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Quantum (q, h)-Bézier surfaces based on bivariate (q, h)-blossoming

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Abstract: We introduce the (q, h)-blossom of bivariate polynomials, and we define the bivariate (q, h)-Bernstein polynomials and (q, h)-Bézier surfaces on rectangular domains using the tensor product. Using the (q, h)-blossom, we construct recursive evaluation algorithms for (q, h)-Bézier surfaces and we derive a dual functional property, a Marsden identity, and a number of other properties for bivariate (q, h)-Bernstein polynomials and (q, h)-Bézier surfaces. We develop a subdivision algorithm for (q, h)-Bézier surfaces with a geometric rate of convergence. Recursive evaluation algorithms for quantum (q, h)-partial derivatives of bivariate polynomials are also derived.

Keywords: Bézier surface, bivariate (q, h)-blossom, bivariate (q, h)-Bernstein basis, Marsden identity, quantum differentiation, recursive evaluation, subdivision

MSC: 41XX, 65D15, 65D17

1 Introduction

The notion of quantum h-, q-, and (q, h)-blossoms of univariate polynomials and the corresponding theories of quantum Bernstein-Bézier curves were introduced in [1–4] by Simeonov, Zafiris, and Goldman, as an extension of the classical blossom, with applications to h- and q-Bézier curves. A non-blossoming approach to proving some related properties and identities is in [5]. Goldman and Simeonov also introduced and investigated quantum Bézier and B-spline curves and their properties using quantum blossoms [6, 7]. Another general analogue is in [8, 9]. The q-Bernstein basis functions were first introduced and studied by G. Phillips and his coauthors [10–13]. An h-version was proposed earlier by Stancu [14, 15] in the context of uniform polynomial approximation of continuous functions. Algorithms bases on polynomial blossoming are elegant and computationally efficient and have great uses and potential in computer-aided design (CAD) and applications [16–20]. The classical notion and basic theory of Bézier curves and splines using polynomial blossoms was introduced by L. Ramshaw [21, 22]. Bézier curves and surfaces were first utilized by the French engineer Pierre Bézier to design and model aerodynamic car shapes for Renault. While the theories of univariate polynomial blossoms and curves have been well-studied and generalized in various directions and in very non-trivial ways, the corresponding multivariate quantum theories for surfaces have received much less attention, despite the fact that modeling by polynomial surfaces is far more important for modeling and CAD. One of the first works on quantum surfaces using polynomial blossom approach is [23].

The main goal of this paper is to introduce a (q, h)-blossom for bivariate polynomials, and extend the main results of [1] to (q, h)-Bézier surfaces. In particular, we will introduce bivariate (q, h)-Bernstein polynomial bases and prove a dual functional property for (q, h)-Bézier surfaces, which will be used to develop recursive evaluation and subdivision algorithm for these surfaces. The importance of this study is due to the

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fact that polynomial surfaces and their Bernstein–Bézier forms have fundamental applications in geometric modeling and CAD [16, 18]. This bivariate (q,h)-Bernstein–Bézier theory is readily generalized in a straightforward way to multivariate polynomials and higher-dimensional surfaces.

The paper is organized as follows. In Section 2 we introduce the relevant notation and terminology, and we define the quantum (q,h)-blossom for polynomials of two variables. We then establish the existence and uniqueness of this (q,h)-blossom. Using the homogeneous analog of the (q,h)-blossom, we derive an explicit formula for quantum (q,h)-partial derivatives in terms of this (q,h)-blossom. In Section 3 we introduce the bivariate (q,h)-Bernstein basis polynomials. We obtain recurrence relations and (q,h)-de Casteljau evaluation algorithms. We construct $n!m!$ affine invariant, recursive evaluation algorithms for polynomials on rectangular domains. Then we show that every bivariate polynomial is a (q,h)-Bézier surface. We end Section 3 by establishing the dual functional property of the bivariate (q,h)-blossom. Section 4 contains various identities, including a generalization of Marsden's identity to bivariate (q,h)-Bernstein polynomials and a partition of unity property. In Section 5 we construct a subdivision algorithm for (q,h)-Bézier surfaces and we prove its geometric rate of convergence. We conclude with Section 6, where we derive recursive evaluation algorithms for quantum (q,h)-partial derivatives of bivariate polynomials, extending analogous results for univariate polynomials from [1, 24].

2 The bivariate (q,h)-blossom

We shall use the following standard q-calculus notation [25]. By

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad k = 0,\dots,n,$$

we shall denote the q-binomial coefficients, where

$$[0]_q! = 1, \qquad [n]_q! = [1]_q \cdots [n]_q, \quad n > 0$$

are the q-factorials, and

$$[n]_q = 1 + q + \cdots + q^{n-1} = \begin{cases} (1-q^n)/(1-q), & q \neq 1 \\ n, & q = 1 \end{cases}$$

are the q-integers.

Next, we recall the definition of the quantum (q,h)-derivative from [1]:

$$D_{q,h}f(t) = \frac{f(qt+h) - f(t)}{(q-1)t + h}, \qquad (q,h) \neq (1,0).$$

Throughout this paper, we assume $q \neq 0, 1$ and $h \neq 0$ are given parameters, and we define $g(t) = qt + h$. The n-fold composition of $g(t)$ is $g^{[0]}(t) = t$, $g^{[n+1]}(t) = (g \circ g^{[n]})(t)$. By induction on n,

$$g^{[n]}(t) = q^n t + [n]_q h, \quad n \in \mathbb{N}. \tag{2.1}$$

We also define

$$g^{-1}(t) = t/q - h/q, \qquad g^{[-n]}(t) = \left(g^{-1}\right)^{[n]}(t). \tag{2.2}$$

The following two properties follow directly from (2.1):

$$g^{[n]}(t) - t = [n]_q(g(t) - t), \qquad g^{[i]}(y) - g^{[i+n]}(x) = q^i(y - g^{[n]}(x)). \tag{2.3}$$

Let S_n denote the set of permutations of $\{1,\dots,n\}$ and let $\mathbb{P}_{n,m}[t,s]$ denote the polynomials of degree n in t and degree m in s.

