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# Two new forms of ordered soft separation axioms

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**Abstract:** The goal of this work is to introduce and study two new types of ordered soft separation axioms, namely soft  $T_i$ -ordered and strong soft  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ). These two types are formulated with respect to the ordinary points and the distinction between them is attributed to the nature of the monotone neighborhoods. We provide several examples to elucidate the relationships among these concepts and to show the relationships associate them with their parametric topological ordered spaces and p-soft  $T_i$ -ordered spaces. Some open problems on the relationships between strong soft  $T_i$ -ordered and soft  $T_i$ -ordered spaces ( $i = 2, 3, 4$ ) are posed. Also, we prove some significant results which associate both types of the introduced ordered axioms with some notions such as finite product soft spaces, soft topological and soft hereditary properties. Furthermore, we describe the shape of increasing (decreasing) soft closed and open subsets of soft regularly ordered spaces; and demonstrate that a condition of strong soft regularly ordered is sufficient for the equivalence between p-soft  $T_1$ -ordered and strong soft  $T_1$ -ordered spaces. Finally, we establish a number of findings that associate soft compactness with some ordered soft separation axioms initiated in this work.

**Keywords:** monotone soft open set, monotone soft neighborhood, soft  $T_i$ -ordered and strong soft  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ )

**MSC:** 54D10, 54D15, 54D30, 54F05

## 1 Introduction

The study of the concept of topological ordered spaces was presented for the first time by Nachbin [1]. He has constructed this concept by adding a partial order relation to the structure of a topological space. With regard to Nachbin's definition of topological ordered spaces, two points can be considered, the first one is that the topology and the partial order relation operate independently of one another, and the second one is that the topological ordered spaces are one of the generalizations of topological spaces. After Nachbin's work, many researchers carried out various studies on ordered spaces (see, for example, [2–5]).

Zadeh [6] introduced the notion of fuzzy sets in 1965 as mathematical instruments for dealing with uncertainties. To put a topological structure to fuzzy set theory, Chang [7] has defined fuzzy topological spaces. Then Katsaras [8] combined a partial order relation and a fuzzy topology to define a fuzzy topological ordered space.

In 1999, the notion of soft sets was proposed by Molodtsov [9] to overcome problems associated with uncertainties, vagueness, impreciseness and incomplete data. This notion includes enough parameters which make it a suitable alternative for the previous mathematical approaches such as fuzzy and rough sets. The useful applications of soft sets to several directions contribute to progress work on it rapidly (see, for example, [10, 11]). The concept of soft topological spaces was introduced by Shabir and Naz in their pioneer work

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[12]. Then many studies on soft topological spaces have been done (see, for example, [13–18]). El-Shafei et al. [19] introduced partial belong and total non-belong relations which are more functional and flexible for theoretical and application studies via the soft set theory and soft topologies. Then they employed these two new notions to present new soft separation axioms, namely  $p$ -soft  $T_i$ -spaces ( $i = 0, 1, 2, 3, 4$ ). The authors of [20–25] have done some amendments for some alleged results on soft axioms. Al-shami and Kočinac [26] explored the equivalence between the extended and enriched soft topologies and has obtained some interesting results related to the parametric topologies. The authors of [27, 28] introduced different types of soft axioms on supra soft topological spaces.

In [29], the authors formulated the concepts of monotone soft sets and soft topological ordered spaces as a new soft structure. They also have utilized the natural belong and total non-belong relations to introduce the notions of  $p$ -soft  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ). In [30] we studied and investigated these notions on supra soft topological ordered spaces.

The topic of soft separation axioms is one of the most significant and interesting in soft topology. In general, soft separation axioms are utilized to obtain more restricted families of soft topological spaces. It turns out, from the previous studies, that there are many points of view to study soft separation axioms. The diversity of these perspectives is attributed to the relations of belong and non-belong that are used in the definitions; and the objects of study, ordinary points or soft points (see, for example, [12, 19, 31–34]). The variety of ordered soft separation axioms will be more extended, because the soft neighborhoods and soft open sets is distinguished according to the partially ordered soft set.

As a contribution of study ordered soft separation axioms, the authors devote this work to defining and investigating two types of ordered soft separation axioms, namely soft  $T_i$ -ordered and strong soft  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ). With the help of examples, we illustrate the relationships among them. Also, we derive their fundamental features such as the finite product of soft  $T_i$ -ordered (resp. strong soft  $T_i$ -ordered) spaces is soft  $T_i$ -ordered (resp. strong soft  $T_i$ -ordered) for  $i = 0, 1, 2$ ; and the property of being a soft  $T_i$ -ordered (strong soft  $T_i$ -ordered) space is a soft topological ordered property for  $i = 0, 1, 2, 3, 4$ . Moreover, we investigate certain properties of them that associated with some notions of soft ordered topology such as soft ordered topological invariant and soft compatibly ordered subspaces. In the end of both Section (3) and Section (4), we discuss some results about the relationships between soft compact spaces and some of the initiated ordered soft separation axioms.

## 2 Preliminaries

This section is allocated to recall some definitions and well known results which we shall utilize them in the next parts of this work.

### 2.1 Soft set

**Definition 2.1.** [9] A pair  $(G, E)$  is said to be a soft set over  $X$  provided that  $G$  is a mapping of a parameters set  $E$  into  $2^X$ .

For short, we use the notation  $G_E$  instead of  $(G, E)$  and we express a soft set  $G_E$  as follows:  $G_E = \{(e, G(e)) : e \in E \text{ and } G(e) \in 2^X\}$ . Also, we use the notation  $S(X_E)$  to denote the collection of all soft sets defined over  $X$  under a set of parameters  $E$ .

**Definition 2.2.** [12, 19] For a soft set  $G_E$  over  $X$  and  $x \in X$ , we say that:

- (i)  $x \in G_E$  if  $x \in G(e)$  for each  $e \in E$ ; and we say that  $x \notin G_E$  if  $x \notin G(e)$  for some  $e \in E$ ;
- (ii)  $x \subseteq G_E$  if  $x \in G(e)$  for some  $e \in E$ ; and we say that  $x \not\subseteq G_E$  if  $x \notin G(e)$  for each  $e \in E$ .

**Definition 2.3.** [11]  $G_E$  over  $X$  is called a null soft set (resp. an absolute soft set) if  $G(e) = \emptyset$  (resp.  $G(e) = X$ ) for each  $e \in E$ ; and it is denoted by  $\tilde{\Phi}$  (resp.  $\tilde{X}$ ).

**Definition 2.4.** [35]  $G_E$  over  $X$  is called a soft point if there are  $e \in E$  and  $x \in X$  such that  $G(e) = \{x\}$  and  $G(b) = \emptyset$  for each  $b \in E \setminus \{e\}$ .

A soft point will be shortly denoted by  $P_e^x$  and we say that  $P_e^x \in G_E$  provided that  $x \in G(e)$ .

**Definition 2.5.** [19]  $G_E$  over  $X$  is said to be stable if there is  $S \subseteq X$  such that  $G(e) = S$  for each  $e \in E$ .

**Definition 2.6.** [10] The relative complement of  $G_E$ , denoted by  $G_E^c$ , is a mapping  $G^c : E \rightarrow 2^X$  defined by  $G^c(e) = X \setminus G(e)$  for each  $e \in E$ .

**Definition 2.7.** [36]  $G_A$  is a soft subset of  $G_B$  if  $A \subseteq B$  and  $G(a) \subseteq F(a)$  for all  $a \in A$ .

**Definition 2.8.** [11] The union of soft sets  $G_A$  and  $F_B$  over  $X$ , denoted by  $G_A \widetilde{\cup} F_B$ , is the soft set  $V_D$ , where  $D = A \cup B$  and a mapping  $V : D \rightarrow 2^X$  is defined as follows:

$$V(d) = \begin{cases} G(d) & : d \in A - B \\ F(d) & : d \in B - A \\ G(d) \cup F(d) & : d \in A \cap B \end{cases}$$

**Definition 2.9.** [10] The intersection of soft sets  $G_A$  and  $F_B$  over  $X$ , denoted by  $G_A \widetilde{\cap} F_B$ , is the soft set  $V_D$ , where  $D = A \cap B \neq \emptyset$ , and a mapping  $V : D \rightarrow 2^X$  is defined by  $V(d) = G(d) \cap F(d)$  for all  $d \in D$ .

**Definition 2.10.** [37] Let  $G_A$  and  $H_B$  be two soft sets over  $X$  and  $Y$ , respectively. Then the cartesian product of  $G_A$  and  $H_B$  is denoted by  $(G \times H)_{A \times B}$  and is defined as  $(G \times H)(a, b) = G(a) \times H(b)$  for each  $a \in A$  and  $b \in B$ .

**Definition 2.11.** [35] A soft mapping  $f_\phi$  of  $S(X_A)$  into  $S(Y_B)$  is a pair of mappings  $f : X \rightarrow Y$  and  $\phi : A \rightarrow B$  such that for soft subsets  $G_K$  and  $H_L$  of  $S(X_A)$  and  $S(Y_B)$ , respectively, we have:

(i)  $f_\phi(G_K) = (f_\phi(G))_B$  is a soft subset of  $S(Y_B)$  such that

$$f_\phi(G)(b) = \begin{cases} \bigcup_{a \in \phi^{-1}(b) \cap K} f(G(a)) & : \phi^{-1}(b) \cap K \neq \emptyset \\ \emptyset & : \phi^{-1}(b) \cap K = \emptyset \end{cases}$$

for each  $b \in B$ ;

(ii)  $f_\phi^{-1}(H_L) = (f_\phi^{-1}(H))_A$  is a soft subset of  $S(X_A)$  such that

$$f_\phi^{-1}(H)(a) = \begin{cases} f^{-1}(H(\phi(a))) & : \phi(a) \in L \\ \emptyset & : \phi(a) \notin L \end{cases}$$

for each  $a \in A$ .

**Definition 2.12.** [35] A soft mapping  $f_\phi : S(X_A) \rightarrow S(Y_B)$  is said to be injective (resp. surjective, bijective) if the two mappings  $f$  and  $\phi$  are injective (resp. surjective, bijective).

