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An admissible Hybrid contraction with an Ulam type stability

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Abstract: In this manuscript, we introduce a new hybrid contraction that unify several nonlinear and linear contractions in the set-up of a complete metric space. We present an example to indicate the genuine of the proved result. In addition, we consider Ulam type stability and well-posedness for this new hybrid contraction.

Keywords: admissible mappings, hybrid contractions, fixed point, metric space

MSC: 47H10, 54H25, 46J10

1 Introduction and preliminaries

In the last three-four decades, there is a blown out in the number of publications in metric fixed point theory. This fact forces researchers to find a way to combine, unify and merge the existing results in a proper way. In this paper, we aim to give an interesting example for this trend. We introduce a new hybrid contraction which not only combine and unify the several existing linear and nonlinear contractions but also extend these results.

Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ such that

(Ψ_1) ψ is non-decreasing;
(Ψ_2) there are $i_0 \in \mathbb{N}$ and $\delta \in (0, 1)$ and a convergent series $\sum_{i=1}^{\infty} v_i$ such that $v_i \geq 0$ and

$$\psi^{i+1}(t) \leq \delta \psi^i(t) + v_i, \tag{1}$$

for $i \geq i_0$ and $t \geq 0$.

Each $\psi \in \Psi$ is called a (c)-comparison function (see [1, 2]).

Lemma 1.1. *[1] If $\psi \in \Psi$, then*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for* $t \geq 0$;
(ii) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
(iii) ψ *is continuous at* 0;
(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *is convergent for* $t \geq 0$.

Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that a mapping $f : X \to X$ is α-orbital admissible ([3]) if

$$\alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, f^2x) \geq 1, \ \forall x \in X. \tag{2}$$

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An α-orbital admissible mapping f is called triangular α-orbital admissible ([3]) if

$$\alpha(\chi, y) \geq 1 \text{ and } \alpha(y, fy) \geq 1 \Rightarrow \alpha(\chi, y) \geq 1, \tag{3}$$

for every $\chi, y \in X$.

Lemma 1.2. *Suppose that for a triangular α-orbital admissible mapping $f : X \to X$ there exists $\chi_0 \in X$ such that $\alpha(\chi_0, f\chi_0) \geq 1$. Then*

$$\alpha(\chi_n, \chi_m) \geq 1, \qquad \text{for all } n, m \in \mathbb{N}, \tag{4}$$

where the sequence $\{\chi_n\}$ is defined by $\chi_{n+1} = f\chi_n$, $n \in \mathbb{N}$.

Definition 1.3. Let $\alpha : X \times X \to [0, \infty)$ be a mapping. The set X is called regular with respect to α if for a sequence $\{\chi_n\}$ in X such that $\alpha(\chi_n, \chi_{n+1}) \geq 1$, for all n and $\chi_n \to \chi \in X$ as $n \to \infty$ we have $\alpha(\chi_n, \chi) \geq 1$ for all n.

2 Main results

We start with a definition of a new notion, namely "admissible hybrid contraction":

Definition 2.1. Let (X, d) be a metric space. A self-mapping f is called an admissible hybrid contraction, if there exist $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(\chi, y)d(f\chi, fy) \leq \psi\left(\mathcal{R}_f^q(\chi, y)\right), \tag{5}$$

where $q \geq 0$ and $\lambda_i \geq 0$, $i = 1, 2, 3, 4, 5$ such that $\sum_{i=1}^5 \lambda_i = 1$ and

$$\mathcal{R}_f^q d(\chi, y) = \begin{cases} \left[\lambda_1 d^q(\chi, y)(\chi, y) + \lambda_2 d^q(\chi, f\chi) + \lambda_3 d^q(y, fy) + \lambda_4 \left(\frac{d(y,fy)(1+d(\chi,f\chi))}{1+d(\chi,y)}\right)^q + \lambda_5 \left(\frac{d(y,f\chi)(1+d(\chi,fy))}{1+d(\chi,y)}\right)^q\right]^{\frac{1}{q}}, \\ \qquad \text{for } q > 0,\ \chi, y \in X \\ [d(\chi, y)]^{\lambda_1} \cdot [d(\chi, f\chi)]^{\lambda_2} \cdot [d(y, fy)]^{\lambda_3} \cdot \left[\frac{d(y,fy)(1+d(\chi,f\chi))}{1+d(\chi,y)}\right]^{\lambda_4} \cdot \left[\frac{d(\chi,fy)+d(y,f\chi)}{2}\right]^{\lambda_5}, \\ \qquad \text{for } q = 0,\ \chi, y \in X \setminus Fix_f(X) \end{cases} \tag{6}$$

(Here $Fix_f(X) = \{\chi \in X : f\chi = \chi\}$.)

