DE GRUYTER Demonstr. Math. 2019; 52:511–522

Research Article Open Access

Alejandro Urieles*, María José Ortega, William Ramírez, and Samuel Vega

New results on the q-generalized Bernoulli polynomials of level m

https://doi.org/10.1515/dema-2019-0039
Received April 9, 2019; accepted September 17, 2019

Abstract: This paper aims to show new algebraic properties from the q-generalized Bernoulli polynomials $B_n^{[m-1]}(x;q)$ of level m, as well as some others identities which connect this polynomial class with the q-generalized Bernoulli polynomials of level m, as well as the q-gamma function, and the q-Stirling numbers of the second kind and the q-Bernstein polynomials.

Keywords: q-generalized Bernoulli polynomials, q-gamma function, q-Stirling numbers, q-Bernstein polynomials

MSC 2010: 33E12

1 Introduction

Fix a fixed $m \in \mathbb{N}$, the generalized Bernoulli polynomials of level m are defined by means of the following generating function [1]

$$\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < 2\pi, \tag{1.1}$$

where the generalized Bernoulli numbers of level m are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$, for all $n \geqslant 0$. We can say that if $m = 1$ in (1.1), then we obtain the definition via a generating function, of the classical Bernoulli polynomials $B_n(x)$ and classical Bernoulli numbers, respectively, i.e. $B_n(x) = B_n^{[0]}(x)$ and $B_n = B_n^{[0]}$, respectively.

The q-analogue of the classical Bernoulli numbers and polynomials were initially investigated by Carlitz [2]. More recently, J. Choi, T. Ernst, D. kim, S. Nalci, C.S. Ryoo [3–8] defined the q-Bernoulli polynomials using different methods and studied their properties. There are numerous recent investigations on q-generalizations of this subject by many others author; see [9–17]. More recently, Mahmudov et al. [18] used the q-Mittag-Leffler function

$$E_{1,m+1}(z;q) := \frac{z^m}{e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!}}, \quad m \in \mathbb{N},$$

to define the generalized q-Apostol Bernoulli numbers and q-Apostol Bernoulli polynomials in x, y of order α and level m using the following generating functions, respectively

$$\left(\frac{z^m}{\lambda e_q^z + T_{m-1,q}(z)}\right)^{\alpha} = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(\lambda) \frac{z^n}{[n]_q!},$$

***Corresponding Author: Alejandro Urieles:** Programa de Matemática Universidad del Atlántico, Barranquilla Colombia; E-mail: alejandrourieles@email.uniatlantico.edu.co
María José Ortega, William Ramírez, Samuel Vega: Departamento de Ciencias Naturales y Exactas Universidad de la Costa, Barranquilla Colombia; E-mail: mortega22@cuc.edu.co, wramirez4@cuc.edu.co, svega@cuc.edu.co

$$\left(\frac{z^m}{\lambda e_q^z - T_{m-1,q}(z)}\right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{z^n}{[n]_q!}, \tag{1.2}$$

where $T_{m-1,q}(z) = \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!}$, $\alpha, \lambda, q \in \mathbb{C}$, $m \in \mathbb{N}$ and $0 < |q| < 1$.

In the present work, we introduce some algebraic properties from the polynomials given in [18] when $\alpha = 1$ and $\lambda = 1$, called q-generalized Bernoulli $B_n^{[m-1]}(x;q)$ of level m, and to research some relations between the q-generalized Bernoulli polynomials of level m and q-gamma function, the q-Stirling numbers of the second kind and the q-Bernstein polynomials.

The paper is organized as follows. Section 2 contains the basic backgrounds about the q-analogue of the generalized Bernoulli polynomials of level m, and some other auxiliary results which we will use throughout the paper. In the Section 3, we introduce some relevant algebraic and differential properties of the q-generalized Bernoulli polynomials of level m. Finally, in Section 4, we show the corresponding relations between q-generalized Bernoulli polynomials of level m and the q-gamma function, as well as the q-Stirling numbers of the second of the kind and the q-Bernstein polynomials.

