



Research Article

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Ulam-Hyers stability of a parabolic partial differential equation

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Abstract: The goal of this paper is to give an Ulam-Hyers stability result for a parabolic partial differential equation. Here we present two types of Ulam stability: Ulam-Hyers stability and generalized Ulam-Hyers-Rassias stability. Some examples are given, one of them being the Black-Scholes equation.

Keywords: parabolic partial differential equation, Ulam-Hyers stability, generalized Ulam-Hyers-Rassias stability, integral inequality, Gronwall lemma, Black-Scholes equation

MSC: 35L70, 45H10, 47H10

1 Introduction

In 1940, in a talk given at Wisconsin University, S.M. Ulam posed the following problem: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" (for more details see [1]). A year later, D.H. Hyers in [2] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. After Hyers' result many papers have been dedicated to this topic, extending Ulam's problem to other functional equations. C. Alsina and R. Ger [3] were the first authors who investigated Ulam-Hyers stability of a differential equation. Generally, we say that a differential equation is Ulam stable if for every approximate solution of the differential equation, there exists an exact solution near it [3]. The results of Alsina and Ger were extended to the stability of various functional equations by D. Cîmpean and D. Popa [4, 5], S.-M. Jung [6], D. Popa and I. Rasa [7, 8], I.A. Rus [9]. Several results on the Hyers-Ulam stability of a variety of ordinary differential equations were also formulated and proved by S.-M. Jung, B. Kim, and Th.M. Rassias [10], S.-M. Jung and Th.M. Rassias [11, 12], H. Rezaei, S.-M. Jung and Th.M. Rassias [13]. The first result proved on the Hyers-Ulam stability of partial differential equations is due to A. Prastaro and Th.M. Rassias [14]. Some results regarding Ulam-Hyers stability of partial differential equations were given by S.-M. Jung [15], N. Lungu and S. Ciplea [16], N. Lungu and D. Popa [17], N. Lungu, C. Craciun [18], N. Lungu and D. Marian [19].

Some recent results regarding stability analysis and their applications were established by H. Khan, A. Khan, T. Abdeljawad and A. Alkhazzan [20], A. Khan, J.F. Gómez-Aguilar, T.S. Khan and H. Khan [21], H. Khan, T. Abdeljawad, M. Aslam, R.A. Khan and A. Khan [22], H. Khan, J.F. Gómez-Aguilar, A. Khan and T.S. Khan [23]. Results regarding fixed point theory and the Ulam stability can be found in [24]. Recent results regarding parabolic partial differential equations were established by J. Lin, S.Y. Reutskiy and J. Lu [25], S.Y. Reutskiy and J. Lin [26].

Following I.A. Rus [27] and N. Lungu, I.A. Rus [28] in this paper we will present two types of Ulam stability for second order nonlinear parabolic partial differential equations: Ulam-Hyers stability as in Definition

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2.1 and generalized Ulam-Hyers-Rassias stability as in Definition 2.3 which are two distinguishable types of stability. More precisely we will consider the following equation:

$$\frac{\partial^2 u}{\partial x^2} = f(x, t, u(x, t), u_x(x, t), u_t(x, t)), -a < x < a, -b < t < b, \quad (1.1)$$

$a, b \in (0, \infty)$, $f \in C((-a, a) \times (-b, b) \times \mathbb{R}^3, \mathbb{R})$, $u \in C^2((-a, a) \times (-b, b), \mathbb{R})$, f analytic.

We mention that for this equation we will study Ulam-Hyers stability (see Theorem 3.1) and generalized Ulam-Hyers-Rassias stability (see Theorem 4.1)

We will apply the theoretical results to the stability of the Black-Scholes equation. The Black-Scholes equation was introduced as a model for financial mathematics. It describes the option price movement and is one of the most important concepts in modern financial theory. In 1973, Black and Scholes published what has come to be known as the Black-Scholes formula [29]. Thousands of traders and investors now use this formula every day to value stock options in markets throughout the world. Robert Merton devised another method to derive the formula that turned out to have very wide applicability; he also generalized the formula in many directions.

The Royal Swedish Academy of Sciences has decided to award the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, 1997, to Robert C. Merton and Myron S. Scholes for a new method to determine the value of derivatives. Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society. In a modern market economy it is essential that firms and households are able to select an appropriate level of risk in their transactions. This takes place on financial markets which redistribute risks towards those agents who are willing and able to assume them. Markets for options and other so-called derivatives are important in the sense that agents who anticipate future revenues or payments can ensure a profit above a certain level or insure themselves against a loss above a certain level. Robert Merton and Myron Scholes, developed this method in close collaboration with Fischer Black, who died in 1995. These three scholars worked on the same problem: option valuation.