**Proposition 2.13.** [35] For a soft mapping  $f_\phi : S(X_A) \rightarrow S(Y_B)$ , we have the following results:

- (i)  $G_A \widetilde{\subseteq} f_\phi^{-1} f_\phi(G_A)$  for each  $G_A \in S(X_A)$ ; and  $f_\phi f_\phi^{-1}(H_B) \widetilde{\subseteq} H_B$  for each  $H_B \in S(Y_B)$ ;
- (ii) If  $f_\phi$  is injective (resp. surjective), then  $G_A = f_\phi^{-1} f_\phi(G_A)$  (resp.  $f_\phi f_\phi^{-1}(H_B) = H_B$ ).

**Proposition 2.14.** [29] Let  $f_\phi : S(X_A) \rightarrow S(Y_B)$  be a soft mapping. Then

- (i) The image of any soft point is a soft point;

(ii) If  $f$  is bijective, then the inverse image of any soft point is a soft point.

**Definition 2.15.** [38] A binary relation  $\preceq$  on  $X \neq \emptyset$  is said to be a partial order relation if it is reflexive, anti-symmetric and transitive. An element  $x \in X$  is said to be the smallest (resp. largest) element of  $X$  if  $x \preceq y$  (resp.  $y \preceq x$ ) for all  $y \in X$ .

Henceforth, a diagonal relation  $\{(x, x) : x \in X\}$  on  $X$  is denoted by  $\triangle$ .

**Definition 2.16.** [29]  $(X, E, \preceq)$  is said to be a partially ordered soft set on  $X \neq \emptyset$  if  $(X, \preceq)$  is a partially ordered set. For two soft points  $P_\alpha^x$  and  $P_\alpha^y$  in  $(X, E, \preceq)$ , we say that  $P_\alpha^x \preceq P_\alpha^y$  if  $x \preceq y$ .

**Definition 2.17.** [29] An increasing operator  $i$  and a decreasing operator  $d$  are two soft maps of  $(S(X_E), \preceq)$  into  $(S(X_E), \preceq)$  defined as follows: for each soft subset  $G_E$  of  $S(X_E)$

- (i)  $i(G_E) = iG_E$ , where  $iG$  is a mapping of  $E$  into  $P(X)$  given by  $iG(e) = i(G(e)) = \{x \in X : b \preceq x \text{ for some } b \in G(e)\}$ ;
- (ii)  $d(G_E) = dG_E$ , where  $dG$  is a mapping of  $E$  into  $P(X)$  given by  $dG(e) = d(G(e)) = \{x \in X : x \preceq b \text{ for some } b \in G(e)\}$ .

**Definition 2.18.** [29] A soft subset  $G_E$  of  $(X, E, \preceq)$  is said to be increasing (resp. decreasing) provided that  $G_E = i(G_E)$  (resp.  $G_E = d(G_E)$ ).

**Theorem 2.19.** [29] The finite product of increasing (resp. decreasing) soft sets is increasing (resp. decreasing).

**Definition 2.20.** [29] A soft map  $f_\phi : (S(X_A), \preceq_1) \rightarrow (S(Y_B), \preceq_2)$  is said to be:

- (i) increasing (resp. decreasing) provided that  $P_\alpha^x \preceq_1 P_\alpha^y$  implies  $f_\phi(P_\alpha^x) \preceq_2 f_\phi(P_\alpha^y)$  (resp.  $f_\phi(P_\alpha^y) \preceq_2 f_\phi(P_\alpha^x)$ );
- (iii) an ordered embedding provided that  $P_\alpha^x \preceq_1 P_\alpha^y$  if and only if  $f_\phi(P_\alpha^x) \preceq_2 f_\phi(P_\alpha^y)$ .

**Theorem 2.21.** [29] Let  $f_\phi : (S(X_A), \preceq_1) \rightarrow (S(Y_B), \preceq_2)$  be a bijective ordered embedding soft mapping. Then the image of each increasing (resp. decreasing) soft set is increasing (resp. decreasing).

## 2.2 Soft topology

**Definition 2.22.** [12] A family  $\tau$  of soft sets over  $X$  under a fixed parameters set  $E$  which contains  $\tilde{X}$  and  $\tilde{\Phi}$  and is closed under finite soft intersection and arbitrary soft union is said to be a soft topology on  $X$ .

The triple  $(X, \tau, E)$  is said to be a soft topological space (briefly, STS). Every member of  $\tau$  is called soft open and its relative complement is called soft closed.

**Proposition 2.23.** [12] If  $(X, \tau, E)$  is an STS, then a family  $\tau_e = \{G(e) : G_E \in \tau\}$  forms a topology on  $X$  for each  $e \in E$ .

The notation  $\tau_e$ , which is given in the proposition above, is said to be a parametric topology and  $(X, \tau_e)$  is said to be a parametric topological space.

**Definition 2.24.** [35] A soft subset  $W_E$  of  $(X, \tau, E)$  is called a soft neighborhood of  $x \in X$  if there exists a soft open set  $G_E$  such that  $x \in G_E \subseteq W_E$ .

**Definition 2.25.** [35, 39] A soft mapping  $f_\phi : (X, \tau, A) \rightarrow (Y, \theta, B)$  is said to be:

- (i) soft continuous if the inverse image of each soft open set is soft open;

- (ii) soft open (resp. soft closed) if the image of each soft open (resp. soft closed) set is soft open (resp. soft closed);
- (iii) a soft homeomorphism if it is bijective, soft continuous and soft open.

**Definition 2.26.** [13] A collection  $\{G_{i_E} : i \in I\}$  of soft open sets is called a soft open cover of  $(X, \tau, E)$  if  $\tilde{X} = \bigcup_{i \in I} G_{i_E}$ . And  $(X, \tau, E)$  is called soft compact if every soft open cover of  $\tilde{X}$  has a finite subcover.

**Proposition 2.27.** [13] Every soft closed subset  $H_E$  of a soft compact space is soft compact.

**Theorem 2.28.** [13] Let  $(X, \tau, A)$  and  $(Y, \theta, B)$  be two STSs. Let  $\Omega = \{G_A \times F_B : G_A \in \tau \text{ and } F_B \in \theta\}$ . Then the family of all arbitrary unions of elements of  $\Omega$  is a soft topology on  $X \times Y$ .

**Definition 2.29.** [1] A triple  $(X, \tau, \preceq)$  is said to be a topological ordered space, where  $\preceq$  and  $\tau$  are respectively a partial order relation and a topology on  $X \neq \emptyset$ .

**Definition 2.30.** [29] A quadrable system  $(X, \tau, E, \preceq)$  is said to be a soft topological ordered space, where  $(X, \tau, E)$  is a soft topological space and  $(X, \preceq)$  is a partially ordered set.

We will write from now on STOS instead of a soft topological ordered space.

**Definition 2.31.** [29] A soft subset  $W_E$  of  $(X, \tau, E, \preceq)$  is said to be an increasing (resp. a decreasing) soft neighborhood of  $x \in X$  if  $W_E$  is increasing (resp. decreasing) and a soft neighborhood of  $x \in X$ .

**Proposition 2.32.** [29] In  $(X, \tau, E, \preceq)$  we find that for each  $e \in E$ , the family  $\tau_e = \{G(e) : G_E \in \tau\}$  with a partial order relation  $\preceq$  form an ordered topology on  $X$ .

**Definition 2.33.** [29] Let  $Y \subseteq X$ . Then  $(Y, \tau_Y, \preceq_Y, E)$  is called a soft ordered subspace of  $(X, \tau, \preceq, E)$  provided that  $(Y, \tau_Y, E)$  is soft subspace of  $(X, \tau, E)$  and  $\preceq_Y = \preceq \cap Y \times Y$ .

**Lemma 2.34.** [29] If  $U_E$  is an increasing (resp. a decreasing) soft subset of  $(X, \tau, \preceq, E)$ , then  $U_E \cap \tilde{Y}$  is an increasing (resp. a decreasing) soft subset of a soft ordered subspace  $(Y, \tau_Y, \preceq_Y, E)$ .

**Definition 2.35.** [29] The product of a finite family of soft topological ordered spaces  $\{(X_i, \tau_i, \preceq_i, E_i) : i \in \{1, 2, \dots, n\}\}$  is an STOS  $(X, \tau, \preceq, E)$ , where  $X = \prod_{i=1}^n X_i$ ,  $\tau$  is the product soft topology on  $X$ ,  $E = \prod_{i=1}^n E_i$  and  $\preceq = \{(x, y) : x, y \in X\}$  such that  $(x_i, y_i) \in \preceq_i$  for every  $i$ .

**Lemma 2.36.** [29] If  $H_{E_1 \times E_2}$  is a decreasing (resp. an increasing) soft closed subset of a soft ordered product space  $(X \times Y, \tau_1 \times \tau_2, E_1 \times E_2, \preceq)$ , then  $H_{E_1 \times E_2} = [G_{E_1}^c \times \tilde{Y}] \cap [\tilde{X} \times F_{E_2}^c]$ , for some increasing (resp. decreasing) soft open sets  $G_{E_1} \in \tau_1$  and  $F_{E_2} \in \tau_2$ .

**Definition 2.37.** [29] A soft ordered subspace  $(Y, \tau_Y, \preceq_Y, E)$  of  $(X, \tau, \preceq, E)$  is called a soft compatibly ordered provided that for each increasing (resp. decreasing) soft closed subset  $H_E$  of  $(Y, \tau_Y, \preceq_Y, E)$ , there exists an increasing (resp. a decreasing) soft closed subset  $H_E^*$  of  $(X, \tau, \preceq, E)$  such that  $H_E = \tilde{Y} \cap H_E^*$ .

**Definition 2.38.** [29] A soft topological ordered property or soft ordered topological invariant is a property of a soft topological ordered space which is invariant under ordered embedding soft homeomorphism mappings.

**Theorem 2.39.** [29]  $(X, \tau, E, \preceq)$  is upper (resp. lower)  $p$ -soft  $T_1$ -ordered if and only if  $(i(x))_E$  (resp.  $(d(x))_E$ ) is soft closed, for all  $x \in X$ .

### 3 Ordered soft separation axioms

In this section, we formulate the concepts of soft  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ) by using monotone soft neighborhoods and establish some of their properties. With the help of illustrative examples, we elucidate the relationship between them; and the interrelations between them and their parametric topological ordered spaces.