Definition 2.1. The (q,h)-blossom of $P(t,s) \in \mathbb{P}[t,s]$ is a function $p(u_1,\dots,u_n;v_1,\dots,v_m;q,h)$, which satisfies the following three (q,h)-blossoming axioms:

• symmetry

$$p(u_1,\dots,u_n;v_1,\dots,v_m;q,h) = p(u_{\sigma_1(1)},\dots,u_{\sigma_1(n)};v_{\sigma_2(1)},\dots,v_{\sigma_2(m)};q,h),$$

for every $\sigma_1 \in S_n$ and $\sigma_2 \in S_m$;

• multiaffine

$$\begin{aligned} p(u_1,\dots,&(1-\alpha)u_k+\alpha z_k,\dots,u_n;v_1,\dots,v_m;q,h)\\ &= (1-\alpha)p(u_1,\dots,u_k,\dots,u_n;v_1,\dots,v_m;q,h)+\alpha p(u_1,\dots,z_k,\dots,u_n;v_1,\dots,v_m;q,h),\\ p(u_1,\dots,&u_n;v_1,\dots,(1-\beta)v_k+\beta w_k,\dots v_m;q,h)\\ &= (1-\beta)p(u_1,\dots,u_n;v_1,\dots,v_k,\dots,v_m;q,h)+\beta p(u_1,\dots,u_n;v_1,\dots,w_k,\dots,v_m;q,h); \end{aligned}$$

• (q,h)-diagonal

$$p(t,g(t),\dots,g^{[n-1]}(t);s,g(s),\dots,g^{[m-1]}(s);q,h) = P(t,s).$$

Let $\varphi_{n,0}(u_1,\dots,u_n) = 1$ and

$$\varphi_{n,k}(u_1,\dots,u_n) = \sum_{1\le i_1<\dots<i_k\le n} u_{i_1}\dots u_{i_k}, \qquad k=1,\dots,n$$

denote the elementary symmetric functions in n variables. Set

$$\varphi_{k,l}^{n,m}(u_1,\dots,u_n;v_1,\dots,v_m) = \varphi_{n,k}(u_1,\dots,u_n)\,\varphi_{m,l}(v_1,\dots,v_m), \tag{2.4}$$

and consider the polynomials

$$\Phi_{k,l}^{n,m}(t,s;q,h) = \Phi_{n,k}(t;q,h)\Phi_{m,l}(s;q,h), \tag{2.5}$$

$k=0,\dots,n$, $l=0,\dots,m$, where

$$\Phi_{n,k}(t;q,h) = \varphi_{n,k}(t,g(t),\dots,g^{[n-1]}(t)), \quad k=0,\dots,n. \tag{2.6}$$

Then $\Phi_{k,l}^{n,m}(t,s;q,h) \in \mathbb{P}_{k,l}[t,s]$ with leading coefficient

$$\varphi_{n,k}(1,q,\dots,q^{n-1})\varphi_{m,l}(1,q,\dots,q^{m-1}) = q^{k(k-1)/2+l(l-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}_q\begin{bmatrix} m\\ l\end{bmatrix}_q \neq 0.$$

Hence, $\{\Phi_{k,l}^{n,m}(t,s;q,h)\}$ is a basis for $\mathbb{P}_{n,m}[t,s]$. Equation (2.5) shows that the (q,h)-blossom of $\Phi_{k,l}^{n,m}(t,s;q,h)$ is $\varphi_{k,l}^{n,m}(u_1,\dots,u_n;v_1,\dots,v_m)$.

Theorem 2.2. (*Existence and uniqueness of the bivariate (q,h)-blossom*) *For every polynomial $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ there exists a unique (q,h)-blossom.*

Proof. Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$. Then there exist unique coefficients $\{c_{k,l}\}$ such that

$$P(t,s) = \sum_{k=0}^{n}\sum_{l=0}^{m} c_{k,l}\Phi_{k,l}^{n,m}(t,s;q,h),$$

and the function

$$p(u_1,\dots,u_n;v_1,\dots,v_m;q,h) = \sum_{k=0}^{n}\sum_{l=0}^{m} c_{k,l}\varphi_{k,l}^{n,m}(u_1,\dots,u_n;v_1,\dots,v_m)$$

is a (q,h)-blossom of $P(t,s)$, since it satisfies the three (q,h)-blossoming axioms.

To prove uniqueness, we assume that $P(t,s)$ has two (q,h)-blossoms $p_\nu(u_1,\dots,u_n;v_1,\dots,v_m;q,h)$, $\nu=1,2$. Then

$$p_\nu(u_1,\dots,u_n;v_1,\dots,v_m;q,h)=\sum_{k=0}^{n}\sum_{l=0}^{m}c_{\nu,k,l}\varphi_{k,l}^{n,m}(u_1,\dots,u_n;v_1,\dots,v_m),$$

for some constants $\{c_{\nu,k,l}\}$, $\nu=1,2$. By the (q,h)-diagonal property

$$P(t,s)=\sum_{k=0}^{n}\sum_{l=0}^{m}c_{\nu,k,l}\Phi_{k,l}^{n,m}(t,s;q,h),\qquad \nu=1,2,$$

implying $c_{1,k,l}=c_{2,k,l}=c_{k,l}$, $k=0,\dots,n$, $l=0,\dots,m$. □

Let $P(t,s)=\sum_{k=0}^{n}\sum_{l=0}^{m}a_{k,l}t^ks^l$. The homogenization of $P(t,s)$ is $P((t,w),(s,z))=w^nz^mP(t/w,s/z)$. Notice that $P((t,1),(s,1))=P(t,s)$. The homogeneous or multilinear bivariate (q,h)-blossom of $P((t,w),(s,z))$ is defined by

$$\begin{aligned}&p((u_1,w_1),\dots,(u_n,w_n);(v_1,z_1),\dots,(v_m,z_m);q,h)\\&:=w_1\dots w_n\,z_1\dots z_m\,p(u_1/w_1,\dots,u_n/w_n;v_1/z_1,\dots,v_m/z_m;q,h),\end{aligned}\tag{2.7}$$

where $p(u_1,\dots,u_n;v_1,\dots,v_m;q,h)$ is the bivariate (q,h)-blossom of $P(t,s)$.