The concept of "admissible hybrid contraction" is inspired from the notion of "interpolative contractions", see e.g. [4–9] The main results of this manuscript is the following theorem:

Theorem 2.2. *Let (X, d) be a complete metric space and let f be an admissible hybrid contraction, Suppose also that:*

- (i) *f is triangular α–orbital admissible;*
- (ii) *there exists $\chi_0 \in X$ such that $\alpha(\chi_0, f\chi_0) \geq 1$;*
- (iii) *either, f is continuous, or*
- (iv) *f^2 is continuous and $\alpha(f\chi, \chi) \geq 1$ for any $\chi \in Fix_{f^2}(X)$.*

Then f has a fixed point.

Proof. Starting from an arbitrary point χ_0 in X we recursively set-up the sequence $\{\chi_n\}$, as $\chi_n = f^n\chi_0$ for all $n \in \mathbb{N}$. Supposing that there exists some $m \in \mathbb{N}$ such that $fv_m = \chi_{m+1} = \chi_m$, we find that χ_m is a fixed point of f and the proof is finished. So, we can presume from now on that $\chi_n \neq \chi_{n-1}$ for any $n \in \mathbb{N}$. Under the assumption (i), f is admissible hybrid contraction, if we substituting in (5) χ by χ_{n-1} and y by χ_n we get

$$\alpha(\chi_{n-1}, \chi_n)d(f\chi_{n-1}, f\chi_n) \leq \psi(\mathcal{R}_f^q(\chi_{n-1}, \chi_n)). \tag{7}$$

Taking into account that f is triangular α–orbital admissible, together with (4) holds and the above inequality becomes

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n) < \psi(\mathcal{R}_f^q(x_{n-1}, x_n). \tag{8}$$

Case 1. For the case $q > 0$ we have

$$\begin{aligned}
\mathcal{R}_f^q(x_{n-1}, x_n) = \;& \Big[\lambda_1 d^q(x_{n-1}, x_n) + \lambda_2 d^q(x_{n-1}, fx_{n-1}) + \lambda_3 d^q(x_n, fx_n) + \lambda_4 \left(\tfrac{d(x_n, fx_n)(1+d(x_{n-1}, fx_{n-1}))}{1+d(x_{n-1}, x_n)}\right)^q \\
&+\lambda_5 \left(\tfrac{d(x_n, fx_{n-1})(1+d(x_{n-1}, fx_n))}{1+d(x_{n-1}, x_n)}\right)^q\Big]^{\frac{1}{q}} \\
= \;& \Big[\lambda_1 d^q(x_{n-1}, x_n) + \lambda_2 d^q(x_{n-1}, x_n) + \lambda_3 d^q(x_n, x_{n+1}) + \lambda_4 \left(\tfrac{d(x_n, x_{n+1})(1+d(x_{n-1}, x_n))}{1+d(x_{n-1}, x_n)}\right)^q \\
&+\lambda_5 \left(\tfrac{d(x_n, x_n)(1+d(x_{n-1}, x_{n+1}))}{1+d(x_{n-1}, x_n)}\right)^q\Big]^{\frac{1}{q}} \\
= \;& \Big[\lambda_1 d^q(x_{n-1}, x_n) + \lambda_2 d^q(x_{n-1}, x_n) + \lambda_3 d^q(x_n, x_{n+1}) + \lambda_4 \left(d(x_n, x_{n+1})\right)^q\Big]^{\frac{1}{q}} \\
= \;& [(\lambda_1 + \lambda_2)d^q(x_{n-1}, x_n) + (\lambda_3 + \lambda_4)d^q(x_n, x_{n+1})]^{1/q},
\end{aligned}$$

and from (8) we get

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n) \\
&< \psi(\mathcal{R}_f^q(x_{n-1}, x_n)) \\
&= \psi([(\lambda_1 + \lambda_2)d^q(x_{n-1}, x_n) + (\lambda_3 + \lambda_4)d^q(x_n, x_{n+1})]^{1/q}).
\end{aligned} \tag{9}$$