**Definition 3.1.** A soft subset  $W_E$  of  $(X, \tau, E, \preceq)$  is said to be:

- (i) partially containing  $x$  provided that  $x \in W_E$ ;
- (ii) partially soft neighborhood of  $x$ , provided that there exists a soft open set  $G_E$  such that  $x \in F_E \subseteq W_E$ ;
- (iii) an increasing (resp. a decreasing) partially soft neighborhood of  $x \in X$  provided that  $W_E$  is an increasing (resp. a decreasing) and partially soft neighborhood of  $x$ .

The following example illustrates the above definition.

**Example 3.2.** Let  $E = \{e_1, e_2\}$  and  $\preceq = \Delta \cup \{(x, z)\}$  be a partial order relation on  $X = \{x, y, z\}$ . Then  $\tau = \{\emptyset, \tilde{X}, G_E\}$  is a soft topology on  $X$ , where  $G_E = \{(e_1, \emptyset), (e_2, \{x\})\}$ . Now, it can be noted that:

- (i) A soft set  $W_E = \{(e_1, \{y\}), (e_2, \{x\})\}$  partially contains  $x$  because  $x \in W_E$ . But  $W_E$  does not partially contain  $z$  because  $z \notin W_E$ ;
- (ii)  $W_E$  is a partially soft neighborhood of  $x$  because  $G_E$  is a soft open set such that  $x \in G_E \subseteq W_E$ ;
- (iii)  $W_E$  is a decreasing partially soft neighborhood of  $x$  because  $W_E$  is decreasing and partially soft neighborhood of  $x$ . On the other hand,  $i(W_E) = \{(e_1, \{y\}), (e_2, \{x, z\})\} \neq W_E$ . Then  $W_E$  is not increasing. Hence,  $W_E$  is not increasing partially soft neighborhood of  $x$ .

**Definition 3.3.**  $(X, \tau, E, \preceq)$  is said to be:

- (i) upper (resp. lower) soft  $T_1$ -ordered if for every  $x \neq y$  in  $X$ , there exists a decreasing (resp. an increasing) soft neighborhood  $W_E$  of  $y$  (resp.  $x$ ) such that  $x \notin W_E$  (resp.  $y \notin W_E$ );
- (iii) soft  $T_0$ -ordered if it is upper soft  $T_1$ -ordered or lower soft  $T_1$ -ordered;
- (iv) soft  $T_1$ -ordered if it is upper soft  $T_1$ -ordered and lower soft  $T_1$ -ordered;
- (v) soft  $T_2$ -ordered if for every  $x \neq y$  in  $X$ , there exist disjoint an increasing soft neighborhood  $W_E$  of  $x$  and a decreasing soft neighborhood  $V_E$  of  $y$ .

**Remark 3.4.** The definition of a  $p$ -soft  $T_2$ -ordered space in [29] reports that for every  $x \neq y$  in  $X$ , there exist two disjoint soft neighborhoods  $W_E$  and  $V_E$  containing  $x$  and  $y$ , respectively. This means that  $y \notin W_E$  and  $x \notin V_E$ . Since  $W_E$  and  $V_E$  are disjoint then  $y \notin W_E$  if and only if  $y \notin W_E$  and  $x \notin V_E$  if and only if  $x \notin V_E$ . So the definitions of soft  $T_2$ -ordered and  $p$ -soft  $T_2$ -ordered spaces are equivalent. Hence, all results concerning  $p$ -soft  $T_2$ -ordered spaces in [29] are still valid for soft  $T_2$ -ordered spaces.

**Proposition 3.5.** Every soft  $T_i$ -ordered space  $(X, \tau, E, \preceq)$  is soft  $T_{i-1}$ -ordered for  $i = 1, 2$ .

*Proof.* The proof follows immediately from Definition (3.3). □

To show that the converse of the above proposition is not always true, we give the following two examples.

**Example 3.6.** Let  $E = \{e_1, e_2\}$  be a set of parameters and  $\preceq = \Delta \cup \{(x, y), (x, z)\}$  be a partial order relation on  $X = \{x, y, z\}$ . We define the soft sets  $\{G_{i_E} : i = 1, 2, 3, 4\}$  as follows:

$$\begin{aligned} G_{1_E} &= \{(e_1, \{y\}), (e_2, \{x, y\})\}; \\ G_{2_E} &= \{(e_1, \{z\}), (e_2, \{x, z\})\}; \end{aligned}$$



$$G_{3_E} = \{(e_1, \{y, z\}), (e_2, X)\} \text{ and} \\ G_{4_E} = \{(e_1, \emptyset), (e_2, \{x\})\}.$$

Then  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, 3, 4\}$  forms a soft topology on  $X$ . Now, for  $y \not\leq x$  and  $y \not\leq z$ , we find that  $W_E = \{(e_1, \{y\}), (e_2, X)\}$  is an increasing soft neighborhood of  $y$  such that  $x \notin W_E$  and  $z \notin W_E$ . Also, for  $z \not\leq x$  and  $z \not\leq y$ , we find that  $W_E = \{(e_1, \{z\}), (e_2, X)\}$  is an increasing soft neighborhood of  $z$  such that  $x \notin W_E$  and  $y \notin W_E$ . Therefore  $(X, \tau, E, \preceq)$  is a lower soft  $T_1$ -ordered space. Hence, it is soft  $T_0$ -ordered. On the other hand, there does not exist a soft neighborhood  $W_E$  of  $x$  such that  $y \notin W_E$  or  $z \notin W_E$ . This means that it is not an upper soft  $T_1$ -ordered space. Hence,  $(X, \tau, E, \preceq)$  is not soft  $T_1$ -ordered.

**Example 3.7.** Let  $E = \{e_1, e_2\}$  be a set of parameters and  $\preceq = \Delta \cup \{(1, 2), (2, 3), (1, 3)\}$  be a partial order relation on  $X = \{1, 2, 3\}$ . The soft sets  $\{G_{i_E} : i = 1, 2, \dots, 9\}$  are defined as follows:

$$\begin{aligned} G_{1_E} &= \{(e_1, \{1\}), (e_2, \{2\})\}; \\ G_{2_E} &= \{(e_1, \{1\}), (e_2, \{1, 2\})\}; \\ G_{3_E} &= \{(e_1, \{1, 2\}), (e_2, \{2\})\}; \\ G_{4_E} &= \{(e_1, \{1, 2\}), (e_2, \{1, 2\})\}; \\ G_{5_E} &= \{(e_1, \{3\}), (e_2, X)\}; \\ G_{6_E} &= \{(e_1, \emptyset), (e_2, \{1, 2\})\}; \\ G_{7_E} &= \{(e_1, \emptyset), (e_2, \{2\})\}; \\ G_{8_E} &= \{(e_1, \emptyset), (e_2, \{1\})\} \text{ and} \\ G_{9_E} &= \{(e_1, \{1, 3\}), (e_2, X)\}. \end{aligned}$$

Then  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 9\}$  forms a soft topology on  $X$ . Now, for  $3 \not\leq 2$  and  $3 \not\leq 1$ , we find that  $G_{5_E}$  is an increasing soft neighborhood of 3 such that  $2 \notin G_{5_E}$  and  $1 \notin G_{5_E}$ ; and  $G_{4_E}$  is a decreasing soft neighborhood of 2 and 1 such that  $3 \notin G_{4_E}$ . Also, for  $2 \not\leq 1$ , we find that  $W_E = \{(e_1, X), (e_2, \{2, 3\})\}$  is an increasing soft neighborhood of 2 and  $G_{2_E}$  is a decreasing soft neighborhood of 1 such that  $1 \notin W_E$  and  $2 \notin G_{2_E}$ . Therefore,  $(X, \tau, E, \preceq)$  is soft  $T_1$ -ordered. In contrast, any soft open set containing 1 intersects any soft open set containing 2. Hence  $(X, \tau, E, \preceq)$  is not soft  $T_2$ -ordered.

**Proposition 3.8.** If  $a$  is the smallest element of a finite upper soft  $T_1$ -ordered space  $(X, \tau, E, \preceq)$ , then there is a decreasing soft neighborhood  $W_E$  of  $a$  such that  $y \notin W_E$  for each  $y \in X \setminus \{a\}$ .

*Proof.* Let  $a$  be the smallest element in  $(X, \preceq)$ . Then  $a \preceq x$  for all  $x \in X$ . Since  $\preceq$  is anti-symmetric, then  $x \not\preceq a$  for all  $x \in X$ . Therefore, there exists a decreasing soft neighborhood  $W_E$  of  $a$  such that  $x \notin W_E$ . Since  $X$  is finite, then  $\bigcap W_E$  is a decreasing soft neighborhood of  $a$  such that  $y \notin W_E$  for each  $y \in X \setminus \{a\}$ .  $\square$

**Proposition 3.9.** If  $a$  is the largest element of a finite lower soft  $T_1$ -ordered space  $(X, \tau, E, \preceq)$ , then there is an increasing soft neighborhood  $W_E$  of  $a$  such that  $y \notin W_E$  for each  $y \in X \setminus \{a\}$ .

*Proof.* The proof is similar to that of Proposition (3.8).  $\square$

**Proposition 3.10.** If  $(X, \tau, E, \preceq)$  is a finite soft  $T_1$ -ordered space, then for each  $x \in X$  there is a soft open set  $G_E$  containing  $x$  such that  $y \notin G_E$  for each  $y \in X \setminus \{x\}$ .

*Proof.* Let  $x \in X$ . Since  $\preceq$  is anti-symmetric, then  $x \not\preceq y$  or  $y \not\preceq x$  for all  $y \in X$ . By hypothesis, there exists an increasing soft neighborhood  $W_E$  of  $x$  or a decreasing soft neighborhood  $W_E$  of  $x$  such that  $y \notin W_E$ . Since  $X$  is finite, then  $\bigcap W_E$  is a soft neighborhood of  $x$  such that  $y \notin W_E$  for each  $y \in X \setminus \{x\}$ . Hence, there is a soft open set  $G_E$  such that  $x \in G_E \subseteq \bigcap W_E$ .  $\square$

**Theorem 3.11.**  $(X, \tau, E, \preceq)$  is soft  $T_2$ -ordered if and only if for all  $(x, y) \notin \preceq$  there are soft open sets  $U_E$  and  $V_E$  containing  $x$  and  $y$ , respectively, such that  $(a, b) \notin \preceq$  for every  $a \in U_E$  and  $b \in V_E$ .