Clearly, this homogeneous (q,h)-blossom is symmetric and linear in the pairs $\{(u_j,w_j)\}_{j=1}^n$ and in the pairs $\{(v_j,z_j)\}_{j=1}^m$. Moreover this blossom has the (q,h)-diagonal property:

$$P((t,w),(s,z))=p((t,w),(g(t,w),w),\dots,(g^{[n-1]}(t,w),w);(s,z),(g(s,z),z),\dots,(g^{[m-1]}(s,z),z);q,h),$$

where $g(x,y)=qx+hy$.

Proposition 2.3. (*Existence and uniqueness of the homogeneous bivariate* (q,h)*-blossom*) *For every polynomial* $P(t,s)\in\mathbb{P}_{n,m}[t,s]$ *there exists a unique homogeneous bivariate* (q,h)*-blossom.*

Proof. This result follows immediately from (2.7) and the existence and uniqueness of the bivariate (q,h)-blossom of $P(t,s)$. □

Remark 2.4. Let $P(t,s)\in\mathbb{P}_{n,m}[t,s]$. The homogeneous (q,h)-blossom of $P(t,s)$ is the homogenization with respect to each variable $u_1,\dots,u_n,v_1,\dots,v_m$ of the bivariate (q,h)-blossom of $P(t,s)$. We shall also denote the homogeneous bivariate (q,h)-blossom of $P(t,s)$ by $p(\tilde u_1,\dots,\tilde u_n;\tilde v_1,\dots,\tilde v_m;q,h)$, where $\tilde u_i=(u_i,w_i)$, $i=1,\dots,n$ and $\tilde v_j=(v_j,z_j)$, $j=1,\dots,m$. We shall identify the blossom values u and v with the homogeneous blossom values $(u,1)$ and $(v,1)$, and denote by $\tilde u^{<k>}$ the k-tuple $(\tilde u,\dots,\tilde u)$.

We define the quantum (q,h)-partial derivatives of a function $F(t,s)$ by

$$\begin{aligned}\frac{\partial_{q,h}}{\partial t}F(t,s)&=\frac{F(qt+h,s)-F(t,s)}{(q-1)t+h},\\ \frac{\partial_{q,h}}{\partial s}F(t,s)&=\frac{F(t,qs+h)-F(t,s)}{(q-1)s+h}.\end{aligned}\tag{2.8}$$

It is clear from (2.8) that these partial derivatives commute.

Theorem 2.5. *Let* $P(t,s)\in\mathbb{P}_{n,m}[t,s]$ *and let* $p(\tilde u_1,\dots,\tilde u_n;\tilde v_1,\dots,\tilde v_m;q,h)$ *be the homogeneous bivariate* (q,h)*-blossom of* $P(t,s)$. *Set* $\delta=(1,0)$. *Then*

$$\begin{aligned}\frac{\partial^r_{1/q,-h/q}}{\partial t^r}\frac{\partial^l_{1/q,-h/q}}{\partial s^l}P(t,s)=&\frac{[n]_q!}{[n-r]_q!}\frac{[m]_q!}{[m-l]_q!}\\&\times p(t,g(t),\dots,g^{[n-r-1]}(t),\delta^{<r>};s,g(s),\dots,g^{[m-l-1]}(s),\delta^{<l>};q,h),\end{aligned}\tag{2.9}$$

$r = 0, \dots, n$, $l = 0, \dots, m$. *Moreover, the homogeneous bivariate* (q, h)*-blossom of* $\frac{\partial^r_{1/q,-h/q}}{\partial t^r}\frac{\partial^l_{1/q,-h/q}}{\partial s^l}P(t, s)$ *is given by*

$$\frac{[n]_q!}{[n-r]_q!}\frac{[m]_q!}{[m-l]_q!}p(\tilde{u}_1, \dots, \tilde{u}_{n-r}, \delta^{<r>}; \tilde{v}_1, \dots, \tilde{v}_{m-l}, \delta^{<l>}; q, h).$$

Proof. Using definitions (2.8) and (2.2), the multilinear and diagonal properties of the homogeneous (q, h)-blossom, and (2.3), we obtain

$$\begin{aligned}\frac{\partial_{1/q,-h/q}}{\partial t}P(t, s) &= \frac{P(g^{-1}(t), s) - P(t, s)}{g^{-1}(t) - t}\\ &= p\left(t, \dots, g^{[n-2]}(t), \left(\frac{g^{-1}(t) - g^{[n-1]}(t)}{g^{-1}(t) - t}, 0\right); s, \dots, g^{[m-1]}(s); q, h\right)\\ &= [n]_q\, p(t, \dots, g^{[n-2]}(t), \delta; s, \dots, g^{[m-1]}(s); q, h).\end{aligned}$$

Since $[n]_q\, p(\tilde{u}_1, \dots, \tilde{u}_{n-1}, \delta, \tilde{v}_1, \dots, \tilde{v}_m; q, h)$ is a symmetric multilinear function, the last equation and Proposition 2.3 show that this function is the homogeneous (q, h)-blossom of $\frac{\partial_{1/q,-h/q}}{\partial t}P(t, s)$. A similar formula holds for $\frac{\partial_{1/q,-h/q}}{\partial s}P(t, s)$, and then (2.9) follows by induction on r and l. □

3 Bivariate (q, h)-Bernstein bases and (q, h)-Bézier surfaces

In this section we extend the definitions and results from [1] (Section 4) to (q, h)-Bernstein basis functions in two variables on rectangular domains.