2 Previous definitions and notations

In this paper, we denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ the sets of natural, nonnegative integer, real, positive real and complex numbers, respectively. The following q-standard definitions and properties can be found in [19–23]. The q-numbers and q-factorial numbers are defined respectively by

$$[z]_q = \frac{1-q^z}{1-q} = \frac{q^z-1}{q-1}, \qquad z \in \mathbb{C}, \quad q \in \mathbb{C}\backslash\{1\}, \quad q^z \neq 1,$$

$$[n]_q! = \prod_{k=1}^{n} [k]_q = [1]_q[2]_q[3]_q \dots [n]_q, \qquad [0]_q! = 1, \quad n \in \mathbb{N}.$$

The q-shifted factorial is defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=o}^{n-1}(1-q^j a), \quad n \in \mathbb{N},$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-q^j a), \quad a, q \in \mathbb{C}; \quad |q| < 1.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad (n, k \in \mathbb{N}_0; 0 \leqslant k \leqslant n).$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0,$$

$$(1-a)_q^n = (a;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k = \prod_{j=0}^{n-1}(1-q^j a).$$

The q-derivative of a function $f(z)$ is defined by

$$D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(qz) - f(z)}{(q-1)z}, \quad 0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}.$$

The q-analogue of the exponential function is defined in two ways

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}, \tag{2.1}$$

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1 + (1-q)q^k z), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.$$

In this sense, we can see that

$$e_q^z \cdot E_q^{-z} = 1,$$
$$e_q^z \cdot E_q^w = e_q^{z+w}.$$

Therefore,

$$D_q e_q^z = e_q^z, \quad D_q E_q^z = E_q^{qz}.$$

Definition 2.1. *For any $t > 0$*

$$\Gamma_q(t) = \int_0^{\infty} x^{t-1} E_q^{-qx} d_q x$$

is called the q-gamma function.

The Jackson's q-gamma function is defined in [20, 24] as follows

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < |q| < 1,$$

replacing x by $n+1$ we have

$$\Gamma_q(n+1) = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}(1-q)^{-n} = (q;q)_n(1-q)^{-n} = [n]_q!, \quad n \in \mathbb{N}.$$

Furthermore, it satisfies the following relations

$$\Gamma_q(1) = 1, \quad \Gamma_q(n) = [n-1]_q!, \quad \Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

Definition 2.2. *[25] For $\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > 0$ and $|q| < 1$ the function $E^{\gamma}_{\alpha,\beta}(z;q)$ is defined as*

$$E^{\gamma}_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n}{(q;q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}.$$

Note that when $\gamma = 1$ the equation above is expressed as

$$E_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)}. \tag{2.2}$$

From (2.2), setting $\alpha = 1$ and $\beta = m+1$, we can deduce that

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+m+1)} = \sum_{n=0}^{\infty} \frac{z^n}{[n+m]_q!} = \frac{1}{z^m} \sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} = \frac{\left(e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!}\right)}{z^m}. \tag{2.3}$$

The q-Stirling number of the first kind $s(n,k)_q$ and the q-Stirling number of the second kind $S(n,k)_q$ are the coefficients in the expansions, (see [26, p.173])

$$
\begin{aligned}
(x)_{n;q} &= \sum_{k=0}^{n} s(n,k)_q x^k, \\
x^n &= \sum_{k=0}^{n} S(n,k)_q (x)_{k,q}, \qquad (2.4)
\end{aligned}
$$

where

$$(x)_{k,q} = \prod_{n=0}^{k-1} (x - [n]_q).$$

Let $C[0,1]$ denote the set of continuous functions on $[0,1]$. For any $f \in C[0,1]$, the q- $\mathbb{B}_n(f;x)$ is called the q-Bernstein operator of order n for f and is defined as (see [15, p.3 Eq. (28)])

$$\mathbb{B}_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x) = \sum_{r=0}^{n} f_r b_{n,r}(x),$$

where $f_r = f([r]_q/[n]_q)$. The q-Bernstein polynomials of degree n or a q-Bernstein basis are defined by

$$b_{n,r}(x) = \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x).$$

We know that $\sum_{k=0}^{n-j} b_{n-j,k}(x) = 1$, and so

$$x^j = \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q x^{j+k} \prod_{t=0}^{n-j-k-1} (1 - q^t x).$$

By using the identity

$$\begin{bmatrix} n-j \\ k-j \end{bmatrix}_q = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q},$$

we have

$$x^j = \sum_{k=j}^{n} \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x). \qquad (2.5)$$

Otherwise, setting $\alpha = \lambda = 1$ in the equation (1.2), we have the following definition:

Definition 2.3. *Let $m \in \mathbb{N}$, $q, z \in \mathbb{C}$, $0 < |q| < 1$. The q-generalized Bernoulli polynomials $B_n^{[m-1]}(x;q)$ of level m are defined in a suitable neighborhood of $z = 0$ by means of the generating function*

$$\left(\frac{z^m}{e_q^z - \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!}} \right) e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x+y;q) \frac{z^n}{[n]_q!}, \quad |z| < 2\pi, \qquad (2.6)$$

where the q-generalized Bernoulli numbers of level m are defined by

$$B_n^{[m-1]}(q) := B_n^{[m-1]}(0;q).$$

Furthermore,

$$
\begin{aligned}
B_n^{[m-1]}(x,y;q) &:= B_n^{[m-1]}(x+y;q), \\
B_n^{[m-1]}(x,0;q) &:= B_n^{[m-1]}(x;q), \\
B_n^{[m-1]}(0,y;q) &:= B_n^{[m-1]}(y;q).
\end{aligned}
$$

The first three q-generalized Bernoulli polynomials of level m (cf. [18, p.7]) are

$$B_0^{[m-1]}(x;q) = [m]_q!,$$
$$B_1^{[m-1]}(x;q) = [m]_q!\left(x - \frac{1}{[m+1]_q}\right),$$
$$B_2^{[m-1]}(x;q) = [m]_q!\left(x^2 - \frac{[2]_q x}{[m+1]_q} + \frac{[2]_q q^{m+1}}{[m+2]_q[m+1]_q^2}\right).$$

Also, the first three q-generalized Bernoulli numbers of level m are

$$B_0^{[m-1]}(q) = [m]_q!,$$
$$B_1^{[m-1]}(q) = -\frac{[m]_q!}{[m+1]_q},$$
$$B_2^{[m-1]}(q) = \frac{[2]_q[m]_q!q^{m+1}}{[m+2]_q[m+1]_q^2}.$$

Definition 2.4. *[14] Let $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$. The q-Bernoulli polynomials in x, y of order α are defined by means of the generating function*

$$\left(\frac{z}{e_q^z - 1}\right)^{\alpha} e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x, y; q)\frac{z^n}{[n]_q!}, \quad |z| < 2\pi, \tag{2.7}$$

where the q-Bernoulli numbers of order α are defined by

$$B_n^{(\alpha)}(q) := B_n^{(\alpha)}(0, 0; q).$$

Furthermore

$$B_n^{(\alpha)}(x, q) := B_n^{(\alpha)}(x, 0; q),$$
$$B_n^{(\alpha)}(y, q) := B_n^{(\alpha)}(0, y; q).$$

3 Properties of the q–generalized Bernoulli polynomials of level m

In this section, we show some properties of the q-generalized Bernoulli polynomials $B_n^{[m-1]}(x;q)$ of level m. We demonstrated the facts for one of them. Obviously, by applying a similar technique, other ones can be determined. The following proposition summarizes some properties of the polynomials $B_n^{[m-1]}(x;q)$. We will only show in details the proofs to (2), (5) and (7).

Proposition 3.1. *Let a fixed $m \in \mathbb{N}$, $n, k \in \mathbb{N}_0$ and $q \in \mathbb{C}$, $0 < |q| < 1$. Let $\left\{B_n^{[m-1]}(x;q)\right\}_{n=0}^{\infty}$ be the sequence of q-generalized Bernoulli polynomials of level m. Then the following statements hold.*

1. *Summation formula. For every $n \geqslant 0$*

$$B_n^{[m-1]}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(q)x^{n-k}. \tag{3.1}$$

2. *For $n \geqslant 1$*

$$\sum_{k=0}^{n} \begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1]}(q) = 0.$$

3. *Addition formulas*

$$B_n^{[m-1]}(x+y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}(n-k)(n-k-1)} B_k^{[m-1]}(x;q) y^{n-k}, \tag{3.2}$$

$$B_n^{[m-1]}(x+y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}(n-k)(n-k-1)} B_k^{[m-1]}(y;q) x^{n-k}, \tag{3.3}$$

$$B_n^{[m-1]}(x+y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(q)(x+y)_q^{n-k},$$

$$B_n^{[m-1]}(x+y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(y;q) x^{n-k}.$$

