The disjointness of the soft neighborhoods $f_\phi(V_E)$ and $f_\phi(W_E)$ finishes the proof. \square

We devote the rest of this section to investigate some findings that associate some given ordered soft separation axioms with soft compactness.

Lemma 3.37. *Let F_E be an increasing (resp. a decreasing) soft open set in a soft regularly ordered space. Then for each $P_e^x \in F_E$ there exists an increasing (resp. a decreasing) soft neighborhood G_E of P_e^x such that $P_e^x \in \overline{G_E} \tilde{\subseteq} F_E$.*

Proof. Suppose that F_E is an increasing soft open set such that $P_e^x \in F_E$. Then $x \notin F_E^c$. Since (X, τ, E, \preceq) is soft regularly ordered, then there exist an increasing soft neighborhood G_E of x and a decreasing soft neighborhood W_E of F_E^c which are disjoint. This automatically means there are soft open sets H_E and L_E such that $x \in H_E \tilde{\subseteq} G_E$ and $F_E^c \tilde{\subseteq} L_E \tilde{\subseteq} W_E$. Thus, $x \in G_E \tilde{\subseteq} W_E \tilde{\subseteq} L_E^c \tilde{\subseteq} F_E$. Hence, $P_e^x \in \overline{G_E} \tilde{\subseteq} L_E^c \tilde{\subseteq} F_E$.

The decreasing case can be proved in a similar manner. \square

Theorem 3.38. *Let H_E be an increasing (resp. a decreasing) soft compact set in a soft regularly ordered space and F_E be a decreasing (resp. an increasing) soft open set containing H_E . Then there exists a decreasing (resp. an increasing) soft neighborhood G_E of H_E such that $\overline{G_E} \tilde{\subseteq} F_E$.*

Proof. Suppose that the given conditions are satisfied. Then for each $P_e^x \in H_E$, we have $P_e^x \in F_E$. Therefore, there is a decreasing soft neighborhood W_{i_E} of P_e^x such that $\overline{W_{i_E}} \tilde{\subseteq} F_E$. Thus, there is a soft open set V_{i_E} containing P_e^x such that $\overline{V_{i_E}} \tilde{\subseteq} \overline{W_{i_E}} \tilde{\subseteq} F_E$. Now, the collection $\{V_{i_E} : P_e^x \in F_E\}$ of soft open sets containing P_e^x forms a soft open cover of H_E . Since H_E is soft compact, then $H_E \tilde{\subseteq} \bigcup_{i=1}^n V_{i_E} \tilde{\subseteq} \bigcup_{i=1}^n W_{i_E}$. Let $G_E = \bigcup_{i=1}^n W_{i_E}$. This is a decreasing soft neighborhood of H_E . Obviously, $H_E \tilde{\subseteq} G_E \tilde{\subseteq} \overline{G_E} \tilde{\subseteq} F_E$.

A similar proof can be given for decreasing case. \square

Corollary 3.39. *Every soft compact and soft regularly ordered space (X, τ, E, \preceq) is soft normally ordered.*

Proof. Suppose that F_{1_E} and F_{2_E} are two disjoint soft closed sets such that F_{1_E} is decreasing and F_{2_E} is increasing. Then $F_{2_E} \tilde{\subseteq} F_{1_E}^c$. Since (X, τ, E, \preceq) is soft compact, then F_{2_E} is soft compact and since (X, τ, E, \preceq) is soft regularly ordered, then there is an increasing soft neighborhood G_E of $F_{1_E}^c$ such that $F_{2_E} \tilde{\subseteq} G_E \tilde{\subseteq} \overline{G_E} \tilde{\subseteq} F_{1_E}^c$. Obviously, $F_{1_E} \tilde{\subseteq} (\overline{G_E})^c$ and $G_E \tilde{\cap} (\overline{G_E})^c = \tilde{\emptyset}$. To prove that $G_E \tilde{\cap} d[(\overline{G_E})^c] = \tilde{\emptyset}$, suppose that there exists an element $x \in G_E$ and $x \in d[(\overline{G_E})^c]$. So there exists an element $y \in (\overline{G_E})^c$ such that $x \preceq y$. This means that $y \in G_E$. But this contradicts the disjointness of G_E and $(\overline{G_E})^c$. Thus, (X, τ, E, \preceq) is soft normally ordered. \square

To show that the converse of the above theorem and corollary fail we give the following example.