If we suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n-1})$, since ψ is a nondecreasing function,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n) \\
&\leq \psi([(\lambda_1 + \lambda_2)d^q(x_{n-1}, x_n) + (\lambda_3 + \lambda_4)d^q(x_n, x_{n+1})]^{1/q}) \\
&\leq \psi([\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)d^q(x_n, x_{n+1})]^{1/q}) \\
&= \psi((\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{1/q}d(x_n, x_{n+1})) \\
&< (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{1/q}d(x_n, x_{n+1}) \\
&\leq d(x_n, x_{n+1}),
\end{aligned} \tag{10}$$

which is a contradiction. Therefore, for every $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n),$$

and the inequality (8) yields

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi([(\lambda_1 + \lambda_2)d^q(x_{n-1}, x_n) + (\lambda_3 + \lambda_4)d^q(x_n, x_{n+1})]^{1/q}) \\
&< \psi([\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)d^q(x_{n-1}, x_n)]^{1/q}) \\
&\leq \psi((\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{1/q}d(x_{n-1}, x_n)) \\
&\leq \psi(d(x_{n-1}, x_n)) < \psi^2(d(x_{n-2}, x_{n-1})) \\
&\dots \\
&< \psi^n(d(x_0, x_1)).
\end{aligned} \tag{11}$$

Let now, $m, p \in \mathbb{N}$ such that $p > m$. By the triangle inequality and since $d(x_m, x_{m+1}) < \psi^m(d(x_0, x_1))$ for any $m \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_m, x_p) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \dots + d(x_{p-1}, x_p) \\
&= \textstyle\sum_{j=m}^{p-1} d(x_j, x_{j+1}) \leq \sum_{j=m}^{p-1} \psi^j(d(x_0, x_1)).
\end{aligned}$$

Since ψ is a c-comparison function the series $\sum_{j=0}^{\infty} \psi^j(d(x_0, x_1))$ is convergent, so that, denoting by $\mathcal{S}_n = \sum_{j=0}^{n} \psi^j(d(x_0, x_1))$ the above inequality becomes:

$$d(x_m, x_p) \leq \mathcal{S}_{p-1} - \mathcal{S}_{m-1},$$

and as $m, p \to \infty$ we get

$$d(x_m, x_p) \to 0, \tag{12}$$

which tells us that $\{x_n\}$ is a Cauchy sequence on a complete metric space, so that, there exists z such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{13}$$

We will prove that this point z is a fixed point of f. If f is continuous, (due to assumption (iii))

$$\lim_{n\to\infty} d(x_{n+1}, fz) = \lim_{n\to\infty} d(x_n, fx_n) = 0,$$

so, we get that $fz = z$, that is, z is a fixed point of f.

In the alternative hypothesis, that f^2 is continuous we have $f^2 z = \lim_{n\to\infty} f^2 x_n = z$ and we want to show that $fz = z$. Supposing that, on the contrary, $fz \neq z$, we have from (5)

$$\begin{aligned}
d(z, fz) = \;& d(f^2 z, fz) \le \alpha(fz, z) d(fz, z) \\
\le\;& \psi(\mathcal{R}_f^q(fz, z)) < \mathcal{R}_f^q(fz, z) \\
=\;& \left[\lambda_1 d^q(fz, z) + \lambda_2 d^q(fz, f^2 z) + \lambda_3 d^q(z, fz) + \lambda_4 \left(\frac{d(z,fz)(1+d(fz,f^2z))}{1+d(fz,z)}\right)^q + \lambda_5 \left(\frac{d(z,f^2z)(1+d(fz,fz))}{1+d(fz,z)}\right)^q\right]^{\frac{1}{q}} \\
=\;& \left[\lambda_1 d^q(fz, z) + \lambda_2 d^q(fz, z) + \lambda_3 d^q(z, fz) + \lambda_4 \left(\frac{d(z,fz)(1+d(fz,z))}{1+d(fz,z)}\right)^q + \lambda_5 \left(\frac{d(z,z)(1+d(fz,fz))}{1+d(fz,z)}\right)^q\right]^{\frac{1}{q}} \\
=\;& \left[(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d^q(fz, z)\right]^{\frac{1}{q}} \\
=\;& \left[(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\right]^{\frac{1}{q}} d(fz, z) \\
\le\;& d(fz, z).
\end{aligned}$$

This is a contradiction, so that $fz = z$.