4. *Inversion formulas*

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1]}(x;q), \tag{3.4}$$

$$y^n = \frac{[n]_q!}{q^{\frac{1}{2}n(n-1)}\,[n+m]_q!} \sum_{k=0}^{n} \begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1]}(y;q), \tag{3.5}$$

$$x^n = \sum_{k=0}^{n} \frac{[n]_q! B_k^{[m-1]}(x;q)}{[k]_q! \Gamma_q(n-k+m+1)}. \tag{3.6}$$

5. *Difference equations*

$$B_n^{[m-1]}(1+y;q) - B_n^{[m-1]}(y;q) = [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q B_k^{[m-1]}(y,q) B_{n-1-k}^{(-1)}(q).$$

6. *Differential relations. For $m \in \mathbb{N}$ and $n, j \in \mathbb{N}_0$, where $0 \leqslant j \leqslant n$, we have*

$$D_q B_{n+1}^{[m-1]}(x;q) = [n+1]_q B_n^{[m-1]}(x;q), \tag{3.7}$$

$$D_q^{(j)} B_n^{[m-1]}(x;q) = \frac{[n]_q!}{[n-j]_q!} B_{n-j}^{[m-1]}(x;q).$$

7. *Integral formulas*

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x;q) d_q x = \frac{B_{n+1}^{[m-1]}(x_1;q) - B_{n+1}^{[m-1]}(x_0;q)}{[n+1]_q}, \quad n \in \mathbb{N}_0, \tag{3.8}$$

$$B_n^{[m-1]}(x;q) = [n]_q \int_0^x B_{n-1}^{[m-1]}(x;q) d_q x + B_n^{[m-1]}(q), \quad n \in \mathbb{N},$$

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x;q) d_q x = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_k^{[m-1]}(q) \left(\frac{x_1^{n+1-k} - x_0^{n+1-k}}{[n+1-k]_q} \right).$$

Proof. To prove (2), we start with (2.1) and (2.6), from which it follows that

$$z^m = \left(\sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left(\sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} \right) = \left(\sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left(\sum_{j=0}^{\infty} \frac{z^{j+m}}{[j+m]_q!} \right),$$

and therefore

$$1 = \left(\sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left(\sum_{j=0}^{\infty} \frac{z^j}{[j+m]_q!} \right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{B_k^{[m-1]}(q)}{[k]_q!}\cdot\frac{z^n}{[n-k+m]_q!}$$
$$= \frac{B_0^{[m-1]}(q)}{[m]_q!} + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{B_k^{[m-1]}(q)}{[k]_q!}\cdot\frac{z^n}{[n-k+m]_q!}.$$

By comparing coefficients of $\frac{z^n}{[n]_q!}$, we have

$$1 = \frac{B_0^{[m-1]}(q)}{[m]_q!} \quad\Rightarrow\quad B_0^{[m-1]}(q) = [m]_q!$$

and

$$\sum_{k=0}^{n}\frac{B_k^{[m-1]}(q)}{[k]_q!\,[n-k+m]_q!} = 0.$$

By multiplying $[n+m]_q!$ on both sides of the equation above, we have

$$\sum_{k=0}^{n}\frac{B_k^{[m-1]}(q)\,[n+m]_q!}{[k]_q!\,[n-k+m]_q!} = 0 \quad\Rightarrow\quad \sum_{k=0}^{n}\begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1]}(q) = 0.$$

□

Proof. Proof of (5). Considering the expression $B_n^{[m-1]}(1+y;q)-B_n^{[m-1]}(y;q)$ and using the generating functions (2.6) and (2.7), we have

$$\boldsymbol{I} := \sum_{n=0}^{\infty}B_n^{[m-1]}(1+y;q)\frac{z^n}{[n]_q!} - \sum_{n=0}^{\infty}B_n^{[m-1]}(y;q)\frac{z^n}{[n]_q!} = \left(\frac{z^m}{e_q^z - \sum_{l=0}^{m-1}\frac{z^l}{[l]_q!}}\right)E_q^{yz}(e_q^z-1)$$
$$= z\sum_{n=0}^{\infty}B_n^{[m-1]}(y;q)\frac{z^n}{[n]_q!}\sum_{n=0}^{\infty}B_n^{(-1)}(q)\frac{z^n}{[n]_q!}.$$