2 Ulam stability

Let $a, b \in (0, \infty)$, $\varepsilon > 0$ and $\varphi \in C((-a, a) \times (-b, b), \mathbb{R}_+)$. We consider the following inequalities

$$\left| \frac{\partial^2 v(x, t)}{\partial x^2} - f(x, t, v(x, t), v_x(x, t), v_t(x, t)) \right| \leq \varepsilon, x \in (-a, a), t \in (-b, b), \quad (2.1)$$

$$\left| \frac{\partial^2 v(x, t)}{\partial x^2} - f(x, t, v(x, t), v_x(x, t), v_t(x, t)) \right| \leq \varphi(x, t), x \in (-a, a), t \in (-b, b), \quad (2.2)$$

$$\left| \frac{\partial^2 v(x, t)}{\partial x^2} - f(x, t, v(x, t), v_x(x, t), v_t(x, t)) \right| \leq \varepsilon \varphi(x, t), x \in (-a, a), t \in (-b, b). \quad (2.3)$$

Definition 2.1. The equation (1.1) is local analytic Ulam-Hyers stable at $(0, 0)$ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each analytic solution v of (2.1) there exists an analytic solution u of (1.1) in a neighborhood V of $(0, 0)$ with:

$$|v(x, t) - u(x, t)| \leq c_f \cdot \varepsilon, \forall (x, t) \in V.$$

Definition 2.2. The equation (1.1) is local analytic Ulam-Hyers-Rassias stable with respect to φ at $(0, 0)$ if there exists a real number $c_{f, \varphi} > 0$ such that for each $\varepsilon > 0$ and for each analytic solution v of (2.3) there exists an analytic solution u of (1.1) in a neighborhood V of $(0, 0)$ with:

$$|v(x, t) - u(x, t)| \leq c_{f, \varphi} \cdot \varepsilon \varphi(x, t), \forall (x, t) \in V.$$

Definition 2.3. The equation (1.1) is generalized local analytic Ulam-Hyers-Rassias stable with respect to φ at $(0, 0)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each analytic solution v of (2.2) there exists an analytic solution u of (1.1) in a neighborhood V of $(0, 0)$ with:

$$|v(x, t) - u(x, t)| \leq c_{f,\varphi} \cdot \varphi(x, t), \forall (x, t) \in V.$$

Remark 2.1. A function v is a solution of (2.1) if and only if there exists a function $g \in C((-a, a) \times (-b, b), \mathbb{R})$ such that

- (i) $|g(x, t)| \leq \varepsilon, \forall x \in (-a, a), \forall t \in (-b, b);$
- (ii) $\frac{\partial^2 v(x, t)}{\partial x^2} = f(x, t, v(x, t), v_x(x, t), v_t(x, t)) + g(x, t), \forall x \in (-a, a), \forall t \in (-b, b).$

Remark 2.2. A function v is a solution of (2.2) if and only if there exists a function $g \in C((-a, a) \times (-b, b), \mathbb{R})$ such that

- (i) $|g(x, t)| \leq \varphi(x, t), \forall x \in (-a, a), \forall t \in (-b, b);$
- (ii) $\frac{\partial^2 v(x, t)}{\partial x^2} = f(x, t, v(x, t), v_x(x, t), v_t(x, t)) + g(x, t), \forall x \in (-a, a), \forall t \in (-b, b).$

Remark 2.3. A function v is a solution of (2.3) if and only if there exists a function $g \in C((-a, a) \times (-b, b), \mathbb{R})$ such that

- (i) $|g(x, t)| \leq \varepsilon \varphi(x, t), \forall x \in (-a, a), \forall t \in (-b, b);$
- (ii) $\frac{\partial^2 v(x, t)}{\partial x^2} = f(x, t, v(x, t), v_x(x, t), v_t(x, t)) + g(x, t), \forall x \in (-a, a), \forall t \in (-b, b).$

Remark 2.4. If v is a solution to the inequality (2.1), then v is a solution to the following integral inequality

$$\left| v(x, t) - v(0, t) - xv_x(0, t) - \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds \right| \leq \varepsilon x^2,$$

for all $\forall x \in (-a, a), \forall t \in (-b, b).$

Indeed, by Remark 2.1 we have that

$$v(x, t) = v(0, t) + xv_x(0, t) + \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds + \int_0^x \int_0^s g(z, t) dz ds.$$

So,

$$\left| v(x, t) - v(0, t) - xv_x(0, t) - \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds \right| \leq \int_0^x \int_0^s |g(z, t)| dz ds \leq \varepsilon x^2.$$

In a similar way we have the following result.