The univariate (q, h)-Bernstein basis functions of degree n on an interval $I = [a, b]$ such that $b \notin \{g^{[j]}(a)\}_{j=0}^{n-1}$ are given by [1]

$$B_i^n(t; I; q, h) = \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{\prod_{j=0}^{i-1}(t - g^{[j]}(a))\prod_{j=0}^{n-i-1}(b - g^{[j]}(t))}{\prod_{j=0}^{n-1}(b - g^{[j]}(a))}, \qquad i = 0, \dots, n.$$

We recall a pair of recurrence relations [1, (4.32)-(4.33)]

$$B_i^n(t; I; q, h) = \alpha_{n,i}(t; I)B_{i-1}^{n-1}(t; I; q, h) + \beta_{n,i}(t; I)B_i^{n-1}(t; I; q, h) \tag{3.1}$$

$$B_i^n(t; I; q, h) = \gamma_{n,i}(t; I)B_{i-1}^{n-1}(t; I; q, h) + \delta_{n,i}(t; I)B_i^{n-1}(t; I; q, h), \tag{3.2}$$

$i = 1, \dots, n$, where

$$\alpha_{n,i}(t; I) = q^{n-i}\frac{t - g^{[i-1]}(a)}{b - g^{[n-1]}(a)}, \qquad \beta_{n,i}(t; I) = \frac{b - g^{[n-i-1]}(t)}{b - g^{[n-1]}(a)},$$

$$\gamma_{n,i}(t; I) = \frac{t - g^{[i-1]}(a)}{b - g^{[n-1]}(a)}, \qquad \delta_{n,i}(t; I) = q^i\frac{b - g^{[n-i-1]}(t)}{b - g^{[n-1]}(a)}.$$

Definition 3.1. The bivariate (q, h)-Bernstein basis functions for the space $\mathbb{P}_{n,m}[t, s]$ on a rectangle $R = I \times J = [a, b] \times [c, d]$ are defined by

$$B_{i,j}^{n,m}(t, s; R; q, h) = B_i^n(t; I; q, h)B_j^m(s; J; q, h), \tag{3.3}$$

$i = 0, \dots, n$, $j = 0, \dots, m$.

Next, we use relations (3.1)–(3.2) to obtain four recurrence relations for the bivariate (q, h)-Bernstein basis functions.

The first recurrence relation is obtained by applying (3.1) to both $B_i^n(t; I; q, h)$ and $B_j^m(s; J; q, h)$. We get

$$\begin{aligned}B_{i,j}^{n,m}(t, s; R; q, h) =& \alpha_{n,i}(t; I)\alpha_{m,j}(s; J)B_{i-1,j-1}^{n-1,m-1}(t, s; R; q, h) + \alpha_{n,i}(t; I)\beta_{m,j}(s; J)B_{i-1,j}^{n-1,m-1}(t, s; R; q, h)\\ &+ \beta_{n,i}(t; I)\alpha_{m,j}(s; J)B_{i,j-1}^{n-1,m-1}(t, s; R; q, h) + \beta_{n,i}(t; I)\beta_{m,j}(s; J)B_{i,j}^{n-1,m-1}(t, s; R; q, h).\end{aligned} \tag{3.4}$$

The second recurrence relation is obtained by applying (3.1) to $B_i^n(t;I;q,h)$ and (3.2) to $B_j^m(s;J;q,h)$. We get

$$\begin{aligned} B_{i,j}^{n,m}(t,s;R;q,h) =& \alpha_{n,i}(t;I)\gamma_{m,j}(s;J)B_{i-1,j-1}^{n-1,m-1}(t,s;R;q,h) + \alpha_{n,i}(t;I)\delta_{m,j}(s;J)B_{i-1,j}^{n-1,m-1}(t,s;R;q,h) \\ &+ \beta_{n,i}(t;I)\gamma_{m,j}(s;J)B_{i,j-1}^{n-1,m-1}(t,s;R;q,h) + \beta_{n,i}(t;I)\delta_{m,j}(s;J)B_{i,j}^{n-1,m-1}(t,s;R;q,h). \end{aligned} \tag{3.5}$$

Applying (3.2) to both $B_i^n(t;I;q,h)$ and $B_j^m(s;J;q,h)$ yields the third recurrence relation

$$\begin{aligned} B_{i,j}^{n,m}(t,s;R;q,h) =& \gamma_{n,i}(t;I)\gamma_{m,j}(s;J)B_{i-1,j-1}^{n-1,m-1}(t,s;R;q,h) + \gamma_{n,i}(t;I)\delta_{m,j}(s;J)B_{i-1,j}^{n-1,m-1}(t,s;R;q,h) \\ &+ \delta_{n,i}(t;I)\gamma_{m,j}(s;J)B_{i,j-1}^{n-1,m-1}(t,s;R;q,h) + \delta_{n,i}(t;I)\delta_{m,j}(s;J)B_{i,j}^{n-1,m-1}(t,s;R;q,h). \end{aligned} \tag{3.6}$$

Another recurrence relation obtained by applying (3.2) to $B_i^n(t;I;q,h)$ and (3.1) to $B_j^m(s;J;q,h)$, follows from the second recurrence relation by interchanging (n,i,t,I) and (m,j,s,J).