Therefore

$$\boldsymbol{I} = \sum_{n=0}^{\infty}B_n^{[m-1]}(y;q)\frac{z^n}{[n]_q!}\sum_{n=1}^{\infty}B_{n-1}^{(-1)}(q)\frac{z^n}{[n-1]_q!}$$
$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}B_k^{[m-1]}(y;q)\frac{z^k}{[k]_q!}B_{n-1-k}^{(-1)}(q)\frac{z^{n-k}}{[n-1-k]_q!}$$
$$= \sum_{n=1}^{\infty}[n]_q\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q B_k^{[m-1]}(y;q)B_{n-1-k}^{(-1)}(q)\frac{z^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty}[n]_q\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q B_k^{[m-1]}(y;q)B_{n-1-k}^{(-1)}(q)\frac{z^n}{[n]_q!}.$$

By comparing coefficients of $\frac{z^n}{[n]_q!}$ on both sides we obtain the result. □

Proof. Proof of (7). From (3.7) we have

$$B_n^{[m-1]}(x;q) = \frac{1}{[n+1]_q}D_qB_{n+1}^{[m-1]}(x;q).$$

Now, by integrating on both sides of the equation above, we get

$$\int_{x_0}^{x_1}B_n^{[m-1]}(x;q)d_qx = \frac{1}{[n+1]_q}\int_{x_0}^{x_1}D_qB_{n+1}^{[m-1]}(x;q)d_qx$$

$$= \frac{1}{[n+1]_q} B_{n+1}^{[m-1]}(x;q)\Big|_{x_0}^{x_1}$$
$$= \frac{B_{n+1}^{[m-1]}(x_1;q) - B_{n+1}^{[m-1]}(x_0;q)}{[n+1]_q}.$$

Setting $x_0 = 0$ and $x_1 = x$ in (3.8), we have

$$\int_0^x B_n^{[m-1]}(x;q)d_qx = \frac{B_{n+1}^{[m-1]}(x;q) - B_{n+1}^{[m-1]}(q)}{[n+1]_q},$$

and so

$$\int_0^x B_{n-1}^{[m-1]}(x;q)d_qx = \frac{B_n^{[m-1]}(x;q) - B_n^{[m-1]}(q)}{[n]_q}.$$

Finally, we get

$$B_n^{[m-1]}(x;q) = [n]_q \int_0^x B_{n-1}^{[m-1]}(x;q)d_qx + B_n^{[m-1]}(q).$$

□

4 Some connection formulas for the polynomials $B_n^{[m-1]}(x+y;q)$

From identities (2.4), (2.5) and Proposition 3.1 we can deduce some interesting algebraic relations between the q-generalized Bernoulli polynomials of level m with the q-gamma function, the q-Stirling numbers of the second kind and the q-Bernstein polynomials.

Proposition 4.1. *For $n, j, k \in \mathbb{N}_0$, $q \in \mathbb{C}$ where $0 < |q| < 1$ and where $m \in \mathbb{N}$, the q-generalized Bernoulli polynomials of level m are related with the q-gamma function by the means of the following identity*

$$B_n^{[m-1]}(x+y;q) = [n]_q! \sum_{k=0}^{n}\sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x;q)}{[k]_q!} \frac{q^{\frac{1}{2}(n-j)(n-j-1)}}{[j]_q!\Gamma_q(n-j-k+m+1)} B_j^{[m-1]}(y;q). \tag{4.1}$$

Proof. By substituting (3.6) in (3.3), we have

$$\begin{aligned} B_n^{[m-1]}(x+y;q) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y;q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x;q)}{[k]_q!\Gamma_q(n-j-k+m+1)} \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x;q)}{[k]_q!} \frac{[n]_q! q^{\frac{1}{2}(n-j)(n-j-1)}}{[j]_q!\Gamma_q(n-j-k+m+1)} B_j^{[m-1]}(y;q) \\ &= [n]_q! \sum_{k=0}^{n}\sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x;q)}{[k]_q!} \frac{q^{\frac{1}{2}(n-j)(n-j-1)}}{[j]_q!\Gamma_q(n-j-k+m+1)} B_j^{[m-1]}(y;q). \end{aligned}$$

