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Sets of p-restriction and p-spectral synthesis

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Abstract: In this paper we investigate the restriction problem. More precisely, we give sufficient conditions for the failure of a set E in $\mathbb{R}^n$ to have the p-restriction property. We also extend the concept of spectral synthesis to $L^p(\mathbb{R}^n)$ for sets of p-restriction when $p > 1$. We use our results to show that there are p-values for which the unit sphere is a set of p-spectral synthesis in $\mathbb{R}^n$ when $n \geqslant 3$.

Keywords: p-spectral synthesis, restriction problem, set of p-restriction, span of translates

MSC: 42B10, 43A15

1 Introduction

Throughout this paper $\mathbb{R}$ will denote the real numbers and $\mathbb{Z}$ will denote the integers. Let $p \in [1, \infty]$ and let n be a positive integer. Indicate by $L^p(\mathbb{R}^n)$ the usual Lebesgue space and denote by $\|\cdot\|_{L^p}$ the usual Banach space norm on $L^p(\mathbb{R}^n)$. Also p' will always represent the conjugate index of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$. For $f \in L^1(\mathbb{R}^n)$ the Fourier transform of f is defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx, \ \xi \in \mathbb{R}^n.$$

The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R}^n)$ and by the Hausdorff-Young inequality, $\mathcal{F}$ can be extended to a continuous operator from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, when $1 < p < 2$. Let E be a closed subset of $\mathbb{R}^n$. If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is continuous on $\mathbb{R}^n$. Consequently, the restriction of $\hat{f}$ to E, which we denote by $\hat{f}\,|_E$, is a well-defined function on E. For $1 < p < 2$ and $f \in L^p(\mathbb{R}^n)$, $\hat{f} \in L^{p'}(\mathbb{R}^n)$. So if E is a set of positive Lebesgue measure we can restrict $\hat{f}$ to E. The interesting question is can $\hat{f}$ be restricted to E when E has Lebesgue measure zero? This question is the heart of the restriction problem, which we will now describe.

For $E \subseteq \mathbb{R}^n$, let $C(E)$ be the set of continuous functions on E and let $C_c(E)$ be the set of functions in $C(E)$ with compact support. Let $L^p(E)$ be the usual Banach space formed with respect to the induced measure $d\sigma$ on E. The norm on $L^p(E)$ will be denoted by $\|\cdot\|_{L^p(E)}$. Recall that the norm on $L^p(\mathbb{R}^n)$ is indicated by $\|\cdot\|_{L^p}$. Let $2 \leqslant n \in \mathbb{Z}$ and let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$. The operator $\mathcal{R}_E \colon \mathcal{S}(\mathbb{R}^n) \to C(E)$ given by

$$\mathcal{R}_E(f) = \hat{f}\,|_E$$

is known as the *restriction operator* associated with E. If $\mathcal{R}_E$ can be extended to a continuous operator from $L^p(\mathbb{R}^n) \to L^q(E)$, then we shall say that $\mathcal{R}_E$ has property $\mathcal{R}(E, p, q)$. Observe that if $1 \leqslant q_1 \leqslant q_2$ and $\mathcal{R}_E$ has property $\mathcal{R}(E, p, q_2)$, then it also has property $\mathcal{R}(E, p, q_1)$. We shall say that E is a set of *p-restriction* if $\mathcal{R}_E$ has property $\mathcal{R}(E, p, 1)$. Note that any closed set in $\mathbb{R}^n$ is a set of 1-restriction. Furthermore, if E is not a set of p-restriction, then $\mathcal{R}_E$ does not have property $\mathcal{R}(E, p, q)$ for any $q \geqslant 1$. The best known result concerning the restriction of $\hat{f}$ to E is the Stein-Tomas theorem: $\mathcal{R}_E$ has property $\mathcal{R}(E, p, 2)$ if and only if $1 \leqslant p \leqslant \frac{2n+2}{n+3}$, where E is a smooth compact hypersurface in $\mathbb{R}^n$ with nonzero Gaussian curvature. In general though it is an extremely difficult problem to determine if $\mathcal{R}_E$ has property $\mathcal{R}(E, p, q)$. A more comprehensive treatment of

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the restriction problem, along with its history, can be found in [1–3], and [4, Chapter 5.4] and the references therein.