Example 3.40. *Consider a partial order relation $\preceq = \triangle$ which is the equality relation on $X = \{x, y\}$ and let $E = \{e_1, e_2\}$ be a parameters set. The collection $\tau = \{\tilde{\emptyset}, \tilde{X}, G_{i_E} : i = 1, 2\}$ is a soft topology on X , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, X)\} \text{ and} \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \emptyset)\}. \end{aligned}$$

Obviously, (X, τ, E, \preceq) is soft normally ordered and soft compact. Also, for every increasing (resp. decreasing) soft compact subset of (X, τ, E, \preceq) and every decreasing (resp. increasing) soft open set F_E containing H_E , there exists a decreasing (resp. an increasing) soft neighborhood G_E of H_E such that $\overline{G_E} \tilde{\subseteq} F_E$. On the other hand, since the soft open sets G_{1_E} and G_{2_E} are not stable, then it follows from Proposition (3.23) that (X, τ, E, \preceq) is not soft regularly ordered.

4 Strong ordered soft separation axioms

The first aim of this section is to define strong ordered soft separation axioms, namely strong soft T_i -ordered spaces ($i = 0, 1, 2, 3, 4$) by using monotone soft open sets in the place of monotone soft neighborhoods. The second aim is to provide some examples to illustrate the relationships between these and the relationships between them and soft T_i -ordered spaces. The third aim is to discuss their main properties and provide some results that associate soft compactness and some initiated strong ordered soft separation axioms.

The following example explains the difference between soft open sets and soft neighborhoods in terms of increasing and decreasing.

Example 4.1. Let $E = \{e_1, e_2\}$ and $\preceq = \Delta \cup \{(x, z), (y, w)\}$ be a partial order relation on $X = \{a, x, y, w, z\}$. The collection $\tau = \{\tilde{\Phi}, \tilde{X}, G_E\}$ is a soft topology on X , where $G_E = \{(e_1, \{a, x\}), (e_2, \{a, w\})\}$. Now, it can be noted that G_E is a soft open set containing a such that $i(G_E) = \{(e_1, \{a, x, z\}), (e_2, \{a, w\})\} \neq G_E$ and $d(G_E) = \{(e_1, \{a, x\}), (e_2, \{a, y, w\})\} \neq G_E$. So that G_E is neither increasing, nor decreasing. On the other hand, $W_E = \{(e_1, \{a, x, z\}), (e_2, \{a, y, w\})\}$ is a monotone soft neighborhood of a because:

- (i) $a \in G_E \tilde{\subseteq} W_E$ and
- (ii) $i(W_E) = W_E$ and $d(G_E) = W_E$.

Also, $U_E = \{(e_1, \{a, x, z\}), (e_2, \{a, w\})\}$ is an increasing soft neighborhood of a , but it is not decreasing and $V_E = \{(e_1, \{a, x\}), (e_2, \{a, y, w\})\}$ is a decreasing soft neighborhood of a , but it is not increasing.

Proposition 4.2.

- (i) Every monotone soft open set containing an element x is a monotone soft neighborhood of x .
- (ii) Every monotone soft open set containing a soft set H_E is a monotone soft neighborhood of H_E .

Proof. Let G_E be a monotone soft open set containing an element x . Then $x \in G_E \subseteq G_E$. Therefore, G_E is a monotone soft neighborhood of x . Also, if G_E is a monotone soft open set containing a soft set H_E . Then $H_E \subseteq G_E \subseteq G_E$. Therefore, G_E is a monotone soft neighborhood of H_E . \square

Example (4.1) demonstrates that the converse of the above proposition fails.

Definition 4.3. (X, τ, E, \preceq) is said to be:

- (i) strong upper (resp. strong lower) soft T_1 -ordered if for every $x \not\preceq y$ in X , there exists a decreasing (resp. an increasing) soft open set W_E containing y (resp. x) such that $x \notin W_E$ (resp. $y \notin W_E$);
- (ii) strong soft T_0 -ordered if it is strong upper soft T_1 -ordered or strong lower soft T_1 -ordered;
- (iii) strong soft T_1 -ordered if it is strong upper soft T_1 -ordered and strong lower soft T_1 -ordered;
- (iv) strong soft T_2 -ordered if for every $x \not\preceq y$ in X , there exist disjoint an increasing soft open set W_E containing x and a decreasing soft open set V_E containing y ;
- (v) strong upper (resp. strong lower) soft regularly ordered if for each increasing (resp. decreasing) soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint soft open sets W_E containing H_E and V_E containing x such that W_E is increasing (resp. decreasing) and V_E is decreasing (resp. increasing);
- (vi) strong soft regularly ordered if it is both strong upper soft regularly ordered and strong lower soft regularly ordered;
- (vii) strong upper (resp. strong lower) soft T_3 -ordered if it is both strong upper (resp. strong lower) soft T_1 -ordered and strong upper (resp. strong lower) soft regularly ordered;
- (viii) strong soft T_3 -ordered if it is both strong upper soft T_3 -ordered and strong lower soft T_3 -ordered;