□

Corollary 4.1. *For $n, j, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we have*

$$B_n^{[m-1]}(x;q) = [n]_q! \sum_{k=0}^{n}\sum_{j=0}^{n-k} \frac{B_j^{[m-1]}(q) B_k^{[m-1]}(x;q)}{[k]_q![j]_q!\Gamma_q(n-j-k+m+1)}. \tag{4.2}$$

Proof. By replacing equation (3.6) in (3.1), we obtain

$$B_n^{[m-1]}(x;q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q B_j^{[m-1]}(q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x;q)}{[k]_q! \Gamma_q(n-j-k+m+1)}$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{B_k^{[m-1]}(x;q)}{[k]_q!} \frac{[n]_q! B_j^{[m-1]}(q)}{[j]_q! \Gamma_q(n-j-k+m+1)}$$
$$= [n]_q! \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{B_j^{[m-1]}(q) B_k^{[m-1]}(x;q)}{[k]_q! [j]_q! \Gamma_q(n-j-k+m+1)}.$$

□

Corollary 4.2. *For $n, j, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we have*

$$B_n^{[m-1]}(x;q) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{[n]_q! B_j^{[m-1]}(q) B_{n-j-k}^{[m-1]}(x;q)}{[k+m]_q! [n-j-k]_q! [j]_q!}. \tag{4.3}$$

Proof. By substituting (3.4) in equation (3.1), we obtain

$$B_n^{[m-1]}(x;q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q B_j^{[m-1]}(q) \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-j-k}^{[m-1]}(x;q)$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{[k]_q!}{[k+m]_q!} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k \end{bmatrix}_q B_j^{[m-1]}(q) B_{n-j-k}^{[m-1]}(x;q)$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{[n]_q! B_j^{[m-1]}(q) B_{n-j-k}^{[m-1]}(x;q)}{[k+m]_q! [n-j-k]_q! [j]_q!}.$$

□

Corollary 4.3. *For $n, j, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$*

$$B_n^{[m-1]}(x+y;q) = [n]_q! \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{B_j^{[m-1]}(x;q) B_k^{[m-1]}(y;q)}{[j]_q! [n-j+m-k]_q! [k]_q!}. \tag{4.4}$$

Proof. By substituting the equation (3.5) in (3.2), we obtain

$$B_n^{[m-1]}(x+y;q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(x;q) \frac{[n-j]_q!}{q^{\frac{1}{2}(n-j)(n-j-1)} [n-j+m]_q!} \sum_{k=0}^{n-j} \begin{bmatrix} n-j+m \\ k \end{bmatrix}_q B_k^{[m-1]}(y;q)$$
$$= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[n-j]_q!}{[n-j+m]_q!} \begin{bmatrix} n-j+m \\ k \end{bmatrix}_q B_j^{[m-1]}(x;q) B_k^{[m-1]}(y;q)$$
$$= [n]_q! \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{B_j^{[m-1]}(x;q) B_k^{[m-1]}(y;q)}{[j]_q! [n-j+m-k]_q! [k]_q!}.$$

□

Proposition 4.2. *For $n, j, k \in \mathbb{N}_0$, $q \in \mathbb{C}$ where $0 < |q| < 1$ and where $m \in \mathbb{N}$, the q-generalized Bernoulli polynomials of level m are related with the q-Stirling numbers of the second kind $S(n,k;q)$ by means of the following identities*

$$B_n^{[m-1]}(x+y;q) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y;q) S(n-j,k;q)(x)_{q;j}, \tag{4.5}$$

$$B_n^{[m-1]}(x;q) = \sum_{j=0}^{n} \sum_{k=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) S(j,k;q)(x)_{q;k}. \tag{4.6}$$

Proof. Proof of (4.5). By replacing (2.4) in (3.3), we have

$$\begin{aligned} B_n^{[m-1]}(x+y;q) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y;q) \sum_{k=0}^{n-j} S(n-j,k;q)(x)_{q;k} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(n-j)(n-j-1)} B_j^{[m-1]}(y;q) S(n-j,k;q)(x)_{q;k}. \end{aligned}$$

□

Proof. Proof of (4.6). By substituting (2.4) in (3.1), we have

$$\begin{aligned} B_n^{[m-1]}(x;q) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) \sum_{k=0}^{j} S(j,k;q)(x)_{q;k} \\ &= \sum_{j=0}^{n} \sum_{k=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) S(j,k;q)(x)_{q;k}. \end{aligned}$$