*Proof.* From Remark (3.4) and Theorem 4.13 in [29], the proof follows.  $\square$

**Proposition 3.12.** *If  $(X, \tau, E, \preceq)$  is soft  $T_2$ -ordered, then all parametric topological ordered spaces  $(X, \tau_e, \preceq)$  are  $T_2$ -ordered.*

*Proof.* The proof follows from Remark (3.4) and Proposition 4.15 in [29].  $\square$

**Corollary 3.13.** *The minimum number of soft open subsets of a finite soft  $T_2$ -ordered space  $(X, \tau, E, \preceq)$  is  $2^{|X|}$  soft open sets.*

The next example clarifies that the converse of Proposition (3.12) fails.

**Example 3.14.** *Consider a partial order relation  $\preceq = \Delta \cup \{(x, y)\}$  on  $X = \{x, y\}$  and let  $E = \{e_1, e_2\}$  be a parameters set. The collection  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2\}$  is a soft topology on  $X$ , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, \{y\})\} \text{ and} \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \{x\})\}. \end{aligned}$$

*Obviously,  $(X, \tau_{e_1}, \preceq)$  and  $(X, \tau_{e_2}, \preceq)$  are  $T_2$ -ordered spaces; notwithstanding,  $(X, \tau, E, \preceq)$  is not a soft  $T_0$ -ordered space.*

Also, the example below combined with the above example illustrate that the soft  $T_i$ -ordered spaces ( $i = 0, 1$ ) and their parametric topological ordered spaces are independent of each other.

**Example 3.15.** *Consider a partial order relation  $\preceq = \Delta \cup \{(x, y)\}$  on  $X = \{x, y, z\}$  and let  $E = \{e_1, e_2, e_3\}$  be a parameters set. The collection  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 7\}$  is a soft topology on  $X$ , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, X), (e_3, X)\}; \\ G_{2_E} &= \{(e_1, X), (e_2, \{y\}), (e_3, X)\}; \\ G_{3_E} &= \{(e_1, X), (e_2, X), (e_3, \{z\})\}; \\ G_{4_E} &= \{(e_1, \{x\}), (e_2, \{y\}), (e_3, X)\}; \\ G_{5_E} &= \{(e_1, \{x\}), (e_2, X), (e_3, \{z\})\}; \\ G_{6_E} &= \{(e_1, X), (e_2, \{y\}), (e_3, \{z\})\} \text{ and} \\ G_{7_E} &= \{(e_1, \{x\}), (e_2, \{y\}), (e_3, \{z\})\}. \end{aligned}$$

*Obviously,  $(X, \tau_{e_1}, \preceq)$ ,  $(X, \tau_{e_2}, \preceq)$  and  $(X, \tau_{e_3}, \preceq)$  are not  $T_0$ -ordered spaces; notwithstanding,  $(X, \tau, E, \preceq)$  is soft  $T_1$ -ordered.*

**Proposition 3.16.** *The property of being a soft  $T_i$ -ordered space is a soft hereditary property for  $i = 0, 1, 2$ .*

*Proof.* From Remark (3.4) and Theorem 4.19 in [29], we obtain the proof in the case of  $i = 2$ .

To prove the proposition in the case of  $i = 1$ , let  $(Y, \tau_Y, E, \preceq_Y)$  be a soft ordered subspace of a soft  $T_1$ -ordered space  $(X, \tau, E, \preceq)$ . For every  $a \not\preceq_Y b \in Y$ , we have  $a \not\preceq b$ . Therefore, there is an increasing soft neighborhood  $W_E$  of  $a$  and a decreasing soft neighborhood  $V_E$  of  $b$ . By setting  $U_E = \tilde{Y} \tilde{\cap} W_E$  and  $G_E = \tilde{Y} \tilde{\cap} V_E$ , we find from Lemma (2.34) that  $U_E$  is an increasing soft neighborhood of  $a$  and  $G_E$  is a decreasing soft neighborhood of  $b$  such that  $b \notin U_E$  and  $a \notin G_E$ . Thus,  $(Y, \tau_Y, \preceq_Y, E)$  is soft  $T_1$ -ordered.

The proof in the case of  $i = 0$  can be done similarly.  $\square$

**Proposition 3.17.** *Every  $p$ -soft  $T_i$ -ordered space is soft  $T_i$ -ordered for  $i = 0, 1$ .*

*Proof.* The proof is complete by observing that  $x \notin G_E$  implies that  $x \notin G_E$  for every  $G_E \subseteq \tilde{X}$ .  $\square$

**Corollary 3.18.**  *$(X, \tau, E, \preceq)$  is upper (resp. lower) soft  $T_1$ -ordered if  $(i(x))_E$  (resp.  $(d(x))_E$ ) is a soft closed set for each  $x \in X$ .*



*Proof.* If  $(i(x))_E$  is a soft closed set for each  $x \in X$ , then it follows from Theorem (2.39) that  $(X, \tau, E, \preceq)$  is upper  $p$ -soft  $T_1$ -ordered. Hence, it is upper soft  $T_1$ -ordered.  $\square$

**Remark 3.19.** It can be seen that the given STOS in Example (3.6) is soft  $T_0$ -ordered, but is not  $p$ -soft  $T_0$ -ordered, since there does not exist a soft neighborhood  $W_E$  of  $y$  such that  $x \notin W_E$ . Also, it can be noted that the given STOS in Example (3.7) is soft  $T_1$ -ordered, but is not  $p$ -soft  $T_1$ -ordered, since there does not exist a soft neighborhood  $W_E$  of 2 such that  $1 \notin W_E$ . Hence, the converse of the above proposition fails.

**Definition 3.20.**  $(X, \tau, E, \preceq)$  is said to be:

- (i) upper (resp. lower) soft regularly ordered if for each increasing (resp. decreasing) soft closed set  $H_E$  and  $x \in X$  such that  $x \notin H_E$ , there exist disjoint soft neighbourhoods  $W_E$  of  $H_E$  and  $V_E$  of  $x$  such that  $W_E$  is increasing (resp. decreasing) and  $V_E$  is decreasing (resp. increasing);
- (ii) soft regularly ordered if it is both upper soft regularly ordered and lower soft regularly ordered;
- (iii) upper (resp. lower) soft  $T_3$ -ordered if it is both upper (resp. lower) soft  $T_1$ -ordered and upper (resp. lower) soft regularly ordered;
- (iv) soft  $T_3$ -ordered if it is both upper soft  $T_3$ -ordered and lower soft  $T_3$ -ordered;
- (v) soft normal ordered if for every two disjoint an increasing soft closed set  $H_{1E}$  and a decreasing soft closed set  $H_{2E}$  there exist a disjoint increasing soft neighbourhood  $V_E$  of  $H_{1E}$  and a decreasing soft neighbourhood  $W_E$  of  $H_{2E}$ ;
- (vi) soft  $T_4$ -ordered if it is soft normally ordered and soft  $T_1$ -ordered.

**Example 3.21.** Let  $(X, \tau, E, \preceq)$  be the soft indiscrete topological space, where  $E$  is an arbitrary set of parameters  $E$  and  $\preceq$  is any partial order relation on  $X$ . Then  $(X, \tau, E, \preceq)$  is lower (upper) soft regularly ordered. So it is soft regularly ordered. Also it is soft normal. But it is not lower (upper) soft  $T_3$ -ordered if  $|X| \geq 2$ . So it is not soft  $T_3$ -ordered. Moreover, it is not soft  $T_4$ -ordered. On the other hand, if  $(X, \tau, E, \preceq)$  is the soft discrete topological space, then it is soft  $T_i$ -ordered for  $i = 3, 4$ .

**Theorem 3.22.**  $(X, \tau, E, \preceq)$  is upper (resp. lower) soft regularly ordered if and only if for all  $x \in X$  and every decreasing (resp. increasing) soft open set  $U_E$  partially containing  $x$ , there is a decreasing (resp. an increasing) partially soft neighbourhood  $V_E$  of  $x$  satisfying  $\overline{V_E} \subseteq U_E$ .

*Proof.* We prove the theorem in the lower soft regularly ordered case. The other case follows similar manner.

*Necessity:* Suppose that  $U_E$  is an increasing soft open set such that  $x \in U_E$ . Then  $U_E^c$  is a decreasing soft closed set and  $x \notin U_E^c$ . Therefore, there is an increasing soft neighbourhood  $V_E$  of  $x$  and a decreasing soft neighbourhood  $W_E$  of  $U_E^c$  such that  $V_E \cap W_E = \tilde{\Phi}$ . Thus, there exists a soft open set  $G_E$  such that  $U_E^c \subseteq G_E \subseteq W_E$ . Since  $V_E \subseteq W_E^c$ , then  $V_E \subseteq W_E^c \subseteq G_E^c \subseteq U_E$  and since  $G_E^c$  is soft closed, then  $\overline{V_E} \subseteq G_E^c \subseteq U_E$ .

*Sufficiency:* Let  $H_E$  be a decreasing soft closed set and  $x \notin H_E$ . Then  $H_E^c$  is an increasing soft open set such that  $x \in H_E^c$ . Therefore, there exists an increasing soft neighbourhood  $V_E$  of  $x$  satisfying  $\overline{V_E} \subseteq H_E^c$ . Obviously,  $H_E \subseteq (\overline{V_E})^c$  and  $(\overline{V_E})^c$  is soft open. Now,  $d(\overline{V_E})^c$  is a decreasing soft neighbourhood of  $H_E$ . Suppose that  $V_E \cap d((\overline{V_E})^c) \neq \tilde{\Phi}$ . Then there are  $x \in X$  and  $e \in E$  such that  $x \in V(e)$  and  $x \in d((\overline{V})^c(e))$ . This implies that there is  $y \in (\overline{V})^c(e)$  such that  $x \preceq y$ . This means that  $y \in V(e)$ . But this contradicts the disjointness of  $V_E$  and  $(\overline{V_E})^c$ . Hence,  $V_E \cap d((\overline{V_E})^c) = \tilde{\Phi}$ .  $\square$

**Proposition 3.23.** Every increasing (decreasing) soft closed or soft open subset of a soft regularly ordered space  $(X, \tau, E, \preceq)$  is stable.