Case 2. For the case $q = 0$ taking $x = x_{n-1}$ and $y = x_n$ we have

$$\begin{aligned}
\mathcal{R}_f^q(x_{n-1}, x_n) = \;& [d(x_{n-1}, x_n)]^{\lambda_1} \cdot [d(x_{n-1}, fx_{n-1})]^{\lambda_2} \cdot [d(x_n, fx_n)]^{\lambda_3} \cdot \left[\frac{d(x_n,fx_n)(1+d(x_{n-1},fx_{n-1}))}{1+d(x_{n-1},x_n)}\right]^{\lambda_4} \cdot \left[\frac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1}))}{2}\right]^{\lambda_5} \\
\le\;& [d(x_{n-1}, x_n)]^{\lambda_1} \cdot [d(x_{n-1}, x_n)]^{\lambda_2} \cdot [d(x_n, x_{n+1})]^{\lambda_3} \cdot \left[\frac{d(x_n,x_{n+1})(1+d(x_{n-1},x_n))}{1+d(x_{n-1},x_n)}\right]^{\lambda_4} \cdot \left[\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_n,x_n))}{2}\right]^{\lambda_5} \\
\le\;& [d(x_{n-1}, x_n)]^{\lambda_1} \cdot [d(x_{n-1}, x_n)]^{\lambda_2} \cdot [d(x_n, x_{n+1})]^{\lambda_3} \cdot \left[\frac{d(x_n,x_{n+1})(1+d(x_{n-1},x_n))}{1+d(x_{n-1},x_n)}\right]^{\lambda_4} \cdot \frac{[d(x_{n-1},x_n)]^{\lambda_5}+[d(x_n,x_{n+1})]^{\lambda_5}}{2} \\
\le\;& [d(x_{n-1}, x_n)]^{\lambda_1+\lambda_2} \cdot [d(x_n, x_{n+1})]^{\lambda_3+\lambda_4} \cdot \frac{[d(x_{n-1},x_n)]^{\lambda_5}+[d(x_n,x_{n+1})]^{\lambda_5}}{2}
\end{aligned}$$

and from (5)

$$d(x_n, x_{n+1}) \le \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \le \psi(\mathcal{R}_f^q(x_{n-1}, x_n)). \tag{14}$$

As in the first case, we have that $d(x_{n-1}, x_n) > d(x_n, x_{n+1})$ since in the contrary case we have a contradiction. Indeed, if we suppose *ad absurdum* that $d(x_{n-1}, x_n) \le d(x_n, x_{n+1})$, we have

$$d(x_n, x_{n+1}) < \psi(\mathcal{R}_f^q(x_{n-1}, x_n)) < \left[d(x_n, x_{n+1}))\right]^{\lambda_1+\lambda_2\lambda_3+\lambda_4+\lambda_5} = d(x_n, x_{n+1}))$$

which is a contradiction. Then from (14) we obtain

$$d(x_n, x_{n+1}) \le \psi(\mathcal{R}_f^q(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)) \tag{15}$$

and inductively we get

$$d(x_n, x_{n+1}) \le \psi^n(d(x_0, x_1)).$$

By using the same arguments as the case $q > 0$ we shall easily obtain that $\{x_n\}$ is a Cauchy sequence in a complete metric space and so, there exists z such that $\lim_{n\to\infty} x_n = z$.

We claim that z is a fixed point of f.
Under the assumption that f is continuous we have

$$\lim_{n\to\infty} d(x_{n+1}, fz) = \lim_{n\to\infty} d(fx_n, fz) = 0,$$

and together with the uniqueness of limit, $fz = z$. Also, if f^2 is continuous, as in case (1) we have that $fz = z$ and then

$$\begin{aligned} d(z, fz) &= d(f^2z, fz) \le \alpha(fz, z)d(f^2z, fz) \le \psi(\mathcal{R}_f^q(f^2z, fz)) \\ &\le \psi([d(z, fz)]^{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5}) < d(z, fz). \end{aligned}$$

This contradiction shows us that $z = fz$.

□

Example. Let $X = [0, 2]$, $d : X \times X \to [0, \infty)$ be the usual metric, $d(x, y) = |x - y|$ for all $x, y \in X$ and the mapping $f : X \to X$ be defined by $f(x) = \begin{cases} 2/3, & \text{if } x \in [0, 1] \\ x/2, & \text{if } x \in (1, 2] \end{cases}$. Consider also a function $\alpha(x, y) = \begin{cases} 2, & \text{if } x, y \in [0, 1] \\ 1, & \text{if } x = 0, y = 2 \\ 0, & \text{otherwise} \end{cases}$ and the comparison function $\psi : [0, \infty) \to [0, \infty)$, $\psi(t) = t/5$. We can easily observe that the assumptions (i) and (ii) are satisfied and since $f^2(x) = 2/3$ is continuous, the assumption (iv) is also verified. For any $x, y \in [0, 1]$ we have $d(fx, fy) = 0$ so, the inequality (5) holds. For $x = 0$ and $y = 2$, we have

$$\begin{aligned} \alpha(0, 2)d(f0, f2) &= \alpha(0, 2)d(2/3, 1) = \tfrac{1}{3} < \tfrac{1}{5}\sqrt{\tfrac{505}{81}} = \tfrac{1}{5}\sqrt{\tfrac{1}{4}(4 + \tfrac{4}{9} + 1 + \tfrac{64}{81})} \\ &= \tfrac{1}{5}\left[\tfrac{1}{4}d^2(0, 2) + \tfrac{1}{4}d^2(0, f0) + +\tfrac{1}{4}d^2(2, f2) + \tfrac{1}{4}\left(\tfrac{d(2,f0)(1+d(0,f2))}{1+d(0,2)}\right)^2\right]^{1/2}. \end{aligned}$$