□

Corollary 4.4. *For $n, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we obtain*

$$\sum_{k=0}^{n} S(n,k)_q (x)_{k,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1]}(x;q).$$

Proposition 4.3. *For $n, j, k \in \mathbb{N}_0, q \in \mathbb{C}$ where $0 < |q| < 1$ and where $m \in \mathbb{N}$ the q-generalized Bernoulli polynomials of level m are related with the q-Bernstein polynomials $b_{n,k}(x;q)$ by means of the following identities*

$$B_n^{[m-1]}(x,q) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) b_{n,k+j}(x;q), \tag{4.7}$$

$$B_n^{[m-1]}(x+y;q) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y;q) b_{n,k+j}(x;q). \tag{4.8}$$

Proof. Proof of (4.7). By replacing (2.5) in (3.1), we have

$$\begin{aligned} B_n^{[m-1]}(x,q) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) \sum_{k=j}^{n} \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x;q) \\ &= \sum_{j=0}^{n} \sum_{k=j}^{n} B_{n-j}^{[m-1]}(q) \begin{bmatrix} k \\ j \end{bmatrix}_q b_{n,k}(x;q) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q B_{n-j}^{[m-1]}(q) b_{n,k+j}(x;q). \end{aligned}$$

□

Proof. Proof of (4.8). By replacing (2.5) in Equation (3.3), we obtain

$$\begin{aligned} B_n^{[m-1]}(x+y;q) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y;q) \sum_{k=j}^{n} \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x;q) \\ &= \sum_{j=0}^{n} \sum_{k=j}^{n} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y;q) b_{n,k}(x;q) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q q^{\frac{1}{2}(j)(j-1)} B_{n-j}^{[m-1]}(y;q) b_{n,k+j}(x;q). \end{aligned}$$

□

Corollary 4.5. *For $n, j \in \mathbb{N}_0$ and $x \in [0,1]$, we have*

$$\sum_{k=j}^{n} \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} b_{n,k}(x;q) = \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{j-k}^{[m-1]}(x;q).$$

Proposition 4.4. *For $n, j, k \in \mathbb{N}_0$ and $n \geqslant j \geqslant k \geqslant 0$, we have*

$$b_{n,k}(x;q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x;q)}{\Gamma_q(j+m+1)}, \tag{4.9}$$

$$b_{n,k}(x;q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x;q)}{[j+m]_q!}. \tag{4.10}$$

Proof. To prove (4.9), we used the following equality [14, Theorem 19, p. 10]

$$\frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} = \sum_{n=k}^{\infty} b_{n,k}(x;q) \frac{z^n}{[n]_q!}.$$

We see that

$$\frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} = \frac{x^k z^k}{[k]_q!} \frac{\left(e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!} \right)}{z^m} \frac{z^m}{\left(e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!} \right)} e_q^z E_q^{-xz}.$$

Next, by using the equations (2.3) and (2.6), we get

$$\begin{aligned} \frac{x^k z^k}{[k]_q!} e_q^z E_q^{-xz} &= \frac{x^k z^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+m+1)} \sum_{j=0}^{\infty} B_j^{[m-1]}(1-x;q) \frac{z^j}{[j]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^k z^k}{[k]_q!} \frac{z^j}{\Gamma_q(j+m+1)} B_{n-j}^{[m-1]}(1-x;q) \frac{z^{n-j}}{[n-j]_q!} \\ &= \sum_{n=k}^{\infty} \sum_{j=0}^{n-k} \frac{x^k}{[k]_q!} \frac{B_{n-k-j}^{[m-1]}(1-x;q)}{\Gamma_q(j+m+1)} \frac{z^n}{[n-k-j]_q!} \\ &= \sum_{n=k}^{\infty} x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x;q)}{\Gamma_q(j+m+1)} \frac{z^n}{[n]_q!}. \end{aligned}$$

By comparing coefficients of $\frac{z^n}{[n]_q!}$ on both sides we obtain

$$b_{n,k}(x;q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x;q)}{\Gamma_q(j+m+1)}.$$

To demostrate (4.10) we used the identity $\Gamma_q(j+m+1) = [j+m]_q!$ and Equation (4.9). Continuing this process, we get

$$b_{n,k}(x;q) = x^k \sum_{j=0}^{n-k} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{[j]_q! B_{n-k-j}^{[m-1]}(1-x;q)}{[j+m]_q!}.$$