*Proof.* Without loss of generality, suppose that  $H_E$  is an increasing soft closed set in a soft regularly ordered space  $(X, \tau, E, \preceq)$  which is not stable. Then there exists  $x \in X$  and  $\alpha, \beta \in E$  such that  $x \in H(\alpha)$  and  $x \notin H(\beta)$ . This means that  $x \notin H_E$ . So for any soft neighborhood  $W_E$  of  $x$  and any soft neighborhood  $V_E$  of  $H_E$ , we obtain

that  $x \in W(\alpha) \cap V(\alpha)$ . Thus, we cannot find disjoint soft neighborhoods of  $x$  and  $H_E$ . This is a contradiction with soft regularly ordered of  $(X, \tau, E, \preceq)$ . Hence,  $H_E$  must be stable.

The proof of the decreasing case can be done similarly.  $\square$

**Corollary 3.24.** *If all increasing (decreasing) soft closed or soft open subsets in  $(X, \tau, E, \preceq)$  are stable, then  $(X, \tau, E, \preceq)$  is  $p$ -soft regularly ordered if and only if it is soft regularly ordered.*

**Proposition 3.25.** *Every soft regularly ordered space is  $p$ -soft regularly ordered.*

*Proof.* Straightforward.  $\square$

The example below shows that the converse of Proposition (3.25) does not hold in general.

**Example 3.26.** *We define the soft sets  $\{G_{i_E} : i = 1, 2, 3, 4\}$  over  $X = \{x, y\}$  with a parameters set  $E = \{e_1, e_2\}$  as follows:*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, \{x\})\}; \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \{y\})\}; \\ G_{3_E} &= \{(e_1, \{y\}), (e_2, \emptyset)\} \text{ and} \\ G_{4_E} &= \{(e_1, X), (e_2, \{x\})\}. \end{aligned}$$

Then  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, 3, 4\}$  is a soft topology on  $X$ . Let  $\preceq = \triangle \cup \{(x, y)\}$  be a partial order relation on  $X = \{x, y\}$ . It can be verified that  $(X, \tau, E, \preceq)$  is a  $p$ -soft regularly ordered space. But it is not soft regularly ordered because an increasing soft open set  $G_{3_E}$  is not stable.

**Remark 3.27.** *In the following we point out that the concepts of soft  $T_3$ -ordered and soft  $T_4$ -ordered spaces are independent of each other.*

- (i) *The given STOS in Example(3.26) is soft  $T_2$ -ordered and soft  $T_4$ -ordered, but it is not soft  $T_3$ -ordered;*
- (ii) *If we consider  $(X, \tau, E, \preceq)$  is STOS such that  $E$  is a singleton set, then  $(X, \tau, E, \preceq)$  is a topological ordered space. So Example 7 in [2] shows that a soft  $T_4$ -ordered space is a proper extension of a soft  $T_3$ -ordered space.*

The following two problems are still open.

**Problem 3.28.** *Is a soft  $T_3$ -ordered space a soft  $T_2$ -ordered space?*

**Problem 3.29.** *Is a soft  $T_3$ -ordered space a  $p$ -soft  $T_3$ -ordered space?*

**Proposition 3.30.** *Every  $p$ -soft  $T_4$ -ordered space  $(X, \tau, E, \preceq)$  is soft  $T_4$ -ordered.*

*Proof.* Straightforward.  $\square$

The converse of Proposition 3.30 fails. We show this in the next example.

**Example 3.31.** *Let  $E = \{e_1, e_2, e_3\}$  and  $\preceq = \triangle \cup \{(x, y)\}$  be a partial order relation on  $X = \{x, y\}$ . The following six soft sets defined as follows.*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}X), (e_2, X), (e_3, X)\}; \\ G_{2_E} &= \{(e_1, \{y\}X), (e_2, X), (e_3, X)\}; \\ G_{3_E} &= \{(e_1, \emptyset), (e_2, X), (e_3, X)\}; \\ G_{4_E} &= \{(e_1, \{x\}), (e_2, \emptyset), (e_3, \emptyset)\}; \\ G_{5_E} &= \{(e_1, \{y\}), (e_2, \emptyset), (e_3, \emptyset)\} \text{ and} \\ G_{6_E} &= \{(e_1, X), (e_2, \emptyset), (e_3, \emptyset)\}. \end{aligned}$$

The collection  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 6\}$  is a soft topology on  $X$ . It can be easily verified that  $(X, \tau, E, \preceq)$  is soft  $T_4$ -ordered. In contrast, we cannot find a soft open set containing  $y$  such that  $x$  does not totally belong to it. Therefore,  $(X, \tau, E, \preceq)$  fails to satisfy a condition of a  $p$ -soft  $T_1$ -ordered space. Thus,  $(X, \tau, E, \preceq)$  is not  $p$ -soft  $T_4$ -ordered.

**Theorem 3.32.** Every soft compatibly ordered subspace  $(Y, \tau_Y, E, \preceq_Y)$  of a soft regularly ordered space  $(X, \tau, E, \preceq)$  is soft regularly ordered.

*Proof.* Suppose that  $H_E$  is an increasing soft closed subset of  $(Y, \tau_Y, E, \preceq_Y)$  such that  $y \notin H_E$ . Because  $(Y, \tau_Y, E, \preceq_Y)$  is soft compatibly ordered subspace of  $(X, \tau, E, \preceq)$ , then there is an increasing soft closed set  $H_E^*$  in  $(X, \tau, E, \preceq)$  such that  $H_E = \tilde{Y} \cap H_E^*$ . By hypothesis, we have a decreasing soft neighborhood  $V_E$  of  $y$  and an increasing soft neighborhood  $W_E$  of  $H_E^*$  such that  $V_E \cap W_E = \tilde{\Phi}$ . From Lemma (2.34) we obtain  $\tilde{Y} \cap V_E$  is a decreasing soft neighborhood of  $y$  and  $\tilde{Y} \cap W_E$  is an increasing soft neighborhood of  $H_E$  in  $(Y, \tau_Y, E, \preceq_Y)$ . The disjointness of  $\tilde{Y} \cap V_E$  and  $\tilde{Y} \cap W_E$  completes the proof that  $(Y, \tau_Y, E, \preceq_Y)$  is upper soft regularly ordered.

In a similar manner it can be proved that  $(Y, \tau_Y, E, \preceq_Y)$  is lower soft regularly ordered. Hence,  $(Y, \tau_Y, E, \preceq_Y)$  is soft regularly ordered.  $\square$

**Corollary 3.33.** Every soft compatibly ordered subspace  $(Y, \tau_Y, E, \preceq_Y)$  of a soft  $T_3$ -ordered space  $(X, \tau, E, \preceq)$  is soft  $T_3$ -ordered.

The proof of the next proposition is easy and thus it is omitted.

**Proposition 3.34.** Every soft closed compatibly ordered subspace of a soft  $T_4$ -ordered space is soft  $T_4$ -ordered.

**Theorem 3.35.** The finite product of soft  $T_i$ -ordered spaces is soft  $T_i$ -ordered for  $i = 0, 1, 2, 3$ .

*Proof.* We only prove the theorem in the case of  $i = 2$ , and the other cases can be proved similarly.

Assume that  $(X, \tau_1, E_1, \preceq_1)$  and  $(Y, \tau_2, E_2, \preceq_2)$  are soft  $T_2$ -ordered spaces and let  $(X \times Y, \tau, E, \preceq)$  be the soft ordered product space of them. Let  $(x_1, y_1) \not\preceq (x_2, y_2) \in X \times Y$ . Then  $x_1 \not\preceq_1 x_2$  or  $y_1 \not\preceq_2 y_2$ . Without loss of generality, say  $x_1 \not\preceq_1 x_2$ . Since  $(X, \tau_1, E_1, \preceq_1)$  is soft  $T_2$ -ordered, then there is an increasing soft neighborhood  $W_{E_1}$  of  $x_1$  and a decreasing soft neighborhood  $V_{E_1}$  of  $x_2$  such that  $x_2 \notin W_{E_1}$  and  $x_1 \notin V_{E_1}$  which are disjoint. Therefore,  $W_{E_1} \times \tilde{Y}$  is an increasing soft neighborhood of  $(x_1, y_1)$  and  $V_{E_1} \times \tilde{Y}$  is a decreasing soft neighborhood of  $(x_2, y_2)$  such that  $(x_2, y_2) \notin [W_{E_1} \times \tilde{Y}]$  and  $(x_1, y_1) \notin [V_{E_1} \times \tilde{Y}]$ . The disjointness of  $W_{E_1} \times \tilde{Y}$  and  $V_{E_1} \times \tilde{Y}$  finishes the proof that  $(X \times Y, \tau, E, \preceq)$  is soft  $T_2$ -ordered.  $\square$

**Theorem 3.36.** The property of being a soft  $T_i$ -ordered space is a soft topological ordered property for  $i = 0, 1, 2, 3, 4$ .

*Proof.* We only prove the theorem in the cases of  $i = 2, 4$ , and the other cases can be proved similarly.

(i) Let  $f_\phi : (X, \tau, A, \preceq_1) \rightarrow (Y, \theta, B, \preceq_2)$  be an ordered embedding soft homeomorphism map such that  $(X, \tau, A, \preceq_1)$  is soft  $T_2$ -ordered. Suppose that  $x \not\preceq_2 y \in Y$ . Then  $P_\beta^x \not\preceq_2 P_\beta^y$  for each  $\beta \in B$ . Since  $f_\phi$  is bijective, then there are  $P_\alpha^a$  and  $P_\alpha^b$  in  $\tilde{X}$  such that  $f_\phi(P_\alpha^a) = P_\beta^x$  and  $f_\phi(P_\alpha^b) = P_\beta^y$  and since  $f_\phi$  is an ordered embedding, then  $P_\alpha^a \not\preceq_1 P_\alpha^b$ . So  $a \not\preceq_1 b$ . By hypothesis, we have an increasing soft neighborhood  $V_E$  of  $a$  and a decreasing soft neighborhood  $W_E$  of  $b$  such that  $V_E \cap W_E = \tilde{\Phi}$ . Since  $f_\phi$  is bijective soft open, then  $f_\phi(V_E)$  and  $f_\phi(W_E)$  are disjoint soft neighborhoods of  $x$  and  $y$ , respectively. From Theorem (2.21), we obtain  $f_\phi(V_E)$  and  $f_\phi(W_E)$  are increasing and decreasing, respectively. Hence, the proof is complete.