- (ix) *strong soft normally ordered if for each disjoint soft closed sets F_E and H_E such that F_E is increasing and H_E is decreasing, there exist disjoint soft open sets G_E containing F_E and U_E containing H_E such that G_E is increasing and U_E is decreasing;*
- (x) *strong soft T_4 -ordered if it is strong soft normally ordered and strong soft T_1 -ordered.*

Proposition 4.4. *Every strong soft T_i -ordered space is soft T_i -ordered for $i = 0, 1, 2, 3, 4$.*

Proof. The proof follows from the fact that every monotone soft open set containing an element x is a monotone soft neighborhood of x and every monotone soft open set containing a soft set H_E is a monotone soft neighborhood of H_E . \square

In what follows, we construct two examples to point out that the converse of the above proposition fails in the cases of $i = 0, 1$. The other cases are still open problems.

Example 4.5. *Let (X, τ, E, \preceq) be the same as in Example (3.6). We point out that this STOS is soft T_0 -ordered. However, it is not strong soft T_0 -ordered, because $y \preceq z$ and there does not exist an increasing soft open set G_E containing x such that $z \notin G_E$.*

Example 4.6. *Let (X, τ, E, \preceq) be the same as in Example (3.7). We point out that this STOS is soft T_1 -ordered. However, it is not strong soft T_1 -ordered, because $2 \preceq 1$ and there does not exist an increasing soft open set G_E containing 2 such that $1 \notin G_E$.*

Problem 4.7. *Is a soft T_i -ordered space a strong soft T_i -ordered space for $i = 2, 3, 4$?*

Theorem 4.8. *Let (X, τ, E, \preceq) be strong soft regularly ordered. Then (X, τ, E, \preceq) is p-soft T_1 -ordered if and only if it is strong soft T_1 -ordered.*

Proof. To prove the "if" part, let $x \preceq y$. Then it follows from Proposition (2.39) that $(i(x))_E$ and $(d(y))_E$ are soft closed sets. So $[(d(y))_E]^c$ is an increasing soft open set containing x and $[(i(x))_E]^c$ is a decreasing soft open set containing y such that $y \notin [(d(y))_E]^c$ and $x \notin [(i(x))_E]^c$. Thus, (X, τ, E, \preceq) is strong soft T_1 -ordered.

To prove the "only if" part, suppose $x \preceq y$ in X . Then there exist an increasing soft open set W_E containing x and a decreasing soft open set V_E containing y such that $y \notin W_E$ and $x \notin V_E$. By Proposition (3.23), W_E and V_E are stable. This means that $y \notin W_E$ and $x \notin V_E$. Thus, (X, τ, E, \preceq) is p-soft T_1 -ordered. \square

Corollary 4.9. *If (X, τ, E, \preceq) is strong soft regularly ordered and upper (resp. lower) strong soft T_1 -ordered, then $(i(x))_E$ (resp. $(d(x))_E$) is soft closed.*

To show that the converse of the above corollary fails, we give the following example.

Example 4.10. *Consider $\preceq = \Delta \cup \{(x, y)\}$ which is a partial order relation on $X = \{x, y\}$ and let $E = \{e_1, e_2\}$ be a parameters set. The collection $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, 3, 4\}$ is a soft topology on X , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, \{x\})\}; \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \{y\})\}; \\ G_{3_E} &= \{(e_1, X), (e_2, \{y\})\} \text{ and} \\ G_{4_E} &= \{(e_1, \{x\}), (e_2, \emptyset)\}. \end{aligned}$$

On the one hand, $(i(x))_E = (d(y))_E = \tilde{X}$, $(d(x))_E = G_{1_E}$ and $(i(y))_E = G_{2_E}$ are soft closed sets. On the other hand, since the soft open sets G_{3_E} and G_{4_E} are not stable, then it follows from Proposition (3.23) that (X, τ, E, \preceq) is not soft regularly ordered. Hence, (X, τ, E, \preceq) is not strong soft regularly ordered.