Definition 3.2. The (q,h)-Bézier surface $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ on $R = I \times J = [a,b] \times [c,d]$ with control points $\{P_{i,j}\}_{i=0,j=0}^{n,m}$ is defined by

$$P(t,s) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{i,j}B_{i,j}^{n,m}(t,s;R;q,h).$$

Each of the four recurrence relations for the bivariate (q,h)-Bernstein basis functions leads to a recursive evaluation algorithm for (q,h)-Bézier surfaces.

– **Algorithm A:** Set
$$\tilde{P}_{i,j}^{0,0} = P_{i,j}, \qquad i = 0,\dots,n,\ j = 0,\dots,m,$$
and for $r = 1,\dots,n$, $l = 1,\dots,m$, define recursively
$$\begin{aligned} \tilde{P}_{i,j}^{r,l} =& \alpha_{n+1-r,i+1}(t,I)\alpha_{m+1-l,j+1}(s,J)\tilde{P}_{i+1,j+1}^{r-1,l-1} + \alpha_{n+1-r,i+1}(t,I)\beta_{m+1-l,j}(s,J)\tilde{P}_{i+1,j}^{r-1,l-1} \\ &+ \beta_{n+1-r,i}(t,I)\alpha_{m+1-l,j+1}(s,J)\tilde{P}_{i,j+1}^{r-1,l-1} + \beta_{n+1-r,i}(t,I)\beta_{m+1-l,j}(s,J)\tilde{P}_{i,j}^{r-1,l-1}, \end{aligned}$$
$i = 0,\dots,n-r$, $j = 0,\dots,m-l$. Then
$$P(t,s) = \sum_{i=0}^{n-r}\sum_{j=0}^{m-l} \tilde{P}_{i,j}^{r,l}B_{i,j}^{n-r,m-l}(t,s;R;q,h).$$
In particular, $P(t,s) = \tilde{P}_{0,0}^{m,n}$.

– **Algorithm B:** Set
$$\hat{P}_{i,j}^{0,0} = P_{i,j}, \qquad i = 0,\dots,n,\ j = 0,\dots,m,$$
and for $r = 1,\dots,n$, $l = 1,\dots,m$, define recursively
$$\begin{aligned} \hat{P}_{i,j}^{r,l} =& \alpha_{n+1-r,i+1}(t,I)\gamma_{m+1-l,j+1}(s,J)\hat{P}_{i+1,j+1}^{r-1,l-1} + \alpha_{n+1-r,i+1}(t,I)\delta_{m+1-l,j}(s,J)\hat{P}_{i+1,j}^{r-1,l-1} \\ &+ \beta_{n+1-r,i}(t,I)\gamma_{m+1-l,j+1}(s,J)\hat{P}_{i,j+1}^{r-1,l-1} + \beta_{n+1-r,i}(t,I)\delta_{m+1-l,j}(s,J)\hat{P}_{i,j}^{r-1,l-1}, \end{aligned}$$
$i = 0,\dots,n-r$, $j = 0,\dots,m-l$. Then
$$P(t,s) = \sum_{i=0}^{n-r}\sum_{j=0}^{m-l} \hat{P}_{i,j}^{r,l}B_{i,j}^{n-r,m-l}(t,s;R;q,h).$$
In particular, $P(t,s) = \hat{P}_{0,0}^{n,m}$.

– **Algorithm C:** Set

$$\check{P}^{0,0}_{i,j} = P_{i,j}, \qquad i = 0,\dots,n,\ j = 0,\dots,m,$$

and for $r = 1,\dots,n$, $l = 1,\dots,m$, define recursively

$$\begin{aligned}\check{P}^{r,l}_{i,j} =& \gamma_{n+1-r,i+1}(t,I)\gamma_{m+1-l,j+1}(s,J)\check{P}^{r-1,l-1}_{i+1,j+1} + \gamma_{n+1-r,i+1}(t,I)\delta_{m+1-l,j}(s,J)\check{P}^{r-1,l-1}_{i+1,j}\\ &+ \delta_{n+1-r,i}(t,I)\gamma_{m+1-l,j+1}(s,J)\check{P}^{r-1,l-1}_{i,j+1} + \delta_{n+1-r,i}(t,I)\delta_{m+1-l,j}(s,J)\check{P}^{r-1,l-1}_{i,j},\end{aligned}$$

$i = 0,\dots,n-r$, $j = 0,\dots,m-l$. Then

$$P(t,s) = \sum_{i=0}^{n-r}\sum_{j=0}^{m-l}\check{P}^{r,l}_{i,j}B^{n-r,m-l}_{i,j}(t,s;R,q,h).$$

In particular, $P(t,s) = \check{P}^{n,m}_{0,0}$.

Another recursive evaluation algorithm is obtained from Algorithm B, by interchanging (n,t,I) and (m,s,J).

Proposition 3.3. *(Recursive evaluation algorithm for the bivariate (q,h)-blossom) Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ and let $p(u_1,\dots,u_n;v_1,\dots,v_m;q,h)$ be the (q,h)-blossom of $P(t,s)$. Set*

$$Q^{0,0}_{i,j} = p(g^{[i]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b);g^{[j]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d);q,h), \tag{3.7}$$

$i = 0,\dots,n$, $j = 0,\dots,m$, and for $r = 0,\dots,n-1$, $l = 0,\dots m-1$, define recursively

$$Q^{r+1,l+1}_{i,j} = (1-\alpha_{r,i})(1-\beta_{l,j})Q^{r,l}_{i,j} + \alpha_{r,i}(1-\beta_{l,j})Q^{r,l}_{i+1,j} + (1-\alpha_{r,i})\beta_{l,j}Q^{r,l}_{i,j+1} + \alpha_{r,i}\beta_{l,j}Q^{r,l}_{i+1,j+1},$$