(ii) Let  $f_\phi : (X, \tau, A, \preceq_1) \rightarrow (Y, \theta, B, \preceq_2)$  be an ordered embedding soft homeomorphism map such that  $(X, \tau, A, \preceq_1)$  is soft normally ordered. Suppose that the two disjoint soft closed sets  $H_E$  and  $F_E$  are increasing and decreasing, respectively. Since  $f_\phi$  is bijective soft continuous, then  $f_\phi^{-1}(H_E)$  and  $f_\phi^{-1}(F_E)$  are disjoint soft closed sets and since  $f_\phi$  is ordered embedding, then  $f_\phi^{-1}(H_E)$  is increasing and  $f_\phi^{-1}(F_E)$  is decreasing. By hy-

pothesis, there are disjoint soft neighborhoods  $V_E$  and  $W_E$  of  $f_\phi^{-1}(H_E)$  and  $f_\phi^{-1}(F_E)$ , respectively, such that  $V_E$  is increasing and  $W_E$  is decreasing. So  $H_E \subseteq f_\phi(V_E)$  and  $F_E \subseteq f_\phi(W_E)$ . It follows by Theorem (2.21) that  $f_\phi(V_E)$  is increasing and  $f_\phi(W_E)$  is decreasing. The disjointness of the soft neighborhoods  $f_\phi(V_E)$  and  $f_\phi(W_E)$  finishes the proof.  $\square$

We devote the rest of this section to investigate some findings that associate some given ordered soft separation axioms with soft compactness.

**Lemma 3.37.** *Let  $F_E$  be an increasing (resp. a decreasing) soft open set in a soft regularly ordered space. Then for each  $P_e^x \in F_E$  there exists an increasing (resp. a decreasing) soft neighborhood  $G_E$  of  $P_e^x$  such that  $P_e^x \in \overline{G_E} \subseteq F_E$ .*

*Proof.* Suppose that  $F_E$  is an increasing soft open set such that  $P_e^x \in F_E$ . Then  $x \notin F_E^c$ . Since  $(X, \tau, E, \preceq)$  is soft regularly ordered, then there exist an increasing soft neighborhood  $G_E$  of  $x$  and a decreasing soft neighborhood  $W_E$  of  $F_E^c$  which are disjoint. This automatically means there are soft open sets  $H_E$  and  $L_E$  such that  $x \in H_E \subseteq G_E$  and  $F_E^c \subseteq L_E \subseteq W_E$ . Thus,  $x \in G_E \subseteq W_E^c \subseteq L_E^c \subseteq F_E$ . Hence,  $P_e^x \in \overline{G_E} \subseteq L_E^c \subseteq F_E$ .

The decreasing case can be proved in a similar manner.  $\square$

**Theorem 3.38.** *Let  $H_E$  be an increasing (resp. a decreasing) soft compact set in a soft regularly ordered space and  $F_E$  be a decreasing (resp. an increasing) soft open set containing  $H_E$ . Then there exists a decreasing (resp. an increasing) soft neighborhood  $G_E$  of  $H_E$  such that  $\overline{G_E} \subseteq F_E$ .*

*Proof.* Suppose that the given conditions are satisfied. Then for each  $P_e^x \in H_E$ , we have  $P_e^x \in F_E$ . Therefore, there is a decreasing soft neighborhood  $W_{i_e}$  of  $P_e^x$  such that  $\overline{W_{i_e}} \subseteq F_E$ . Thus, there is a soft open set  $V_{i_e}$  containing  $P_e^x$  such that  $\overline{V_{i_e}} \subseteq \overline{W_{i_e}} \subseteq F_E$ . Now, the collection  $\{V_{i_e} : P_e^x \in F_E\}$  of soft open sets containing  $P_e^x$  forms a soft open cover of  $H_E$ . Since  $H_E$  is soft compact, then  $H_E \subseteq \bigcup_{i=1}^n V_{i_e} \subseteq \bigcup_{i=1}^n \overline{W_{i_e}}$ . Let  $G_E = \bigcup_{i=1}^n W_{i_e}$ . This is a decreasing soft neighborhood of  $H_E$ . Obviously,  $H_E \subseteq G_E \subseteq \overline{G_E} \subseteq F_E$ .

A similar proof can be given for decreasing case.  $\square$

**Corollary 3.39.** *Every soft compact and soft regularly ordered space  $(X, \tau, E, \preceq)$  is soft normally ordered.*

*Proof.* Suppose that  $F_{1_E}$  and  $F_{2_E}$  are two disjoint soft closed sets such that  $F_{1_E}$  is decreasing and  $F_{2_E}$  is increasing. Then  $F_{2_E} \subseteq F_{1_E}^c$ . Since  $(X, \tau, E, \preceq)$  is soft compact, then  $F_{2_E}$  is soft compact and since  $(X, \tau, E, \preceq)$  is soft regularly ordered, then there is an increasing soft neighborhood  $G_E$  of  $F_{1_E}^c$  such that  $F_{2_E} \subseteq G_E \subseteq \overline{G_E} \subseteq F_{1_E}^c$ . Obviously,  $F_{1_E} \subseteq (\overline{G_E})^c$  and  $G_E \cap (\overline{G_E})^c = \emptyset$ . To prove that  $G_E \cap d[(\overline{G_E})^c] = \emptyset$ , suppose that there exists an element  $x \in G_E$  and  $x \in d[(\overline{G_E})^c]$ . So there exists an element  $y \in (\overline{G_E})^c$  such that  $x \preceq y$ . This means that  $y \in G_E$ . But this contradicts the disjointness of  $G_E$  and  $(\overline{G_E})^c$ . Thus,  $(X, \tau, E, \preceq)$  is soft normally ordered.  $\square$

To show that the converse of the above theorem and corollary fail we give the following example.

**Example 3.40.** *Consider a partial order relation  $\preceq = \triangle$  which is the equality relation on  $X = \{x, y\}$  and let  $E = \{e_1, e_2\}$  be a parameters set. The collection  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_e} : i = 1, 2\}$  is a soft topology on  $X$ , where*

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, X)\} \text{ and} \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \emptyset)\}. \end{aligned}$$

Obviously,  $(X, \tau, E, \preceq)$  is soft normally ordered and soft compact. Also, for every increasing (resp. decreasing) soft compact subset of  $(X, \tau, E, \preceq)$  and every decreasing (resp. increasing) soft open set  $F_E$  containing  $H_E$ , there exists a decreasing (resp. an increasing) soft neighborhood  $G_E$  of  $H_E$  such that  $\overline{G_E} \subseteq F_E$ . On the other hand, since the soft open sets  $G_{1_E}$  and  $G_{2_E}$  are not stable, then it follows from Proposition (3.23) that  $(X, \tau, E, \preceq)$  is not soft regularly ordered.

## 4 Strong ordered soft separation axioms

The first aim of this section is to define strong ordered soft separation axioms, namely strong soft  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ) by using monotone soft open sets in the place of monotone soft neighborhoods. The second aim is to provide some examples to illustrate the relationships between these and the relationships between them and soft  $T_i$ -ordered spaces. The third aim is to discuss their main properties and provide some results that associate soft compactness and some initiated strong ordered soft separation axioms.

The following example explains the difference between soft open sets and soft neighborhoods in terms of increasing and decreasing.

**Example 4.1.** Let  $E = \{e_1, e_2\}$  and  $\preceq = \Delta \cup \{(x, z), (y, w)\}$  be a partial order relation on  $X = \{a, x, y, w, z\}$ . The collection  $\tau = \{\tilde{\Phi}, \tilde{X}, G_E\}$  is a soft topology on  $X$ , where  $G_E = \{(e_1, \{a, x\}), (e_2, \{a, w\})\}$ . Now, it can be noted that  $G_E$  is a soft open set containing  $a$  such that  $i(G_E) = \{(e_1, \{a, x, z\}), (e_2, \{a, w\})\} \neq G_E$  and  $d(G_E) = \{(e_1, \{a, x\}), (e_2, \{a, y, w\})\} \neq G_E$ . So that  $G_E$  is neither increasing, nor decreasing. On the other hand,  $W_E = \{(e_1, \{a, x, z\}), (e_2, \{a, y, w\})\}$  is a monotone soft neighborhood of  $a$  because:

- (i)  $a \in G_E \subseteq W_E$  and
- (ii)  $i(W_E) = W_E$  and  $d(W_E) = W_E$ .

Also,  $U_E = \{(e_1, \{a, x, z\}), (e_2, \{a, w\})\}$  is an increasing soft neighborhood of  $a$ , but it is not decreasing and  $V_E = \{(e_1, \{a, x\}), (e_2, \{a, y, w\})\}$  is a decreasing soft neighborhood of  $a$ , but it is not increasing.

### Proposition 4.2.

- (i) Every monotone soft open set containing an element  $x$  is a monotone soft neighborhood of  $x$ .
- (ii) Every monotone soft open set containing a soft set  $H_E$  is a monotone soft neighborhood of  $H_E$ .

*Proof.* Let  $G_E$  be a monotone soft open set containing an element  $x$ . Then  $x \in G_E \subseteq G_E$ . Therefore,  $G_E$  is a monotone soft neighborhood of  $x$ . Also, if  $G_E$  is a monotone soft open set containing a soft set  $H_E$ . Then  $H_E \subseteq G_E \subseteq G_E$ . Therefore,  $G_E$  is a monotone soft neighborhood of  $H_E$ .  $\square$

Example (4.1) demonstrates that the converse of the above proposition fails.