Proposition 4.11. *The following three concepts are equivalent if (X, τ, E, \preceq) is strong soft regularly ordered:*

- (i) strong soft T_2 -ordered;
- (ii) strong soft T_1 -ordered;
- (iii) strong soft T_0 -ordered.

Proof. The directions (i)→(iii) are obvious.

To prove that (iii)→(i), let $x \preceq y \in X$. Since (X, τ, E, \preceq) is strong soft T_0 -ordered, then it is strong lower soft T_1 -ordered or strong upper soft T_1 -ordered. Say, it is strong upper soft T_1 -ordered. It follows, by the above corollary, that $(i(x))_E$ is an increasing soft closed set. Since $y \notin (i(x))_E$ and (X, τ, E, \preceq) is strong soft regularly ordered, then there exist disjoint soft open sets W_E and V_E containing $(i(x))_E$ and y , respectively, such that W_E is increasing and V_E is decreasing. Hence, the proof is complete. \square

Corollary 4.12. *The following concepts are equivalent if (X, τ, E, \preceq) is strong lower (resp. strong upper) soft regularly ordered:*

- (i) strong soft T_2 -ordered;
- (ii) strong soft T_1 -ordered;
- (iii) strong lower (resp. strong upper) soft T_1 -ordered.

Corollary 4.13. *Every strong soft T_i -ordered space (X, τ, E, \preceq) is strong soft T_{i-1} -ordered for $i = 1, 2, 3$.*

The converse of the above corollary need not be true in general as demonstrated in the next three examples.

Example 4.14. *The given STOS in Example (3.7) is strong soft T_0 -ordered, however, it is not strong soft T_1 -ordered.*

Example 4.15. *Let $E = \{e_1, e_2\}$ be a parameters set. Let $\preceq = \Delta \cup \{(2, 3)\}$ be a partial order relation on the set of natural numbers \mathbb{N} . Then $\tau = \{\tilde{\Phi}, G_E \tilde{\subseteq} \tilde{\mathbb{N}} : G_E^c \text{ is finite}\}$ is a soft topology on \mathbb{N} . Now, we have the following cases:*

- (i) since $3 \not\preceq 2$, then $U_E = \{(e_1, \mathbb{N} \setminus \{2\}), (e_2, \mathbb{N} \setminus \{2\})\}$ is an increasing soft open set containing 3 and $V_E = \{(e_1, \mathbb{N} \setminus \{3\}), (e_2, \mathbb{N} \setminus \{3\})\}$ is a decreasing soft open set containing 2 such that $2 \notin U_E$ and $3 \notin V_E$;
- (ii) since $x \not\preceq 2$ for all $x \neq 3$, then $U_E = \{(e_1, \mathbb{N} \setminus \{2\}), (e_2, \mathbb{N} \setminus \{2\})\}$ is an increasing soft open set containing x and $V_E = \{(e_1, \mathbb{N} \setminus \{x, 3\}), (e_2, \mathbb{N} \setminus \{x, 3\})\}$ is a decreasing soft open set containing x such that $2 \notin U_E$ and $x \notin V_E$;
- (iii) since $2 \not\preceq x$, then $U_E = \{(e_1, \mathbb{N} \setminus \{x\}), (e_2, \mathbb{N} \setminus \{x\})\}$ is an increasing soft open set containing 2 and $V_E = \{(e_1, \mathbb{N} \setminus \{2, 3\}), (e_2, \mathbb{N} \setminus \{2, 3\})\}$ is a decreasing soft open set containing x such that $x \notin U_E$ and $2 \notin V_E$;
- (iv) since $3 \not\preceq x$ for all $x \neq 2$, then $U_E = \{(e_1, \mathbb{N} \setminus \{2, x\}), (e_2, \mathbb{N} \setminus \{2, x\})\}$ is an increasing soft open set containing 3 and $V_E = \{(e_1, \mathbb{N} \setminus \{3\}), (e_2, \mathbb{N} \setminus \{3\})\}$ is a decreasing soft open set containing x such that $x \notin U_E$ and $3 \notin V_E$;
- (v) since $x \not\preceq 3$, then $U_E = \{(e_1, \mathbb{N} \setminus \{2, 3\}), (e_2, \mathbb{N} \setminus \{2, 3\})\}$ is an increasing soft open set containing x and $V_E = \{(e_1, \mathbb{N} \setminus \{x\}), (e_2, \mathbb{N} \setminus \{x\})\}$ is a decreasing soft open set containing 3 such that $3 \notin U_E$ and $x \notin V_E$.