$i = 0,\dots,n-r-1$, $j = 0,\dots,m-l-1$, where

$$\alpha_{r,i} = \frac{u_{r+1} - g^{[i+r]}(a)}{g^{[i]}(b) - g^{[i+r]}(a)}, \qquad \beta_{l,j} = \frac{v_{l+1} - g^{[j+l]}(c)}{g^{[j]}(d) - g^{[j+l]}(c)}.$$

Then

$$\begin{aligned}Q^{r,l}_{i,j}(u_1,\dots,u_r;v_1,\dots,v_l;q,h) = p(&g^{[i+r]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b),u_1,\dots,u_r;\\ &g^{[j+l]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d),v_1,\dots,v_l;q,h),\end{aligned} \tag{3.8}$$

$i = 0,\dots,n-r$, $r = 0,\dots,n$, $j = 0,\dots,m-l$, $l = 0,\dots,m$. In particular,

$$Q^{n,m}_{0,0} = p(u_1,\dots,u_n;v_1,\dots,v_m;q,h).$$

Proof. This result follows by induction on r and l. □

Proposition 3.4. *Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ and let $p(u_1,\dots,u_n;v_1,\dots,v_m;q,h)$ be the (q,h)-blossom of $P(t,s)$. There exist $n!m!$ affine-invariant, recursive evaluation algorithms for $P(t,s)$ defined recursively as follows.*

Let $\sigma_1 \in S_n$ and $\sigma_2 \in S_m$. Set $P^{0,0}_{i,j} = Q^{0,0}_{i,j}$, $i = 0,\dots,n$, $j = 0,\dots,m$, as in (3.7). Then for $r = 0,\dots,n-1$, $l = 0,\dots,m-1$, define

$$P^{r+1,l+1}_{i,j}(t,s) = (1-\gamma_{r,i})(1-\delta_{l,j})P^{r,l}_{i,j}(t,s) + \gamma_{r,i}(1-\delta_{l,j})P^{r,l}_{i+1,j}(t,s) + (1-\gamma_{r,i})\delta_{l,j}P^{r,l}_{i,j+1}(t,s) + \gamma_{r,i}\delta_{l,j}P^{r,l}_{i+1,j+1}(t,s),$$

$i = 0,\dots,n-r-1$, $j = 0,\dots,m-l-1$, where

$$\gamma_{r,i} = \frac{g^{[\sigma_1(r+1)-1]}(t) - g^{[i+r]}(a)}{g^{[i]}(b) - g^{[i+r]}(a)}, \qquad \delta_{l,j} = \frac{g^{[\sigma_2(l+1)-1]}(s) - g^{[j+l]}(c)}{g^{[j]}(d) - g^{[j+l]}(c)}.$$

Then

$$\begin{aligned}P^{r,l}_{i,j}(t,s) = p(&g^{[i+r]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b),g^{[\sigma_1(1)-1]}(t),\dots,g^{[\sigma_1(r)-1]}(t);\\ &g^{[j+l]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d),g^{[\sigma_2(1)-1]}(s),\dots,g^{[\sigma_2(l)-1]}(s);q,h),\end{aligned} \tag{3.9}$$

$i = 0,\dots,n-r$, $r = 0,\dots,n$, and $j = 0,\dots,m-l$, $l = 0,\dots,m$. In particular,

$$P^{n,m}_{0,0}(t,s) = P(t,s).$$

Proof. This result follows by substituting $u_r = g^{[\sigma_1(r)-1]}(t)$, $r = 1, \dots, n$ and $v_l = g^{[\sigma_2(l)-1]}(s)$, $l = 1, \dots, m$, in the recursive evaluation algorithm of Proposition 3.3. □

Theorem 3.5. (*Every bivariate polynomial is a (q,h)-Bézier surface over the rectangle $R = [a,b] \times [c,d]$*) *Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ and let $p(u_1,\dots,u_n; v_1,\dots,v_m; q,h)$ be the (q,h)-blossom of $P(t,s)$. Then*

$$\begin{aligned} P(t,s) = \sum_{i=0}^{n}\sum_{j=0}^{m} p(&g^{[i]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b); \\ &g^{[j]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d);q,h)B_{i,j}^{n,m}(t,s;R;q,h). \end{aligned} \tag{3.10}$$

Proof. In terms of the polynomial basis defined with (2.5)-(2.6), we have

$$P(t,s) = \sum_{k=0}^{n}\sum_{l=0}^{m} c_{k,l}\Phi_{k,l}^{n,m}(t,s;q,h).$$

By Theorem 2.2 the (q,h)-blossom of $P(t,s)$ is given by

$$p(u_1,\dots,u_n;v_1,\dots,v_m;q,h) = \sum_{k=0}^{n}\sum_{l=0}^{m} c_{k,l}\varphi_{k,l}^{n,m}(u_1,\dots,u_n,v_1,\dots,v_m). \tag{3.11}$$

By (2.6) and [1, Theorem 4.6]

$$\Phi_{n,k}(t;q,h) = \sum_{i=0}^{n} \varphi_{n,k}(g^{[i]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b))B_i^n(t;[a,b];q,h),$$
$$\Phi_{m,l}(s;q,h) = \sum_{j=0}^{m} \varphi_{m,l}(g^{[j]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d))B_j^m(s;[c,d];q,h).$$