In this paper we will only be concerned with the case where $\mathcal{R}_E$ has property $\mathcal{R}(E, p, 1)$, that is E is a set of p-restriction. Set

$$J(E) = \{f \in \mathcal{S}(\mathbb{R}^n) \mid \hat{f}\,|_E = 0\},$$

and if E is a set of p-restriction define

$$I^p(E) = \{f \in L^p(\mathbb{R}^n) \mid \hat{f}\,|_E = 0\}.$$

This paper was inspired by the paper [5] where these spaces were investigated for the case when E is the unit sphere S^{n-1} in $\mathbb{R}^n$. Our first main result is:

Theorem 1.1. *Let E be a smooth compact submanifold of codimension k in $\mathbb{R}^n$. If there exists $f \in C_c(\mathbb{R}^n)$ such that $\hat{f}$ vanishes on E, then E is not a set of p-restriction for $\frac{2n}{n+k} \leqslant p \in \mathbb{R}$.*

Thus $\mathcal{R}_E$ does not have property $\mathcal{R}(E, p, q)$ when $p \geqslant \frac{2n}{n+k}$ and $q \geqslant 1$. If E is a hypersurface, then the lower bound for p becomes $\frac{2n}{n+1}$. We are able to improve this lower bound for hypersurfaces with the constant relative nullity condition, which we now define. Let U be an open set in $\mathbb{R}^{n-1}$ and let $F = \{(x, \phi(x)) \mid x \in U\}$ be a smooth hypersurface in $\mathbb{R}^n$. If the Hessian matrix

$$\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)$$

of ϕ has constant rank $n-1-\nu$ on U, where $0 \leqslant \nu \leqslant n-1$, then we say that ϕ has *constant relative nullity* ν. A smooth hypersurface E of $\mathbb{R}^n$ is said to have *constant relative nullity* ν if every localization F of E has constant relative nullity ν. If $\nu = n-1$, then E is a hyperplane. It is known that hyperplanes are sets of p-restriction only if $p = 1$. Thus we will only consider hypersurfaces with $0 \leqslant \nu \leqslant n-2$. Note that $\nu = 0$ for S^{n-1}.

Theorem 1.2. *Let $2 \leqslant n \in \mathbb{Z}$ and let E be a smooth compact hypersurface in $\mathbb{R}^n$ with constant relative nullity ν, for $0 \leqslant \nu \leqslant n-2$. If $\frac{2(n-\nu)}{n-\nu+1} \leqslant p \in \mathbb{R}$, then E is not a set of p-restriction.*

Let $f \in L^p(\mathbb{R}^n)$ and let $y \in \mathbb{R}^n$. The translate of f by y, which we write as f_y, is the function $f_y(x) = f(x-y)$, where $x \in \mathbb{R}^n$. For $f \in L^p(\mathbb{R}^n)$, let $T^p[f]$ be the closed subspace of $L^p(\mathbb{R}^n)$ spanned by f and its translates. The zero set $Z(f)$ of $f \in L^1(\mathbb{R}^n)$ is defined by

$$Z(f) = \{\xi \in \mathbb{R}^n \mid \hat{f}(\xi) = 0\}.$$

In Section 3 we will see that if $Z(f)$ is a set of p-restriction, then $T^p[f] \neq L^p(\mathbb{R}^n)$. If $X \subset L^p(X)$, then $\overline{X}^p$ will denote the closure of X in $L^p(\mathbb{R}^n)$.