Thus, (X, τ, E, \preceq) is strong soft T_1 -ordered. In contrast, one can note that it is not strong soft T_2 -ordered.

Example 4.16. *The given STOS in Example (3.26) is strong soft T_2 -ordered, however, it is not strong soft T_3 -ordered.*

Remark 4.17. *In the following, we point out that the concepts of strong soft T_3 -ordered and strong soft T_4 -ordered spaces are independent of each other.*

- (i) The given STOS in Example(3.26) is strong soft T_4 -ordered, but it is not strong soft T_3 -ordered;
- (ii) If we consider (X, τ, E, \preceq) is an STOS such that E is a singleton and \preceq is an equality relation, then (X, τ, E, \preceq) is a topological space. So Niemytzki space in general topology shows that a strong soft T_4 -ordered space is a proper extension of a strong soft T_3 -ordered space.

The proofs of Theorem (4.18) and Theorem (4.19) below are similar to the proofs of Theorem (3.35) and Theorem (3.36) respectively.

Theorem 4.18. *A finite product of strong soft T_i -ordered spaces is strong soft T_i -ordered for $i = 0, 1, 2$.*

Theorem 4.19. *The property of being a strong soft T_i -ordered space is a soft topological ordered property for $i = 0, 1, 2, 3, 4$.*

We devote the rest of this section to investigate some findings that associate some strong ordered soft separation axioms with soft compactness.

Lemma 4.20. *Let F_E be an increasing (resp. a decreasing) soft open subset in a strong soft regularly ordered space. Then for each $P_e^x \in F_E$, there exists an increasing (resp. a decreasing) soft open set G_E containing P_e^x such that $P_e^x \in \overline{G_E} \subsetneq F_E$.*

Proof. The proof is similar to that of Lemma (3.37). □

Theorem 4.21. *Let H_E be an increasing (resp. a decreasing) soft compact subset in a strong soft regularly ordered space and F_E be a decreasing (resp. an increasing) soft open set containing H_E . Then there exists a decreasing (resp. an increasing) soft open set G_E such that $H_E \subsetneq G_E \subsetneq \overline{G_E} \subsetneq F_E$.*

Proof. The proof is similar to that of Theorem (3.38). □

Corollary 4.22. *Every soft compact strong soft regularly ordered space is strong soft normally ordered.*

Proof. The proof is similar to that of Corollary (3.39). □

Remark 4.23.

- (i) If a partial order relation is diagonal, then a soft topological ordered space can be viewed as a soft topological space. In this case the concepts of soft T_i -ordered spaces and strong soft T_i -ordered spaces for $i = 0, 1, 2, 3, 4$, are equivalent;
- (ii) If a set of parameters is a singleton, then a soft topological ordered space can be viewed as a topological ordered space. In this case the notations \notin and $\not\in$ are equivalent. Hence, the concepts of p -soft T_i -ordered spaces, soft T_i -ordered spaces and T_i -ordered spaces for $i = 0, 1, 2, 3, 4$, are equivalent.

Proposition 4.24. *Every strong soft T_i -ordered space (X, τ, E, \preceq) is a soft T_i -space, for $i = 0, 1, 2$.*

Proof. The proof follows directly from the definitions of strong soft T_i -ordered and soft T_i -spaces. □

Remark 4.25. *To confirm that the converse of the above proposition fails, we consider E is a singleton and then we suffice with the examples introduced in [2]. Also, by considering E is a singleton, Example 3 in [2] shows that the concepts of strong soft T_i -ordered and soft T_i -spaces ($i = 3, 4$) are independent of each other.*

In conclusion, we give Figure 1 to illustrate the relationships among some types of ordered soft separation axioms.