**Definition 4.3.**  $(X, \tau, E, \preceq)$  is said to be:

- (i) strong upper (resp. strong lower) soft  $T_1$ -ordered if for every  $x \not\preceq y$  in  $X$ , there exists a decreasing (resp. an increasing) soft open set  $W_E$  containing  $y$  (resp.  $x$ ) such that  $x \notin W_E$  (resp.  $y \notin W_E$ );
- (ii) strong soft  $T_0$ -ordered if it is strong upper soft  $T_1$ -ordered or strong lower soft  $T_1$ -ordered;
- (iii) strong soft  $T_1$ -ordered if it is strong upper soft  $T_1$ -ordered and strong lower soft  $T_1$ -ordered;
- (iv) strong soft  $T_2$ -ordered if for every  $x \not\preceq y$  in  $X$ , there exist disjoint an increasing soft open set  $W_E$  containing  $x$  and a decreasing soft open set  $V_E$  containing  $y$ ;
- (v) strong upper (resp. strong lower) soft regularly ordered if for each increasing (resp. decreasing) soft closed set  $H_E$  and  $x \in X$  such that  $x \notin H_E$ , there exist disjoint soft open sets  $W_E$  containing  $H_E$  and  $V_E$  containing  $x$  such that  $W_E$  is increasing (resp. decreasing) and  $V_E$  is decreasing (resp. increasing);
- (vi) strong soft regularly ordered if it is both strong upper soft regularly ordered and strong lower soft regularly ordered;
- (vii) strong upper (resp. strong lower) soft  $T_3$ -ordered if it is both strong upper (resp. strong lower) soft  $T_1$ -ordered and strong upper (resp. strong lower) soft regularly ordered;
- (viii) strong soft  $T_3$ -ordered if it is both strong upper soft  $T_3$ -ordered and strong lower soft  $T_3$ -ordered;

- (ix) strong soft normally ordered if for each disjoint soft closed sets  $F_E$  and  $H_E$  such that  $F_E$  is increasing and  $H_E$  is decreasing, there exist disjoint soft open sets  $G_E$  containing  $F_E$  and  $U_E$  containing  $H_E$  such that  $G_E$  is increasing and  $U_E$  is decreasing;
- (x) strong soft  $T_4$ -ordered if it is strong soft normally ordered and strong soft  $T_1$ -ordered.

**Proposition 4.4.** Every strong soft  $T_i$ -ordered space is soft  $T_i$ -ordered for  $i = 0, 1, 2, 3, 4$ .

*Proof.* The proof follows from the fact that every monotone soft open set containing an element  $x$  is a monotone soft neighborhood of  $x$  and every monotone soft open set containing a soft set  $H_E$  is a monotone soft neighborhood of  $H_E$ .  $\square$

In what follows, we construct two examples to point out that the converse of the above proposition fails in the cases of  $i = 0, 1$ . The other cases are still open problems.

**Example 4.5.** Let  $(X, \tau, E, \preceq)$  be the same as in Example (3.6). We point out that this STOS is soft  $T_0$ -ordered. However, it is not strong soft  $T_0$ -ordered, because  $y \not\preceq z$  and there does not exist an increasing soft open set  $G_E$  containing  $x$  such that  $z \notin G_E$ .

**Example 4.6.** Let  $(X, \tau, E, \preceq)$  be the same as in Example (3.7). We point out that this STOS is soft  $T_1$ -ordered. However, it is not strong soft  $T_1$ -ordered, because  $2 \not\preceq 1$  and there does not exist an increasing soft open set  $G_E$  containing 2 such that  $1 \notin G_E$ .

**Problem 4.7.** Is a soft  $T_i$ -ordered space a strong soft  $T_i$ -ordered space for  $i = 2, 3, 4$ ?

**Theorem 4.8.** Let  $(X, \tau, E, \preceq)$  be strong soft regularly ordered. Then  $(X, \tau, E, \preceq)$  is  $p$ -soft  $T_1$ -ordered if and only if it is strong soft  $T_1$ -ordered.

*Proof.* To prove the "if" part, let  $x \not\preceq y$ . Then it follows from Proposition (2.39) that  $(i(x))_E$  and  $(d(y))_E$  are soft closed sets. So  $[(d(y))_E]^c$  is an increasing soft open set containing  $x$  and  $[(i(x))_E]^c$  is a decreasing soft open set containing  $y$  such that  $y \notin [(d(y))_E]^c$  and  $x \notin [(i(x))_E]^c$ . Thus,  $(X, \tau, E, \preceq)$  is strong soft  $T_1$ -ordered.

To prove the "only if" part, suppose  $x \not\preceq y$  in  $X$ . Then there exist an increasing soft open set  $W_E$  containing  $x$  and a decreasing soft open set  $V_E$  containing  $y$  such that  $y \notin W_E$  and  $x \notin V_E$ . By Proposition (3.23),  $W_E$  and  $V_E$  are stable. This means that  $y \notin W_E$  and  $x \notin V_E$ . Thus,  $(X, \tau, E, \preceq)$  is  $p$ -soft  $T_1$ -ordered.  $\square$

**Corollary 4.9.** If  $(X, \tau, E, \preceq)$  is strong soft regularly ordered and upper (resp. lower) strong soft  $T_1$ -ordered, then  $(i(x))_E$  (resp.  $(d(x))_E$ ) is soft closed.

To show that the converse of the above corollary fails, we give the following example.

**Example 4.10.** Consider  $\preceq = \Delta \cup \{(x, y)\}$  which is a partial order relation on  $X = \{x, y\}$  and let  $E = \{e_1, e_2\}$  be a parameters set. The collection  $\tau = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, 3, 4\}$  is a soft topology on  $X$ , where

$$\begin{aligned} G_{1_E} &= \{(e_1, \{x\}), (e_2, \{x\})\}; \\ G_{2_E} &= \{(e_1, \{y\}), (e_2, \{y\})\}; \\ G_{3_E} &= \{(e_1, X), (e_2, \{y\})\} \text{ and} \\ G_{4_E} &= \{(e_1, \{x\}), (e_2, \emptyset)\}. \end{aligned}$$

On the one hand,  $(i(x))_E = (d(y))_E = \tilde{X}$ ,  $(d(x))_E = G_{1_E}$  and  $(i(y))_E = G_{2_E}$  are soft closed sets. On the other hand, since the soft open sets  $G_{3_E}$  and  $G_{4_E}$  are not stable, then it follows from Proposition (3.23) that  $(X, \tau, E, \preceq)$  is not soft regularly ordered. Hence,  $(X, \tau, E, \preceq)$  is not strong soft regularly ordered.

**Proposition 4.11.** The following three concepts are equivalent if  $(X, \tau, E, \preceq)$  is strong soft regularly ordered:



- (i) strong soft  $T_2$ -ordered;
- (ii) strong soft  $T_1$ -ordered;
- (iii) strong soft  $T_0$ -ordered.

*Proof.* The directions (i)  $\rightarrow$  (iii) are obvious.

To prove that (iii)  $\rightarrow$  (i), let  $x \not\preceq y \in X$ . Since  $(X, \tau, E, \preceq)$  is strong soft  $T_0$ -ordered, then it is strong lower soft  $T_1$ -ordered or strong upper soft  $T_1$ -ordered. Say, it is strong upper soft  $T_1$ -ordered. It follows, by the above corollary, that  $(i(x))_E$  is an increasing soft closed set. Since  $y \notin (i(x))_E$  and  $(X, \tau, E, \preceq)$  is strong soft regularly ordered, then there exist disjoint soft open sets  $W_E$  and  $V_E$  containing  $(i(x))_E$  and  $y$ , respectively, such that  $W_E$  is increasing and  $V_E$  is decreasing. Hence, the proof is complete.  $\square$

**Corollary 4.12.** *The following concepts are equivalent if  $(X, \tau, E, \preceq)$  is strong lower (resp. strong upper) soft regularly ordered:*

- (i) strong soft  $T_2$ -ordered;
- (ii) strong soft  $T_1$ -ordered;
- (iii) strong lower (resp. strong upper) soft  $T_1$ -ordered.

**Corollary 4.13.** *Every strong soft  $T_i$ -ordered space  $(X, \tau, E, \preceq)$  is strong soft  $T_{i-1}$ -ordered for  $i = 1, 2, 3$ .*

The converse of the above corollary need not be true in general as demonstrated in the next three examples.

**Example 4.14.** *The given STOS in Example (3.7) is strong soft  $T_0$ -ordered, however, it is not strong soft  $T_1$ -ordered.*

**Example 4.15.** *Let  $E = \{e_1, e_2\}$  be a parameters set. Let  $\preceq = \triangle \cup \{(2, 3)\}$  be a partial order relation on the set of natural numbers  $\mathbb{N}$ . Then  $\tau = \{\tilde{\Phi}, G_E \subseteq \tilde{N} : G_E^c \text{ is finite}\}$  is a soft topology on  $\mathbb{N}$ . Now, we have the following cases:*

- (i) *since  $3 \not\preceq 2$ , then  $U_E = \{(e_1, \mathbb{N} \setminus \{2\}), (e_2, \mathbb{N} \setminus \{2\})\}$  is an increasing soft open set containing 3 and  $V_E = \{(e_1, \mathbb{N} \setminus \{3\}), (e_2, \mathbb{N} \setminus \{3\})\}$  is a decreasing soft open set containing 2 such that  $2 \notin U_E$  and  $3 \notin V_E$ ;*
- (ii) *since  $x \not\preceq 2$  for all  $x \neq 3$ , then  $U_E = \{(e_1, \mathbb{N} \setminus \{2\}), (e_2, \mathbb{N} \setminus \{2\})\}$  is an increasing soft open set containing  $x$  and  $V_E = \{(e_1, \mathbb{N} \setminus \{x, 3\}), (e_2, \mathbb{N} \setminus \{x, 3\})\}$  is a decreasing soft open set containing  $x$  such that  $2 \notin U_E$  and  $x \notin V_E$ ;*
- (iii) *since  $2 \not\preceq x$ , then  $U_E = \{(e_1, \mathbb{N} \setminus \{x\}), (e_2, \mathbb{N} \setminus \{x\})\}$  is an increasing soft open set containing 2 and  $V_E = \{(e_1, \mathbb{N} \setminus \{2, 3\}), (e_2, \mathbb{N} \setminus \{2, 3\})\}$  is a decreasing soft open set containing  $x$  such that  $x \notin U_E$  and  $2 \notin V_E$ ;*
- (iv) *since  $3 \not\preceq x$  for all  $x \neq 2$ , then  $U_E = \{(e_1, \mathbb{N} \setminus \{2, x\}), (e_2, \mathbb{N} \setminus \{2, x\})\}$  is an increasing soft open set containing 3 and  $V_E = \{(e_1, \mathbb{N} \setminus \{3\}), (e_2, \mathbb{N} \setminus \{3\})\}$  is a decreasing soft open set containing  $x$  such that  $x \notin U_E$  and  $3 \notin V_E$ ;*
- (v) *since  $x \not\preceq 3$ , then  $U_E = \{(e_1, \mathbb{N} \setminus \{2, 3\}), (e_2, \mathbb{N} \setminus \{2, 3\})\}$  is an increasing soft open set containing  $x$  and  $V_E = \{(e_1, \mathbb{N} \setminus \{x\}), (e_2, \mathbb{N} \setminus \{x\})\}$  is a decreasing soft open set containing 3 such that  $3 \notin U_E$  and  $x \notin V_E$ .*

Thus,  $(X, \tau, E, \preceq)$  is strong soft  $T_1$ -ordered. In contrast, one can note that it is not strong soft  $T_2$ -ordered.