We will now briefly review the concept of spectral synthesis in $L^1(\mathbb{R}^n)$. Suppose I is a closed ideal in $L^1(\mathbb{R}^n)$ and define the zero set of I by

$$Z(I) = \bigcap_{f \in I} Z(f).$$

Let E be a closed set in $\mathbb{R}^n$, then $I^1(E)$ is a closed ideal in $L^1(\mathbb{R}^n)$ with zero set E. In fact, $I^1(E)$ is the largest closed ideal in $L^1(\mathbb{R}^n)$ whose zero set is E. Now let

$$k(E) = \{f \in \mathcal{S}(\mathbb{R}^n) \mid \hat{f} = 0 \text{ on a neighborhood of } E\}.$$

Then

$$k(E) \subseteq J(E) \subseteq L^p(\mathbb{R}^n)$$

and $\overline{k(E)}^1$ is the smallest closed ideal in $L^1(\mathbb{R}^n)$ with zero set E. The set E is known as a set of spectral synthesis if $\overline{k(E)}^1 = I^1(E)$. A more detailed account of spectral synthesis can be found in [6][7, Chapter 7]. Extending the concept of spectral synthesis to $L^p(\mathbb{R}^n)$ for $p > 1$ falls short since the analog to $I^1(E)$, $I^p(E)$, is not well-defined for closed sets of Lebesgue measure zero in $\mathbb{R}^n$. However, for sets of p-restriction $I^p(E)$ is well-defined, which allows us to extend the idea of spectral synthesis to $L^p(\mathbb{R}^n)$ for sets E of p-restriction. We shall say that a set E of p-restriction is a set of *p-spectral synthesis* if

$$\overline{k(E)}^p = I^p(E).$$

We can now state:

Theorem 1.3. *Let $2 \leqslant n \in \mathbb{Z}$ and let E be a smooth compact hypersurface in $\mathbb{R}^n$ with constant relative nullity ν, $0 \leqslant \nu \leqslant n-2$. If E is a set of p-restriction for some p that satisfies one of the following:*

1. $\frac{2(n-\nu)}{n+3-\nu} \leqslant p < 2$ *and* $0 \leqslant \nu < n-3$
2. $1 < p < 2$ *for* $n-3 \leqslant \nu < n-1$,

then E is a set of p-spectral synthesis.

It is known that S^1 is a set of spectral synthesis in $\mathbb{R}^2$ [8], but S^{n-1} is not a set of spectral synthesis in $\mathbb{R}^n$ for $n \geqslant 3$ [7, Chapter 7.3]. We will use Theorem 1.3 to show that there are p-values where S^{n-1} is a set of p-spectral synthesis when $n \geqslant 3$.

This paper is organized as follows: In Section 2 we give some background and results that will be needed for this paper. In Section 3 we will prove Theorems 1.1 and 1.2 by linking them to the problem of determining when $T^p[f]$ is dense in $L^p(\mathbb{R}^n)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{f} = 0$ on E. In Section 4 we prove Theorem 1.3, and use the theorem to show that there are p-values for which the unit sphere S^{n-1} is a set of p-spectral synthesis in $\mathbb{R}^n$ for $n \geqslant 3$.

2 Preliminaries

In this section we will give some results that will be used in the sequel. The convolution of two measurable functions f and g on $\mathbb{R}^n$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

Let $1 < p < 2$. Each $\phi \in L^{p'}(\mathbb{R}^n)$ defines a bounded linear functional T_ϕ on $L^p(\mathbb{R}^n)$ via

$$T_\phi(f) = \int_{\mathbb{R}^n} f(-x)\phi(x)\,dx.$$

Sometimes we will write $\langle f, \phi \rangle$ in place of $T_\phi(f)$. For closed subspaces X in $L^p(\mathbb{R}^n)$,

$$\mathrm{Ann}(X) = \{\phi \in L^{p'}(\mathbb{R}^n) \mid T_\phi(f) = 0 \text{ for all } f \in X\},$$

will denote the annihilator of X in $L^{p'}(\mathbb{R}^n)$. The following characterization of $\mathrm{Ann}(X)$ when X is a translation-invariant subspace of $L^p(\mathbb{R}^n)$ will be needed later.