Remark 2.5. If v is a solution to the inequality (2.2), then v is a solution to the following integral inequality

$$\left| v(x, t) - v(0, t) - xv_x(0, t) - \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds \right| \leq \int_0^x \int_0^s \varphi(z, t) dz ds.$$

Remark 2.6. If v is a solution to the inequality (2.3), then v is a solution to the following integral inequality

$$\left| v(x, t) - v(0, t) - xv_x(0, t) - \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds \right| \leq \varepsilon \int_0^x \int_0^s \varphi(z, t) dz ds.$$

3 Ulam-Hyers stability of equation (1.1)

The first result of this paper is the following.

Theorem 3.1. *We suppose that*

- (i) $0 < a < \infty, 0 < b < \infty$;
- (ii) $f \in C((-a, a) \times (-b, b) \times \mathbb{R}^3, \mathbb{R})$ is analytic;
- (iii) there exists $L_f > 0$ such that

$$|f(x, t, u_1, p_1, q_1) - f(x, t, u_2, p_2, q_2)| \leq L_f |u_1 - u_2|,$$

for all $x \in (-a, a), t \in (-b, b)$ and $u_1, u_2, p_1, p_2, q_1, q_2 \in \mathbb{R}$.

Then:

- (a) for analytic functions $\alpha, \beta \in C((-b, b), \mathbb{R})$ the equation (1.1) has a unique analytic solution in a neighborhood of $(0, 0)$ with

$$\begin{cases} u(0, t) = \alpha(t) \\ u_x(0, t) = \beta(t), \end{cases}$$

- (b) the equation (1.1) is local analytic Ulam-Hyers stable at $(0, 0)$.

Proof. (a) This is a known consequence from existence and uniqueness theorems (Cauchy-Kovalevskaja theorem [30]).

(b) Let v be an analytic solution to the inequality (2.1). Denote by u the unique analytic solution in a neighborhood V of $(0, 0)$ to the equation (1.1) which satisfies the conditions

$$\begin{cases} u(0, t) = v(0, t) \\ u_x(0, t) = v_x(0, t). \end{cases}$$

From Remark 2.4 and condition (iii) we have that

$$\begin{aligned} |v(x, t) - u(x, t)| &\leq \left| v(x, t) - v(0, t) - xv_x(0, t) - \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds \right| \\ &\quad + \int_0^x \int_0^s |f(z, t, v(z, t), v_x(z, t), v_t(z, t)) - f(z, t, u(z, t), u_x(z, t), u_t(z, t))| dz ds \\ &\leq \varepsilon x^2 + L_f \int_0^x \int_0^s |v(z, t) - u(z, t)| dz ds, \forall (x, t) \in V. \end{aligned}$$

From Gronwall lemma (see [31], p. 6) we have

$$|v(x, t) - u(x, t)| \leq a^2 e^{L_f a^2} \cdot \varepsilon = c_f \cdot \varepsilon,$$

where $c_f = a^2 e^{L_f a^2}$.

So, the equation (1.1) is local Ulam-Hyers stable at $(0, 0)$. □

4 Generalized Ulam-Hyers-Rassias stability of equation (1.1)

In what follows we consider the equation (1.1) and the inequality (2.2).

Theorem 4.1. *We suppose that*

- (i) $0 < a < \infty, 0 < b < \infty$;
(ii) $f \in C((-a, a) \times (-b, b) \times \mathbb{R}^3, \mathbb{R})$ is analytic;
(iii) there exists $L_f \in C^1((-a, a) \times (-b, b), \mathbb{R}_+)$ such that

$$|f(x, t, u_1, p_1, q_1) - f(x, t, u_2, p_2, q_2)| \leq L_f(x, t) |u_1 - u_2|,$$

for all $x \in (-a, a)$, $t \in (-b, b)$ and $u_1, u_2, p_1, p_2, q_1, q_2 \in \mathbb{R}$;

- (iv) there exists $\lambda_\varphi > 0$ such that

$$\int_0^x \int_0^s \varphi(z, t) dz ds \leq \lambda_\varphi \cdot \varphi(x, t), \forall x \in (-a, a), t \in (-b, b);$$

- (v) $\varphi : (-a, a) \times (-b, b) \rightarrow \mathbb{R}_+$ is increasing;
(vi) $e^{\int_0^x \int_0^s L_f(z, t) dz ds}$ is convergent and there exists a real number M such that $e^{\int_0^x \int_0^s L_f(z, t) dz ds} \leq M, \forall x \in (-a, a), \forall t \in (-b, b)$.