**Example 4.16.** *The given STOS in Example (3.26) is strong soft  $T_2$ -ordered, however, it is not strong soft  $T_3$ -ordered.*

**Remark 4.17.** *In the following, we point out that the concepts of strong soft  $T_3$ -ordered and strong soft  $T_4$ -ordered spaces are independent of each other.*



- (i) The given STOS in Example(3.26) is strong soft  $T_4$ -ordered, but it is not strong soft  $T_3$ -ordered;
- (ii) If we consider  $(X, \tau, E, \preceq)$  is an STOS such that  $E$  is a singleton and  $\preceq$  is an equality relation, then  $(X, \tau, E, \preceq)$  is a topological space. So Niemytzki space in general topology shows that a strong soft  $T_4$ -ordered space is a proper extension of a strong soft  $T_3$ -ordered space.

The proofs of Theorem (4.18) and Theorem (4.19) below are similar to the proofs of Theorem (3.35) and Theorem (3.36) respectively.

**Theorem 4.18.** A finite product of strong soft  $T_i$ -ordered spaces is strong soft  $T_i$ -ordered for  $i = 0, 1, 2$ .

**Theorem 4.19.** The property of being a strong soft  $T_i$ -ordered space is a soft topological ordered property for  $i = 0, 1, 2, 3, 4$ .

We devote the rest of this section to investigate some findings that associate some strong ordered soft separation axioms with soft compactness.

**Lemma 4.20.** Let  $F_E$  be an increasing (resp. a decreasing) soft open subset in a strong soft regularly ordered space. Then for each  $P_e^x \in F_E$ , there exists an increasing (resp. a decreasing) soft open set  $G_E$  containing  $P_e^x$  such that  $P_e^x \in \widetilde{G_E} \subseteq F_E$ .

*Proof.* The proof is similar to that of Lemma (3.37). □

**Theorem 4.21.** Let  $H_E$  be an increasing (resp. a decreasing) soft compact subset in a strong soft regularly ordered space and  $F_E$  be a decreasing (resp. an increasing) soft open set containing  $H_E$ . Then there exists a decreasing (resp. an increasing) soft open set  $G_E$  such that  $H_E \subseteq \widetilde{G_E} \subseteq F_E$ .

*Proof.* The proof is similar to that of Theorem (3.38). □

**Corollary 4.22.** Every soft compact strong soft regularly ordered space is strong soft normally ordered.

*Proof.* The proof is similar to that of Corollary (3.39). □

**Remark 4.23.** (i) If a partial order relation is diagonal, then a soft topological ordered space can be viewed as a soft topological space. In this case the concepts of soft  $T_i$ -ordered spaces and strong soft  $T_i$ -ordered spaces for  $i = 0, 1, 2, 3, 4$ , are equivalent;

(ii) If a set of parameters is a singleton, then a soft topological ordered space can be viewed as a topological ordered space. In this case the notations  $\not\subseteq$  and  $\not\supseteq$  are equivalent. Hence, the concepts of  $p$ -soft  $T_i$ -ordered spaces, soft  $T_i$ -ordered spaces and  $T_i$ -ordered spaces for  $i = 0, 1, 2, 3, 4$ , are equivalent.

**Proposition 4.24.** Every strong soft  $T_i$ -ordered space  $(X, \tau, E, \preceq)$  is a soft  $T_i$ -space, for  $i = 0, 1, 2$ .

*Proof.* The proof follows directly from the definitions of strong soft  $T_i$ -ordered and soft  $T_i$ -spaces. □

**Remark 4.25.** To confirm that the converse of the above proposition fails, we consider  $E$  is a singleton and then we suffice with the examples introduced in [2]. Also, by considering  $E$  is a singleton, Example 3 in [2] shows that the concepts of strong soft  $T_i$ -ordered and soft  $T_i$ -spaces ( $i = 3, 4$ ) are independent of each other.

In conclusion, we give Figure 1 to illustrate the relationships among some types of ordered soft separation axioms.

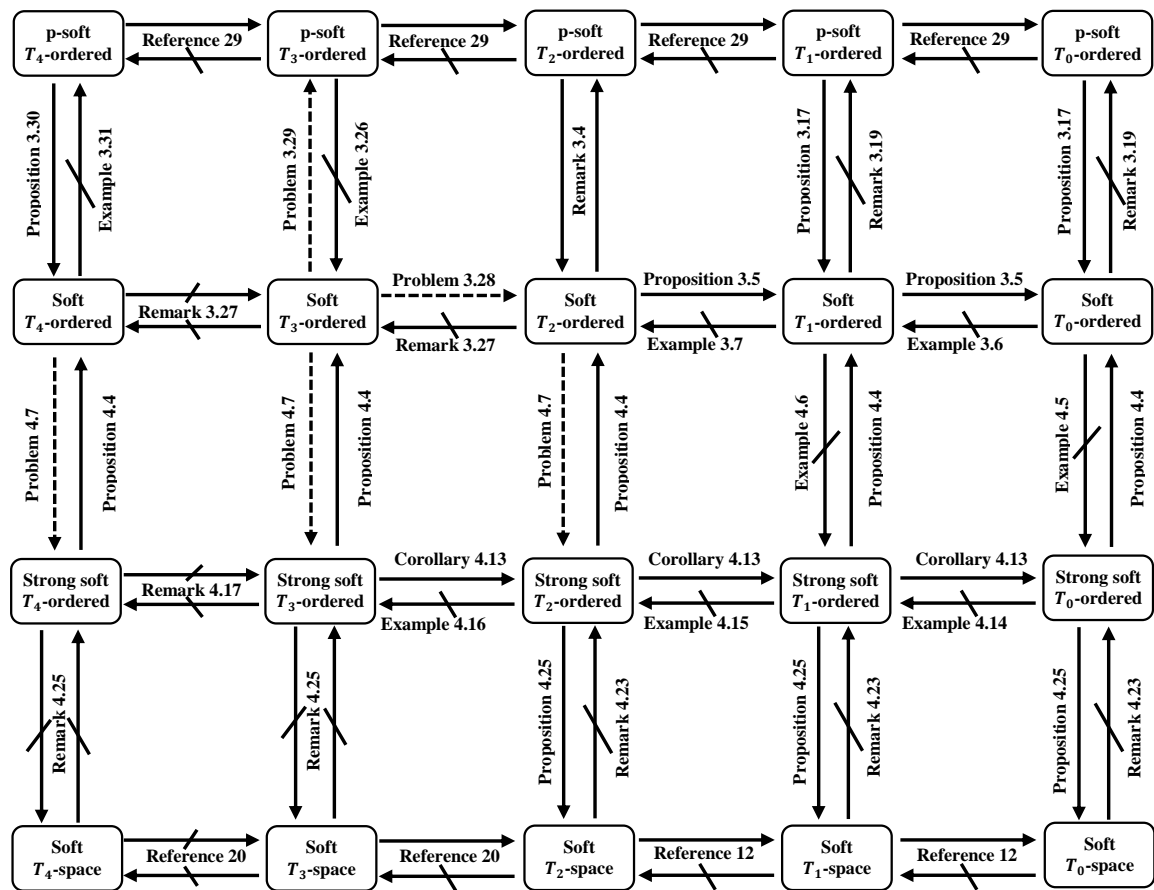


Figure 1: The relationships among some types of ordered soft separation spaces.

## 5 Conclusion and future work

By combining a partial order relation and a topology on a non-empty set, Nachbin [1] defined the topological ordered space. Similarly, Al-shami et al. [29] defined the soft topological ordered space. Studying soft separation axioms via soft topological spaces is a significant topic because they help establish a wider family which can be easily applied to classify the objects under study. We demonstrate in the last paragraph of introduction the reasons for doing many studies via soft separation axioms and the variety of these studies will be more via ordered soft separation axioms. Throughout this work, we use the notions of monotone soft neighborhoods and monotone soft open sets to present soft  $T_i$ -ordered and strong soft  $T_i$ -ordered spaces, respectively, for  $i = 0, 1, 2, 3, 4$ . These two types are formulated with respect to the ordinary points. We establish several results such as strong soft  $T_i$ -ordered spaces is strictly finer than soft  $T_i$ -ordered spaces and support this result with number of interesting examples. Also, we discuss the relationships which associate the soft  $T_i$ -ordered (strong soft  $T_i$ -ordered) spaces with p-soft  $T_i$ -ordered spaces and soft  $T_i$ -spaces. In Theorem (4.8), we give a condition that satisfies the equivalence between p-soft  $T_1$ -ordered and strong soft  $T_1$ -ordered spaces. In the end of Section (3) and Section (4), we present a number of results that associate soft compactness with some of the initiated ordered soft separation axioms. Some open problems on the relationship between strong soft  $T_i$ -ordered and soft  $T_i$ -ordered spaces ( $i = 2, 3, 4$ ) are posed.

To extend this study, one can generalize the initiated concepts on supra soft topological spaces [40]. All these results will provide a base to researchers who want to work in the soft ordered topology field and will

help to establish a general framework for applications in practical fields.

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