Proposition 2.1. *Let X be a translation-invariant subspace of $L^p(\mathbb{R}^n)$. Then $\phi \in \mathrm{Ann}(X)$ if and only if $f * \phi = 0$ for all $f \in X$.*

Proof. Observe that for $f \in L^p(\mathbb{R}^n)$ and $\phi \in L^{p'}(\mathbb{R}^n)$

$$f * \phi(x) = \int_{\mathbb{R}^n} f(x-y)\phi(y)\,dy = \int_{\mathbb{R}^n} f_{-x}(-y)\phi(y)\,dy = T_\phi(f_{-x}).$$

It follows from the translation invariance of X that $f * \phi = 0$ for all $f \in X$ if and only if $\phi \in \mathrm{Ann}(X)$. □

The space $L^p(\mathbb{R}^n)$ is a $L^1(\mathbb{R}^n)$-module since $f * g \in L^p(\mathbb{R}^n)$ whenever $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. The following proposition will not be used in the paper, but we record it here for its independent interest.

Proposition 2.2. *If E is a set of p-restriction, then $I^p(E)$ is a $L^1(\mathbb{R}^n)$-submodule of $L^p(\mathbb{R}^n)$.*

Proof. A modification of the proof of [7, Theorem 7.1.2] will show that a closed translation-invariant subspace of $L^p(\mathbb{R}^n)$ is translation invariant if and only if it is a $L^1(\mathbb{R}^n)$-submodule of $L^p(\mathbb{R}^n)$. The proposition now follows since $I^p(E)$ is a closed translation-invariant subspace of $L^p(\mathbb{R}^n)$. □

It is well known that the Fourier transform is an isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, with the inverse Fourier transform given by

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i(\xi \cdot x)}\, d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. A continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ is known as a *temperate distribution*. A nice property of temperate distributions is that the Fourier transform can be extended to them. In fact, the Fourier transform defines an isomorphism on the temperate distributions. Indeed, if T is a tempered distribution, then $\widehat{T}$ is the tempered distribution given by

$$\widehat{T}(f) = T(\widehat{f})$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The inverse Fourier transform $\check{T}$ of a temperate distribution T is defined by

$$\check{T}(f) = T(\check{f}),$$

where $f \in \mathcal{S}(\mathbb{R}^n)$. Since elements of $L^p(\mathbb{R}^n)$ are temperate distributions, we can define the Fourier transform $\widehat{f}$ for $f \in L^p(\mathbb{R}^n)$ in the distributional sense when $p > 2$. *For the rest of this paper, distribution will mean temperate distribution.*

We shall write $\operatorname{supp}(\psi)$ to indicate the support of ψ, where depending on the context, ψ is a function, measure, or distribution.

We conclude this section with a result that will be needed later.

Proposition 2.3. *If E is a compact subset of $\mathbb{R}^n$, then there exists an $f \in \mathcal{S}(\mathbb{R}^n)$ for which $Z(f) = E$.*

Proof. Let B be an open ball containing E and let $x \in B \backslash E$. The Whitney extension theorem produces a smooth function $f_x \colon B \to \mathbb{R}$ such that $f_x = 0$ on E and $f_x > 0$ at x. For the purpose of this proof only, f_x will mean the function defined above instead of the translate of f. For each $x \in B \backslash E$ there exists an open ball B_x for which f_x is positive on B_x. Now choose a countable subcover B_{x_n} of $B \backslash E$. Let

$$a_n = n^{-2}[\sup_B(f_{x_n})]^{-1}.$$

Then

$$g = \sum_{n=1}^{\infty} a_n f_{x_n}$$

is a smooth function on B. Let B_1 be an open ball satisfying $E \subseteq B_1 \subseteq B$. Denote by h the smooth function obtained by multiplying g by a smooth function that equals one on B_1 and zero on $\mathbb{R}^n \backslash B$. Set $F = h + s$ where $s \in \mathcal{S}(\mathbb{R}^n)$ that is zero on B_1 and positive on $\mathbb{R}^n \backslash \overline{B_1}$, where $\overline{B_1}$ is the closure of B_1. Thus $F \in \mathcal{S}(\mathbb{R}^n)$ and can be expressed as $\widehat{f}$ for some $f \in \mathcal{S}(\mathbb{R}^n)$. The proof of the proposition is now complete since $F^{-1}(0) = E$. □