Therefore

$$\begin{aligned} P(t,s) &= \sum_{k=0}^{n}\sum_{l=0}^{m} c_{k,l}\sum_{i=0}^{n}\sum_{j=0}^{m}\varphi_{n,k}(g^{[i]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b)) \\ &\quad\times \varphi_{m,l}(g^{[j]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d))B_i^n(t;[a,b];q,h)B_j^m(s;[c,d];q,h) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,j}^{n,m}(t,s;R;q,h)\sum_{k=0}^{n}\sum_{l=0}^{m} c_{k,l} \\ &\quad\times \varphi_{k,l}^{n,m}(g^{[i]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b);g^{[j]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d)), \end{aligned} \tag{3.12}$$

where we used (2.4) and (3.3). By (3.11) the expression in the last two lines reduces to the right-hand side of (3.10). □

Corollary 3.6. *The (q,h)-Bernstein basis functions $\{B_{i,j}^{n,m}(t,s;R;q,h)\}_{i,j=0}^{n,m}$ on any rectangle $R = [a,b]\times[c,d]$ form a basis for $\mathbb{P}_{n,m}[t,s]$.*

Corollary 3.7. (*Dual Functional Property of the bivariate (q,h)-blossom*) *Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ be a (q,h)-Bézier surface on $[a,b]\times[c,d]$ with control points $\{P_{i,j}\}$ and (q,h)-blossom $p(u_1,\dots,u_n;v_1,\dots,v_m;q,h)$. Then*

$$P_{i,j} = p(g^{[i]}(a),\dots,g^{[n-1]}(a),b,\dots,g^{[i-1]}(b);g^{[j]}(c),\dots,g^{[m-1]}(c),d,\dots,g^{[j-1]}(d);q,h), \tag{3.13}$$

$i = 0,\dots,n$, $j = 0,\dots,m$.

Proof. This result follows from Theorem 3.5 and Corollary 3.6. □

Corollary 3.8. (*Endpoint Interpolation Property*)

$$P_{0,0} = P(a,c), \qquad P_{0,m} = P(a,d), \qquad P_{n,0} = P(b,c), \qquad P_{n,m} = P(b,d).$$

Proof. These relations follow immediately from Corollary 3.7 and the diagonal property of the bivariate (q, h)-blossom. □

Proposition 3.9. *Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ be a (q,h)-Bézier surface on $[a,b] \times [c,d]$ with control points $\{P_{i,j}\}$. Let $P^{r,l}_{i,j}$, $r = 0,\ldots,n$, $l = 0,\ldots,m$, $i = 0,\ldots,n-r$, and $j = 0,\ldots,m-l$, be the nodes in the recursive evaluation algorithm for $P(t,s)$ using the identity permutations. Set $R_{i,j} = [g^{[i]}(a), g^{[i]}(b)] \times [g^{[j]}(c), g^{[j]}(d)]$. Then*

$$P^{r,l}_{i,j}(t,s) = \sum_{\nu=0}^{r}\sum_{\mu=0}^{l} P_{i+\nu,j+\mu} B^{r,l}_{\nu,\mu}(t,s;R_{i,j};q,h). \tag{3.14}$$

Proof. By Propositions 3.3 and 3.4, the (q,h)-blossom of $P^{r,l}_{i,j}(t,s) \in \mathbb{P}_{r,l}[t,s]$ is

$$Q^{r,l}_{i,j}(u_1,\ldots,u_r; v_1,\ldots,v_l; q,h).$$

By the dual functional property on $R_{i,j}$, (3.8), and (3.13)

$$\begin{aligned}
P^{r,l}_{i,j}(t,s) &= \sum_{\nu=0}^{r}\sum_{\mu=0}^{l} Q^{r,l}_{i,j}(g^{[i+\nu]}(a),\ldots,g^{[i+r-1]}(a), g^{[i]}(b),\ldots,g^{[i+\nu-1]}(b);\\
&\quad g^{[j+\mu]}(c),\ldots,g^{[j+l-1]}(c), g^{[j]}(d),\ldots,g^{[j+\mu-1]}(d); q,h) B^{r,l}_{\nu,\mu}(t,s;R_{i,j};q,h)\\
&= \sum_{\nu=0}^{r}\sum_{\mu=0}^{l} p(g^{[i+\nu]}(a),\ldots,g^{[n-1]}(a), b,\ldots,g^{[i+\nu-1]}(b);\\
&\quad g^{[j+\mu]}(c),\ldots,g^{[m-1]}(c), d,\ldots,g^{[j+\mu-1]}(d); q,h) B^{r,l}_{\nu,\mu}(t,s;R_{i,j};q,h)\\
&= \sum_{\nu=0}^{r}\sum_{\mu=0}^{l} P_{i+\nu,j+\mu} B^{r,l}_{\nu,\mu}(t,s;R_{i,j};q,h).
\end{aligned}$$

□

4 Identities and properties of the bivariate (q, h)-Bernstein bases

In this section we give bivariate analogs of several standard identities of the (q, h)-Bernstein bases [1, 3, 4]. Let $R = [a,b] \times [c,d]$.

Proposition 4.1. (*Bivariate (q,h)-Marsden's Identity*)

$$\begin{aligned}
\prod_{i=0}^{n-1}\prod_{j=0}^{m-1}(x - g^{[i]}(t))(y - g^{[j]}(s)) &= \sum_{i=0}^{n}\sum_{j=0}^{m}\left\{\prod_{\nu=i}^{n-1}(x - g^{[\nu]}(a))\prod_{\nu=0}^{i-1}(x - g^{[\nu]}(b))\prod_{\nu=j}^{m-1}(y - g^{[\nu]}(c))\prod_{\nu=0}^{j-1}(y - g^{[\nu]}(d))\right\}\\
&\quad \times B^{n,m}_{i,j}(t,s;R;q,h).
\end{aligned} \tag{4.1}$$