Then the equation (1.1) is generalized local analytic Ulam-Hyers-Rassias stable with respect to φ at $(0, 0)$.

Proof. Let v be an analytic solution to the inequality (2.2). Using the Cauchy-Kovalevskaja theorem, we denote by u the unique analytic solution in a neighborhood V of $(0, 0)$ to the equation (1.1) which satisfies the conditions

$$\begin{cases} u(0, t) = v(0, t) \\ u_x(0, t) = v_x(0, t). \end{cases}$$

We have

$$u(x, t) = v(0, t) + xu_x(0, t) + \int_0^x \int_0^s f(z, t, u(z, t), u_x(z, t), u_t(z, t)) dz ds$$

and

$$\left| v(x, t) - v(0, t) - xv_x(0, t) - \int_0^x \int_0^s f(z, t, v(z, t), v_x(z, t), v_t(z, t)) dz ds \right| \leq \int_0^x \int_0^s \varphi(z, t) dz ds \leq \lambda_\varphi \cdot \varphi(x, t).$$

From the above relations we have

$$|v(x, t) - u(x, t)| \leq \lambda_\varphi \cdot \varphi(x, t) + \int_0^x \int_0^s L_f(z, t) |v(z, t) - u(z, t)| dz ds, \forall (x, t) \in V.$$

From Gronwall lemma (see [31], p. 6) it follows

$$|v(x, t) - u(x, t)| \leq \left[\lambda_\varphi e^{\int_0^x \int_0^s L_f(z, t) dz ds} \right] \varphi(x, t) \leq c_{f, \varphi} \cdot \varphi(x, t),$$

where $c_{f, \varphi} = \lambda_\varphi M$.

So, the equation (1.1) is generalized local analytic Ulam-Hyers-Rassias stable with respect to φ at $(0, 0)$. \square

5 Applications

Example 5.1. We consider the parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = u(x, t), -a < x < a, -b < t < b, a, b \in \mathbb{R}_+. \quad (5.1)$$

This equation is local analytic Ulam-Hyers stable at $(0, 0)$.

Indeed, we apply Theorem 3.1 with

$$f(x, t, u(x, t), u_x(x, t), u_t(x, t)) = u, -a < x < a, -b < t < b. \quad (5.2)$$

This function is analytic. We also have

$$|f(x, t, u_1, p_1, q_1) - f(x, t, u_2, p_2, q_2)| \leq |u_1 - u_2|,$$

for all $x \in (-a, a)$, $t \in (-b, b)$ and $u_1, u_2, p_1, p_2, q_1, q_2 \in \mathbb{R}$, hence condition iii) from Theorem 3.1 is satisfied. Consequently equation (5.1) is local analytic Ulam-Hyers stable at $(0, 0)$.

Example 5.2. We consider the nonlinear Black-Scholes equation [32]

$$\frac{\partial^2 v}{\partial s^2} = \frac{2}{\sigma^2 s^2} \left[rv(s, t) - rs \frac{\partial v}{\partial s} - \frac{\partial v}{\partial t} \right], \quad (5.3)$$

$\Omega = \{(s, t) \mid s \in (s_1, s_2), t \in (t_1, t_2), s > 0\}$, $v \in C^2(\Omega)$, where $v(s, t)$ represents the price of the derivative financial product. The independent variables (s, t) are the share price underlying assets and time, respectively. The constants σ and r are the volatility of the underlying asset and the risk-free interest rate, respectively.

This equation is of parabolic type. By the conditions of Theorem 4.1, the Black-Scholes equation is generalized local analytic Ulam-Hyers-Rassias stable at (x_0, t_0) , $x_0 \in (s_1, s_2)$, $t_0 \in (t_1, t_2)$.

Indeed, we have $f(s, t, v(s, t), v_s(s, t), v_t(s, t)) = \frac{2}{\sigma^2 s^2} \left[rv(s, t) - rs \frac{\partial v}{\partial s} - \frac{\partial v}{\partial t} \right]$ which is analytic on Ω , if $s > 0$. We can translate the point $(0, 0)$ from Theorem 4.1 into (x_0, t_0) . Using Cauchy-Kovalevskaja theorem at (x_0, t_0) and a similar proof to the proof of Theorem 4.1, if the conditions of Theorem 4.1 are satisfied on Ω , we deduce that the Black-Scholes equation is generalized local analytic Ulam-Hyers-Rassias stable at (x_0, t_0) .

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