3 Proofs of Theorems 1.1 and 1.2

Let E be a compact set in $\mathbb{R}^n$ with induced measure $d\sigma$. Suppose E has the p-restriction property. This is equivalent to the existence of a constant C that depends on p and n and satisfies

$$\|\widehat{f}\|_{L^1(E)} \leqslant C\|f\|_{L^p} \tag{3.1}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Condition (3.1) is equivalent to the dual condition

$$\|\widecheck{Fd\sigma}\|_{L^{p'}} \leqslant C\|F\|_{L^\infty(E)} \tag{3.2}$$

for all smooth functions F on E, and where $\widecheck{Fd\sigma}$ is the inverse Fourier transform of the measure $Fd\sigma$. Recall that the inverse Fourier transform of a finite Borel measure $d\mu$ is

$$\widecheck{d\mu}(x) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)}\, d\mu(\xi),$$

where $x \in \mathbb{R}^n$. Setting $F \equiv 1$ on E we see from (3.2) that $\widecheck{d\sigma} \in L^{p'}(\mathbb{R}^n)$. We record this as:

Lemma 3.1. *Let $1 < p < 2$ and let E be a compact set in $\mathbb{R}^n$ with induced measure $d\sigma$. If E is a set of p-restriction, then $\widecheck{d\sigma} \in L^{p'}(\mathbb{R}^n)$.*

Proposition 3.2. *Let $1 < p < 2$ and let E be a compact subset of $\mathbb{R}^n$ with induced measure $d\sigma$. If E is a set of p-restriction, then $I^p(E) \neq L^p(E)$.*

Proof. Let $f \in I^p(E)$ and let $\phi(x) = \widecheck{d\sigma}(x)$. By Lemma 3.1, $\phi(x) \in L^{p'}(\mathbb{R}^n)$. Let (f_n) be a sequence of Schwartz functions that satisfy $\|f_n - f\|_{L^p} \to 0$. For $x \in \mathbb{R}^n$,

$$|f * \phi(x)| = \lim_{n\to\infty} |f_n * \phi(x)| = \lim_{n\to\infty} \left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)} \widehat{f_n}(\xi) d\sigma(\xi)\right|.$$

Because $\widehat{f}\,|_E = 0$ we obtain

$$\begin{aligned} \lim_{n\to\infty} \left|\int_E e^{2\pi i(x\cdot\xi)} \widehat{f_n}(\xi) d\sigma(\xi)\right| &\leqslant \lim_{n\to\infty} \int_E |(\widehat{f} - \widehat{f_n})(\xi) e^{2\pi i(x\cdot\xi)}| d\sigma(\xi) \\ &\leqslant \lim_{n\to\infty} \|\widehat{f} - \widehat{f_n}\|_{L^1(E)}. \end{aligned}$$

Since $\mathcal{R}_E$ has property $\mathcal{R}(E, p, 1)$, $\lim_{n\to\infty} \|\widehat{f} - \widehat{f_n}\|_{L^1(E)} = 0$. Hence, $f * \phi = 0$ due to supp $d\sigma \subseteq E$ and ϕ is a nonzero element in $\operatorname{Ann}(I^p(E))$ by Proposition 2.1. Thus $I^p(E) \neq L^p(\mathbb{R}^n)$. □

The following corollary to Proposition 3.2, which is crucial for the proofs of Theorems 1.1 and 1.2, gives a useful criterion in terms of $T^p[f]$ to determine when E is not a set of p-restriction.