□

References

[1] Natalini P., Bernardini A., A generalization of the Bernoulli polynomials, J. Appl. Math., 2003, 3, 155–163
[2] Carlitz L., q-Bernoulli numbers and polynomials, Duke Math., 1948, 15, 987–1000
[3] Choi J., Anderson P., Srivastava H. M., Carlitz's q-Bernoulli and q-Euler numbers and polynomials and a class of q-Hurwitz zeta functions, Appl. Math. Comput., 2009, 215, 1185–1208
[4] Ernst T., q-Bernoulli and q-Euler polynomials, an umbral approach, Int. J. Difference Equ., 2006, 1, 31–80
[5] Hegazi A. S., Mansour M., A note on q-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys., 2006, 13(1), 9–18
[6] Kim D., Kim M.-S., A note on Carlitz q-Bernoulli numbers and polynomials, Adv. Difference Equ., 2012, 2012:44
[7] Quintana Y., Ramírez W., Urieles A., Generalized Apostol-type polynomials matrix and its algebraic properties, Math. Rep., 2019, 21, 249–264
[8] Ryoo C. S., A note on q-Bernoulli numbers and polynomials, Appl. Math. Lett, 2017, 20(5), 524–531
[9] Garg M., Alha S., A new class of q-Apostol-Bernoulli polynomials of order α, Revi. Tecn. URU, 2014, 6, 67–76
[10] Hernandes P., Quintana Y., Urieles A., About extensions of generalized Apostol-type polynomials, Res. Math., 2015, 68, 203–225
[11] Kurt B., A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_n^2(x, y)$, Appl. Math. Sci., 2010, 4(47), 2315–2322
[12] Kurt B., Some relationships between the generalized Apostol-Bernoulli and Apostol-Euler polynomials, Turk. Jou. Ana. Num. The., 2013, 1(1), 54–58
[13] Luo Q.-M., Guo B.-N., Qi F., Debnath L., Generalizations of Bernoulli numbers and polynomials, Int. J. Math. Math. Sci., 2003, 59, 3769–3776
[14] Mahmudov N. I., On a class of q-Bernoulli and q-Euler polynomials, Adv. Difference Equ., 2013, 1, 108–125
[15] Ramírez W., Castilla L., Urieles A., An extended generalized q-extensions for the Apostol type polynomials, Abstr. Appl. Anal., 2018, Article ID 2937950, DOI: 10.1155/2018/2937950
[16] Tremblay R., Gaboury S., Fugere J., A further generalization of Apostol-Bernoulli polynomials and related polynomials, Hon. Math. Jou., 2012, 34, 311–326
[17] Quintana Y., Ramírez W., Urieles A., On an operational matrix method based on generalized Bernoulli polynomials of level m, Calcolo, 2018, 55, 30
[18] Mahmudov N. I., Eini Keleshteri M., q-extensions for the Apostol type polynomials, J. Appl. Math., 2014, Article ID 868167, http://dx.doi.org/10.1155/2014/868167
[19] Ernst T., The history of q-calculus and a new method, Licentiate Thesis, Dep. Math. Upps. Unive., 2000
[20] Gasper G., Rahman M., Basic Hypergeometric Series, Cambr. Univ. Press, 2004
[21] Kac V., Cheung P., Quantum Calculus, Springer-Verlag New York, 2002
[22] Araci S., Duran U., Acikgoz M., (p, q)-Volkenborn integration, J. Number Theory, 2017, 171, 18–30
[23] Araci S., Duran U., Acikgoz M., Srivastava H. M., A certain (p, q)-derivative operator and associated divided differences, J. Ineq. Appl., 2016, 2016:301, DOI: 10.1186/s13660-016-1240-8
[24] Srivastava H. M., Choi J., Zeta and q-zeta functions and associated series and integrals, Editorial Elsevier, Boston, 2012, DOI: 10.1016/C2010-0-67023-4
[25] Sharma S., Jain R., On some properties of generalized q-Mittag Leffler, Math. Aeterna, 2014, 4(6), 613–619
[26] Ernst T., A comprehensive treatment of q-calculus, Birkhäuser, 2012
[27] Ostrovska S., q-Bernstein polynomials and their iterates, J. Approx. Theory, 2003, 123(2), 232–255