In all other cases, $\alpha(x, y) = 0$ and (5) is obviously satisfied. Thus, letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \frac{1}{4}$, $\lambda_4 = 0$ and $q = 2$ we obtain that f is an admissible hybrid contraction which satisfies the assumptions (i), (ii), (iv) of Theorem 2.2 and then $x = 0$ is the fixed point of f.

Theorem 2.3. *Let (X, d) be a complete metric space and let f be an admissible hybrid contraction, Suppose also that:*

1. *f is triangular α–orbital admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \ge 1$;*
3. *(X, d) is regular with respect to α.*

Then f possesses a fixed point.

Proof. Following the lines in the proof of Theorem 2.2, we already know that for any $q \ge 0$, the sequence $\{x_n\}$ is Cauchy, and due to the completeness of the metric space (X, d), there exists a point z such that $\lim_{n\to\infty} d(x_n, z) = 0$. Since the space X is regular with respect to α, inequality (5) together with the triangular inequality gives us

$$d(z, fz) \le d(z, x_{n+1}) + d(x_{n+1}, fz) \le \alpha(x_n, z)d(fx_n, fz) \le \psi(\mathcal{R}_f^q(x_n, z)) \le \mathcal{R}_f^q(x_n, z). \tag{16}$$

Again, we have to consider two separate cases. For the case $p > 0$,

$$\begin{aligned} \mathcal{R}_f^q(x_n, z) = & \left[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, fx_n) + \lambda_3 d^q(z, fz) + \lambda_4\left(\tfrac{d(z,fz)(1+d(x_n,fx_n))}{1+d(x_n,z)}\right)^q + \lambda_5\left(\tfrac{d(z,fx_n)(1+d(x_n,fz))}{1+d(x_n,z)}\right)^q\right]^{\frac{1}{q}} \\ = & \left[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, x_{n+1}) + \lambda_3 d^q(z, fz) + \lambda_4\left(\tfrac{d(z,fz)(1+d(x_n,x_{n+1}))}{1+d(x_n,z)}\right)^q + \lambda_5\left(\tfrac{d(z,x_{n+1})(1+d(x_n,fz))}{1+d(x_n,z)}\right)^q\right]^{\frac{1}{q}}. \end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{R}_f^q(x_n, z) = (\lambda_3 + \lambda_4)d(z, fz)$, letting $n \to \infty$ in (16) we obtain $d(z, fz) < d(z, fz)$ which implies that $fz = z$.
Similarly, for the case $q = 0$, we get $\lim_{n\to\infty} \mathcal{R}_f^q(x_n, z) = 0$ and then $d(z, fz) = 0$.

□

Corollary 2.4. *Let (X, d) be a complete metric space and the functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$. Let f be a self map on X such that:*

(i) f is triangular α–orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;
(iii) either, f is continuous, or
(iv) f^2 is continuous and $\alpha(fx, x) \geq 1$ for any $x \in Fix_{f^2}(X)$.

If one of the below conditions (c_1)-(c_3) is satisfied, then f has a fixed point $z \in X$, that is, $fz = z$.

(c_1) *$\alpha(x, y)d(x, y) \leq \psi(\mathcal{A}_f^q(x, y))$, where a_1, a_2, a_3, a_4 are non-negative real such that $a_1 + a_2 + a_3 + a_4 = 1$ and*

$$\mathcal{A}_f^q d(x, y) = \begin{cases} \left[a_1 d^q(x, y)(x, y) + a_2 d^q(x, fx) + a_3 d^q(y, fy) + a_4 \left(\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right)^q\right]^{\frac{1}{q}}, & \text{for } q > 0, x, y \in X \\ [d(x, y)]^{a_1} \cdot [d(x, fx)]^{a_2} \cdot [d(y, fy)]^{a_3} \cdot \left[\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right]^{a_4}, & \text{for } q = 0, x, y \in X \setminus Fix_f(X) \end{cases} \tag{17}$$