Corollary 3.3. *Let $1 < p < 2$ and let E be a compact subset of $\mathbb{R}^n$. If there exists an $f \in L^1(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$ for which $E \subseteq Z(f)$ and $T^p[f] = L^p(\mathbb{R}^n)$, then E is not a set of p-restriction.*

Proof. Assume E is a set of p-restriction. Then by Proposition 3.2 $I^p(E) \neq L^p(\mathbb{R}^n)$. Since $Z(f_y) = Z(f)$ for all $y \in \mathbb{R}^n$, $T^p[f] \subseteq I^p(E)$ which contradicts our hypothesis $T^p[f] = L^p(\mathbb{R}^n)$. □

3.1 Proof of Theorem 1.1

It was shown in [9, Corollary 1] that if $f \in C_c(\mathbb{R}^n)$ and if k is the minimal codimension of $Z(f)$ in $\mathbb{R}^n$, then $T^p[f] = L^p(\mathbb{R}^n)$ whenever $2n/(n+k) \leqslant p < \infty$. Theorem 1.1 now follows by combining this fact with Corollary 3.3.

3.2 Proof of Theorem 1.2

We will prove Theorem 1.2 by proving a more general theorem. We start with a definition. Let E be a closed subset of $\mathbb{R}^n$. We shall say that E is *p-thin* if the only distribution T that satisfies supp $T \subseteq E$ and $\check{T} \in L^p(\mathbb{R}^n)$ is $T = 0$.

Theorem 3.4. *Let $1 < p < 2$ and let E be a compact subset of $\mathbb{R}^n$. If E is p'-thin, then E is not a set of p-restriction.*

Proof. Assume that E is p'-thin. Let $f \in \mathcal{S}(\mathbb{R}^n)$ with $Z(f) = E$. The theorem will follow from Corollary 3.3 if we can show $T^p[f] = L^p(\mathbb{R}^n)$. Suppose instead that $T^p[f] \neq L^p(\mathbb{R}^n)$. By Proposition 2.1 there exists a nonzero $\phi \in L^{p'}(\mathbb{R}^n)$ for which $f * \phi = 0$, which implies $\operatorname{supp} \hat{\phi} \subseteq Z(f)$ because $\widehat{f * \phi} = \hat{f}\hat{\phi}$. Due to our assumption E is p'-thin, $\phi = 0$, a contradiction. Hence $T^p[f] = L^p(\mathbb{R}^n)$. □

It was shown in [10, Theorem 1] that if a set E satisfies the hypothesis of Theorem 1.2 then E is p'-thin. Therefore, E is not a set of p-restriction and Theorem 1.2 is proved.

4 *p*-spectral synthesis

We start with a definition. Let E be a k-dimensional submanifold in $\mathbb{R}^n$ with induced Lebesgue measure $d\sigma$. We shall say that E has the *p-approximate property* if for each distribution T with $\operatorname{supp} T \subseteq E$ and $\check{T} \in L^p(\mathbb{R}^n)$, we can find a sequence of measures T_j on E, absolutely continuous with respect to $d\sigma$, such that $\|\check{T}_j - \check{T}\|_{L^p} \to 0$ as $j \to \infty$. Our results on sets of p-spectral synthesis are an immediate consequence of previous work by Guo on sets that have the p-approximate property [11, 12]. In fact, it is stated in [11] that the p-approximate property is a variation of the spectral synthesis property. Sets with the p-restriction property allows us to make this statement more transparent. Specifically, Theorem 4.1 will show that p-spectral synthesis follows from the p'-approximate property for submanifolds with the p-restriction property.

4.1 Proof of Theorem 1.3

Suppose E is a k-dimensional submanifold of $\mathbb{R}^n$ and let $d\sigma$ be the induced Lebesgue measure on E. Also assume that E is a set of p-restriction for some $1 < p < 2$. Let Φ denote the closed subspace of $L^{p'}(\mathbb{R}^n)$ generated by

$$\{\widecheck{Fd\sigma} \mid F \text{ is smooth on } E\}.$$

The next result will be needed in the proof of Theorem 1.3.