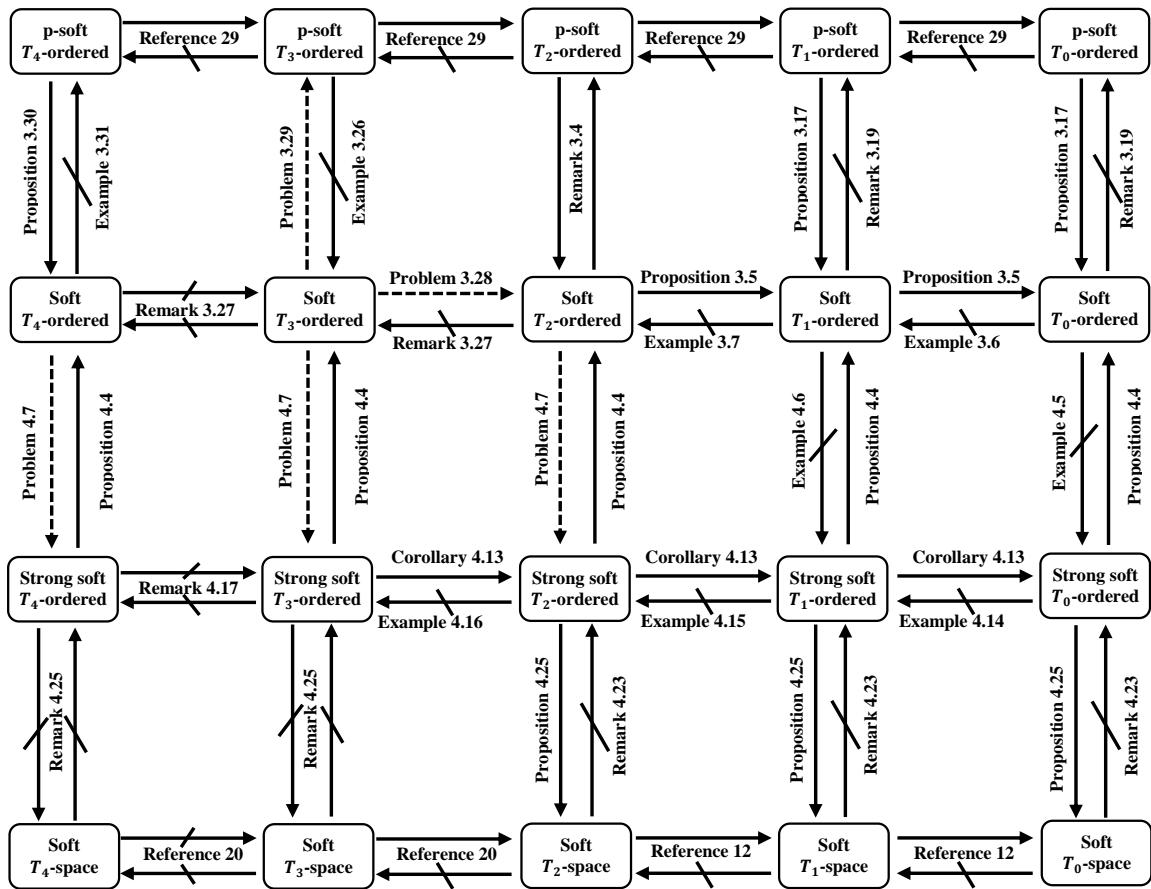


Figure 1: The relationships among some types of ordered soft separation spaces.

5 Conclusion and future work

By combining a partial order relation and a topology on a non-empty set, Nachbin [1] defined the topological ordered space. Similarly, Al-shami et al. [29] defined the soft topological ordered space. Studying soft separation axioms via soft topological spaces is a significant topic because they help establish a wider family which can be easily applied to classify the objects under study. We demonstrate in the last paragraph of introduction the reasons for doing many studies via soft separation axioms and the variety of these studies will be more via ordered soft separation axioms. Throughout this work, we use the notions of monotone soft neighborhoods and monotone soft open sets to present soft T_i -ordered and strong soft T_i -ordered spaces, respectively, for $i = 0, 1, 2, 3, 4$. These two types are formulated with respect to the ordinary points. We establish several results such as strong soft T_i -ordered spaces is strictly finer than soft T_i -ordered spaces and support this result with number of interesting examples. Also, we discuss the relationships which associate the soft T_i -ordered (strong soft T_i -ordered) spaces with p-soft T_i -ordered spaces and soft T_i -spaces. In Theorem (4.8), we give a condition that satisfies the equivalence between p-soft T_1 -ordered and strong soft T_1 -ordered spaces. In the end of Section (3) and Section (4), we present a number of results that associate soft compactness with some of the initiated ordered soft separation axioms. Some open problems on the relationship between strong soft T_i -ordered and soft T_i -ordered spaces ($i = 2, 3, 4$) are posed.

To extend this study, one can generalize the initiated concepts on supra soft topological spaces [40]. All these results will provide a base to researchers who want to work in the soft ordered topology field and will

help to establish a general framework for applications in practical fields.

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