(c_2) *$\alpha(x, y)d(x, y) \leq \psi(\mathcal{B}_f^q(x, y))$, where b_1, b_2, b_3 are non-negative real such that $b_1 + b_2 + b_3 = 1$ and*

$$\mathcal{B}_f^q d(x, y) = \begin{cases} \left[b_1 d^q(x, y)(x, y) + b_2 d^q(x, fx) + b_3 d^q(y, fy)\right]^{\frac{1}{q}}, & \text{for } q > 0, x, y \in X \\ [d(x, y)]^{b_1} \cdot [d(x, fx)]^{b_2} \cdot [d(y, fy)]^{b_3}, & \text{for } q = 0, x, y \in X \setminus Fix_f(X). \end{cases} \tag{18}$$

(c_3) *$\alpha(x, y)d(x, y) \leq \psi(\mathcal{B}_f^q(x, y))$, where c_1, c_2 are non-negative real numbers such that $c_1 + c_2 = 1$ and*

$$\mathcal{C}_f^q d(x, y) = \begin{cases} \left[c_1 d^q(x, fx) + c_2 d^q(y, fy)\right]^{\frac{1}{q}}, & \text{for } q > 0, x, y \in X \\ [d(x, fx)]^{c_1} \cdot [d(y, fy)]^{c_2}, & \text{for } q = 0, x, y \in X \setminus Fix_f(X). \end{cases} \tag{19}$$

We can get a series of corollaries, considering in Corollary 2.4 by assigning $\psi \in \Psi$ properly, for example, by taking $\psi(t) = kt$ for any $t \geq 0$ with $k \in [0, 1)$, and/or $\alpha(x, y) = 1$ or both. Since it is apparent we skip the details.

Theorem 2.5. *If in Theorems 2.2 and 2.3, in the case $q > 0$, we assume supplementary that*

$$\alpha(x, y) \geq 1$$

for any $x, y \in Fix_f(X)$ then the fixed point of f is unique.

Proof. Let $v \in X$ be another fixed point of f, different from z. By replacing in (5), and taking into account the additional hypotheses, we have

$$\begin{aligned} d(z, v) \leq \alpha(z, v)(fz, fv) \leq \psi(\mathcal{R}_f^q(z, v)) < \mathcal{R}_f^q(z, v) = & \left[\lambda_1 d^q(z, v)\lambda_2 d^q(z, fz) + \lambda_3 d^q(v, fv) + \right. \\ & \left. +\lambda_4 \left(\frac{d(v,fv)(1+d(z,fz))}{1+d(z,v)}\right)^q + \lambda_5 \left(\frac{d(v,fz)(1+d(z,fv))}{1+d(z,v)}\right)^q\right]^{\frac{1}{q}} \\ = & \left[\lambda_1 d^q(z, v) + \lambda_2 d^q(z, z) + \lambda_3 d^q(v, v) + \right. \\ & \left. +\lambda_4 \left(\frac{d(v,v)(1+d(z,z))}{1+d(z,v)}\right)^q + \lambda_5 \left(\frac{d(v,z)(1+d(z,v))}{1+d(z,v)}\right)^q\right]^{\frac{1}{q}} \\ = & \; d(z, v)(\lambda_1 + \lambda_5)^{1/q} \leq d(z, v), \end{aligned}$$

which is a contradiction. Thus, $z = v$, so that f possesses exactly one fixed point. □

Example. Let $X = \{a, b, c, d\}$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$, $d(x, x) = 0$ for any $x, y \in X$ and

$$d(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(a, b), (b, c), (c, d)\} \\ 2, & \text{if } (x, y) \in \{(a, c), (b, d)\} \\ 3, & \text{if } (x, y) \in \{(a, d)\} \end{cases}$$

On metric space $(\mathcal{X}, d)$ let us define the self-mapping f by $f(a) = f(b) = a$, $f(c) = d$, $f(d) = b$. Consider also a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, where $\alpha(x, a) = \alpha(a, x) = 3$ for any $x \in \mathcal{X}$, $\alpha(b, d) = 1$, $\alpha(x, y) = 0$ otherwise and the comparison function $\psi : [0, \infty) \to [0, \infty)$, $\psi(t) = \sqrt[4]{\frac{3}{4}}t$. Since neither f, nor f^2 are continuous, Theorem 2.2 cannot be applied. On the other hand, is easy to see that f is triangular α–orbital admissible and also the assumptions (2), (3) from Theorem 2.3 are satisfied. Considering $q = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$ and $\lambda_5 = 0$ and taking into account the definition of function α, we remark that the only interesting case is for $x = b$ and $y = d$. We have in this case:

$$\begin{aligned}\alpha(b, d)d(fb, fd) = d(a, b) = \; & 1 < \sqrt{2} = \sqrt[4]{\tfrac{3}{4}} \cdot 2^{1/4} \cdot 1 \cdot 2^{1/4} \cdot (\tfrac{4}{3})^{1/4} \\ = \; & \sqrt[4]{\tfrac{3}{4}}[d(b, d)]^{\lambda_1} \cdot [d(b, fb)]^{\lambda_2} \cdot [d(d, fd)]^{\lambda_3} \cdot [\tfrac{d(d,fd)(1+d(b,fb))}{1+d(b,d)}]^{\lambda_4} \\ = \; & \psi\left([d(b, d)]^{\lambda_1} \cdot [d(b, fb)]^{\lambda_2} \cdot [d(d, fd)]^{\lambda_3} \cdot [\tfrac{d(d,fd)(1+d(b,fb))}{1+d(b,d)}]^{\lambda_4}\right).\end{aligned}$$

Consequently, the map f has a fixed point, that is $x = a$.

3 Ulam type stability

Considered as a type of data dependence, the notion of Ulam stability was started by Ulam [10, 11] and deveped by Hyers [12], Rassias [13], etc In this section we investigate the general Ulam type stability in sense of a fixed point problem.

Suppose that $f : \mathcal{X} \to \mathcal{X}$ is a self-mapping on a metric space $(\mathcal{X}, d)$. The fixed point problem

$$x = fx, \tag{20}$$

has the general Ulam type stability if and only if there exists an increasing function $\rho : [0, \infty) \leftrightarrow [0, \infty)$, continuous at 0 with $\rho(0) = 0$ such that for every $\varepsilon > 0$ and for each $y^* \in \mathcal{X}$ which satisfies the inequality

$$d(y^*, fy^*) \leq \varepsilon \tag{21}$$

there exists a solution $z \in \mathcal{X}$ of (20) such that

$$d(z, y^*) \leq \rho(\varepsilon). \tag{22}$$

In case that for $C > 0$, we consider $\rho(t) = Ct$ for all $t \geq 0$ then the fixed point equation (20) is said to be Ulam type stable.

On a metric space $(\mathcal{X}, d)$, the fixed point problem (20), where $f : \mathcal{X} \to \mathcal{X}$, is said to be well-posed if the following assumptions are satisfy:

1. f has a unique fixed point z in $\mathcal{X}$;
2. $d(x_n, z) = 0$ for each sequence $\{x_n\} \in \mathcal{X}$ such that $\lim_{n\to\infty} d(x_n, fx_n) = 0$.

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete metric space. If we add the condition $\lambda_1 + \lambda_5 < \frac{1}{c^2(q)}$, where $c(q) = \max\{1, 2^{q-1}\}$, to the assumptions of Theorem 2.5, then the following affirmations hold:*

(i) the fixed point equation (20) is Ulam-Hyers stable if $\alpha(u, v) \geq 1$ for any u, v satisfying the inequality (21);

(ii) the fixed point equation (20) is well-posed if $\alpha(x_n, z) \geq 1$ for any sequence $\{x_n\} \in \mathcal{X}$ such that $\lim_{n\to\infty} d(x_n, fx_n) = 0$ and $Fix_f(\mathcal{X}) = z$.

Proof. (i) Since from Theorem 2.5 we know that there is an unique $z \in \mathcal{X}$ such that $fz = z$, let $y^* \in \mathcal{X}$ such that

$$d(y^*, fy^*) \leq \varepsilon, \text{ for } \varepsilon > 0.$$

Obvious, z verifies (21) so we have that $\alpha(y^\star, z) \geq 1$ and then by using the triangular inequality we get