Theorem 4.1. *Let $1 < p < 2$ and let E be a compact, smooth k-dimensional submanifold of $\mathbb{R}^n$ and assume that E has the p-restriction property. Let $d\sigma$ be the induced measure on E. If E has the p'-approximate property, then E is a set of p-spectral synthesis.*

Proof. Since E is a set of p-restriction, $\widecheck{d\sigma} \in L^{p'}(\mathbb{R}^n)$ by Lemma 3.1. We begin by showing $\Phi \subseteq \operatorname{Ann}(I^p(E))$. Let $\phi \in \Phi$. We can assume that $\phi = \widecheck{d\mu}$, where $d\mu = Fd\sigma$ for some smooth function F on E. Let $f \in I^p(E)$, using the argument from Proposition 3.2 we obtain that $f * \phi = 0$, which implies that $\phi \in \operatorname{Ann}(I^p(E))$ by Proposition 2.1.

Now let $\phi \in \operatorname{Ann}(\overline{k(E)}^p)$. Since $\operatorname{supp}(\hat{\phi}) \subseteq E$ and E has the p'-approximate property, there exists a sequence of measures $F_n d\sigma$, where F_n is smooth on E, such that $\|\widecheck{F_n d\sigma} - \phi\|_{L^{p'}} \to 0$. Thus $\phi \in \Phi$, which implies $\operatorname{Ann}(\overline{k(E)}^p) \subseteq \operatorname{Ann}(I^p(E))$. Clearly, $\operatorname{Ann}(I^p(E)) \subseteq \operatorname{Ann}(\overline{k(E)}^p)$. Therefore, E is a set of p-spectral synthesis. □

Now suppose E satisfies the hypotheses of Theorem 1.3 and recall that ν denotes the constant relative nullity of E. In [12, Theorem 2] it was proved that E has the p'-approximate property for $2 \leqslant p' \leqslant 2(n-\nu)/(n-3-\nu)$ when $1 \leqslant \nu < n-3$ and $2 < p' < \infty$ when $n-3 \leqslant \nu < n-1$. Furthermore, it was shown in [11, Theorem 1] that when $\nu = 0$, E has the p'-approximate property for $2 < p' < 2n/(n-3)$ when $n > 3$ and for $2 < p' < \infty$ when $n = 2$ or $n = 3$. Theorem 1.3 now follows from combining these last two sentences with Theorem 4.1.

4.2 p-spectral synthesis and the unit sphere

We mentioned in the Introduction that for $n \geqslant 3$, S^{n-1} is not a set of spectral synthesis. Using Theorem 1.3 we will be able to show that there are p-values for which S^{n-1} is a set of p-spectral synthesis. Guo proved in [11, Theorem 1] that S^2 has the p'-approximate property when $p' > 2$ and $n = 3$; and S^{n-1} has the p'-approximate property for $2 < p' \leqslant \frac{2n}{n-3}$ when $n \geqslant 4$. It follows from the Stein-Tomas theorem that S^{n-1} is a set of p-restriction for $1 \leqslant p \leqslant \frac{2n+2}{n+3}$. Consequently, Theorem 4.1 yields that S^2 is a set of p-spectral synthesis for $1 < p \leqslant \frac{4}{3}$ and for $n \geqslant 4$, S^{n-1} is a set of p-spectral synthesis for $\frac{2n}{n+3} \leqslant p \leqslant \frac{2n+2}{n+3}$. The upper bound in this inequality is probably not sharp, in fact it would not surprise us if it is $\frac{2n}{n+1}$. However, the lower bound is sharp. Indeed, [12, Lemma 2.3(ii)] tells us that a distribution ϕ can be constructed for which $\phi \in L^{p'}(\mathbb{R}^n)$ for $p' > \frac{2n}{n-3}$, and $\operatorname{supp}(\widehat{\phi}) \subseteq S^{n-1}$ that satisfies for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle f, \phi \rangle = \int_{S^{n-1}} \frac{\partial \widehat{f}}{\partial x_n}\, d\sigma.$$

Since there exists $f \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{f}\,|_{S^{n-1}} = 0$ and $\frac{\partial \widehat{f}}{\partial x_n}\,|_{S^{n-1}} \neq 0$, $\phi \notin \operatorname{Ann}(I^p(S^{n-1}))$. Thus, for $n \geqslant 4$, S^{n-1} is not a set of p-spectral synthesis for $1 \leqslant p < \frac{2n}{n+3}$.

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