$$\begin{aligned}
d(z, y^\star) \leq d(fz, fy^\star) + d(fy^\star, y^\star) \leq\ & \alpha(y^\star, z)d(fy^\star, fz) + d(fy^\star, y^\star)\\
\leq\ & \psi(\mathcal{R}_f^d(y^\star, z)) + d(fy^\star, y^\star) < \mathcal{R}_f^d(y^\star, z) + d(fy^\star, y^\star)\\
\leq\ & \Big[\lambda_1 d^q(z, y^\star) + \lambda_2 d^q(y^\star, fy^\star) + \lambda_3 d^q(z, fz) + \lambda_4 \left(\tfrac{d(z,fz)(1+d(y^\star,fy^\star))}{1+d(y^\star,z)}\right)^q\\
& +\lambda_5 \left(\tfrac{d(z,fy^\star)(1+d(y^\star,fz))}{1+d(y^\star,z)}\right)^q\Big]^{\frac{1}{q}} + d(fy^\star, y^\star)\\
=\ & \Big[\lambda_1 d^q(z, y^\star) + \lambda_2 d^q(y^\star, fy^\star) + \lambda_3 d^q(z, z)+\\
& +\lambda_4 \left(\tfrac{d(z,z)(1+d(y^\star,fy^\star))}{1+d(y^\star,z)}\right)^q + \lambda_5 \left(\tfrac{d(z,fy^\star)(1+d(y^\star,z))}{1+d(y^\star,z)}\right)^q\Big]^{\frac{1}{q}} + d(fy^\star, y^\star)\\
\leq\ & \left[\lambda_1 d^q(z, y^\star) + \lambda_2 \varepsilon^q + \lambda_5 d^q(z, fy^\star)\right]^{\frac{1}{q}} + \varepsilon\\
\leq\ & \left[\lambda_1 d^q(z, y^\star) + \lambda_2 \varepsilon^q + \lambda_5 \left(d(z, y^\star) + d(y^\star, fy^\star)\right)^q\right]^{\frac{1}{q}} + \varepsilon\\
\leq\ & \left[\lambda_1 d^q(z, y^\star) + \lambda_2 \varepsilon^q + \lambda_5 \left(d(z, y^\star) + \varepsilon\right)^q\right]^{\frac{1}{q}} + \varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d^q(z, y^\star) &\leq c(q)\left[\lambda_1 d^q(z, y^\star) + \lambda_2 \varepsilon^q + \lambda_5 \left(d(z, y^\star) + \varepsilon\right)^q + \varepsilon^q\right]\\
&\leq c(q)\left[\lambda_1 d^q(z, y^\star) + \lambda_2 \varepsilon^q + \lambda_5 c(q)\left(d^q(z, y^\star) + \varepsilon^q\right) + \varepsilon^q\right],
\end{aligned}$$

where $c(q) = \max\left\{1, 2^{q-1}\right\}$. By simple calculation, from the above inequality we have

$$d^q(z, y^\star) \leq \frac{(1 + \lambda_2 + c(q)\lambda_5)c(q)}{1 - c(q)\lambda_1 - c^2(q)\lambda_5}\varepsilon^q,$$

which is equivalent with

$$d(z, y^\star) \leq C\varepsilon,$$

where $C = \left(\frac{(1+\lambda_2+c(q)\lambda_5)c(q)}{1-c(q)\lambda_1-c^2(q)\lambda_5}\right)^{\frac{1}{q}}$, for any $q > 0$ and $\lambda_1, \lambda_5 \in [0, 1)$ such that $\lambda_1 + \lambda_5 < \frac{1}{c^2(q)}$.
(ii) Taking into account the supplementary condition and since $Fix_f(X) = z$ we have

$$\begin{aligned}
d(x_n, z) &\leq d(x_n, fx_n) + d(fx_n, fz)\\
&\leq d(x_n, fx_n) + \alpha(x_n, z)d(fx_n, fz)\\
&\leq d(x_n, fx_n) + \psi(\mathcal{R}_f^d(x_n, z))\\
&< d(x_n, fx_n) + \mathcal{R}_f^d(x_n, z)\\
&\leq \Big[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, fx_n) + \lambda_3 d^q(z, fz)\\
&\quad +\lambda_4 \left(\tfrac{d(z,fz)(1+d(x_n,fx_n))}{1+d(x_n,z)}\right)^q + \lambda_5 \left(\tfrac{d(z,fx_n)(1+d(x_n,fz))}{1+d(x_n,z)}\right)^q\Big]^{\frac{1}{q}} + d(x_n, fx_n)\\
&= \left[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, fx_n) + \lambda_5 d^q(z, fx_n)\right]^{\frac{1}{q}} + d(x_n, fx_n)\\
&\leq \left[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, fx_n) + \lambda_5 (d(z, x_n) + d(x_n, fx_n))^q\right]^{\frac{1}{q}} + d(x_n, fx_n)\\
&\leq \left[\lambda_1 d^q(x_n, z) + \lambda_2 d^q(x_n, fx_n) + \lambda_5 c(q)(d^q(z, x_n) + d^q(x_n, fx_n))\right]^{\frac{1}{q}} + d(x_n, fx_n),
\end{aligned}$$

or,

$$d(x_n, z)^q \leq \frac{(1 + \lambda_2 + c(q)\lambda_5)c(q)}{1 - c(q)\lambda_1 - c^2(q)\lambda_5} d^q(x_n, fx_n).$$

Letting $n \to \infty$ in the above inequality and keeping in mind that $\lim_{n\to\infty} d(x_n, fx_n) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n, z) = 0$$

that is, the fixed point equation (20) is well-posed.

□

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