Proof. This identity follows from the dual functional property applied to the polynomial on the left-hand side of (4.1) whose bivariate (q,h)-blossom in s and t is

$$\prod_{i=1}^{n}\prod_{j=1}^{m}[(x - u_i)(y - v_j)].$$

□

Proposition 4.2. (*Partition of Unity and Representation of Linear Functions*)

$$1 = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,j}^{n,m}(t, s; R; q, h), \tag{4.2}$$

$$t = \sum_{i=0}^{n}\sum_{j=0}^{m} \left(\frac{q^i[n-i]_q a + [i]_q b}{[n]_q}\right) B_{i,j}^{n,m}(t, s; R; q, h), \tag{4.3}$$

$$s = \sum_{i=0}^{n}\sum_{j=0}^{m} \left(\frac{q^j[m-j]_q c + [j]_q d}{[m]_q}\right) B_{i,j}^{n,m}(t, s; R; q, h). \tag{4.4}$$

Proof. Equations (4.2)–(4.4) follow immediately from [1, Proposition 5.2] and (3.3). □

Corollary 4.3. (*Affine Invariance*) *(q, h)-Bézier surfaces are affine invariant.*

Proof. Let $P(t, s) \in \mathbb{P}_{n,m}[t, s]$ be a (q, h)-Bézier surface on R with control points $\{P_{i,j}\}$, and let L be a linear transformation, and v be a vector. Then by Proposition 4.2 we have

$$\sum_{i=0}^{n}\sum_{j=0}^{m}(L\,P_{i,j} + v)B_{i,j}^{n,m}(t, s; R; q, h) = L\sum_{i=0}^{n}\sum_{j=0}^{m} P_{i,j}B_{i,j}^{n,m}(t, s; R; q, h) + v\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,j}^{n,m}(t, s; R; q, h) = L\,P(t, s) + v.$$

□

Proposition 4.4. (*Change of Basis Formula*) *Let $\{\Phi_{k,l}^{n,m}(t, s; q, h)\}$ be the polynomials defined by* (2.5)–(2.6). *Then*

$$\Phi_{k,l}^{n,m}(t, s; q, h) = \sum_{i=0}^{n}\sum_{j=0}^{m} \varphi_{k,l}^{n,m}(g^{[i]}(a), \dots, g^{[n-1]}(a), b, \dots, g^{[i-1]}(b);$$
$$g^{[j]}(c), \dots, g^{[m-1]}(c), d, \dots, g^{[j-1]}(d); q, h)B_{i,j}^{n,m}(t, s; R; q, h),$$

$k = 0, \dots, n$, $l = 0, \dots, m$.

Proof. The relation follows from the fact that the bivariate (q, h)-blossom of $\Phi_{k,l}^{n,m}(t, s; q, h)$ is $\varphi_{k,l}^{n,m}(u_1, \dots, u_n; v_1, \dots, v_m)$ and Theorem 3.5. □

Theorem 4.5. (*Lagrange Interpolation and Lagrange Basis Functions*)
Let $P(t, s) \in \mathbb{P}_{n,m}[t, s]$. Then

$$P(t, s) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,j}^{n,m}(t, s; [a, g^{[n]}(a)] \times [c, g^{[m]}(c)]; q, h)P(g^{[i]}(a), g^{[j]}(c)).$$

Thus the (q, h)-Bernstein basis functions $\left\{B_{i,j}^{n,m}(t, s; [a, g^{[n]}(a)] \times [c, g^{[m]}(c)]; q, h)\right\}$ are the Lagrange basis functions for nodes $a, g(a), \dots, g^{[n]}(a), c, g(c), \dots, g^{[m]}(c)$.

Proof. By the dual functional property and the diagonal property of the bivariate (q, h)-blossom, the control points of $P(t, s)$ on $[a, g^{[n]}(a)] \times [c, g^{[m]}(c)]$ are

$$P_{i,j} = p(g^{[i]}(a), \dots, g^{[i+n-1]}(a); g^{[j]}(c), \dots, g^{[j+m-1]}(c); q, h) = P(g^{[i]}(a), g^{[j]}(c)),$$

$i = 0, \dots, n$, $j = 0, \dots, m$. □

5 A subdivision algorithm for (q, h)-Bézier surfaces

In this section we present a de Casteljau subdivision algorithm for (q, h)-Bézier surfaces and we establish its rate of convergence. This algorithm is a 2D-analog of the subdivision algorithms for (q, h)-Bézier curves in [1, 3, 4].

Proposition 5.1. *(Subdivision Algorithm for (q, h)-Bézier Surfaces) Let $P(t, s) \in \mathbb{P}_{n,m}[t, s]$ be a (q, h)-Bézier surface on $[a, b] \times [c, d]$ with control points $\{P_{i,j}\}$ and (q, h)-blossom $p(u_1, \dots, u_n; v_1, \dots, v_m; q, h)$. Fix $x \in (a, b)$ and $y \in (c, d)$, such that*

$$x \notin \{g^{[j]}(a), g^{[-j]}(b)\}_{j=0}^{n-1} \qquad \text{and} \qquad y \notin \{g^{[j]}(c), g^{[-j]}(d)\}_{j=0}^{m-1}. \tag{5.1}$$

(Left-Lower (q, h)-subdivision) A control polygon for $P(t, s)$ over the rectangle $[a, x] \times [c, y]$ is generated by selecting $\sigma_1(k) = k$, $k = 1, \dots, n$ and $\sigma_2(l) = l$, $l = 1, \dots, m$ in Proposition 3.4. Then

$$P(t, s) = \sum_{k=0}^{n} \sum_{l=0}^{m} P_{k,l}^{LL} B_{k,l}^{n,m}(t, s; [a, x] \times [c, y]; q, h), \tag{5.2}$$

where

$$P_{k,l}^{LL} = p(g^{[k]}(a), \dots, g^{[n-1]}(a), x, \dots, g^{[k-1]}(x); g^{[l]}(c), \dots, g^{[m-1]}(c), y, \dots, g^{[l-1]}(y); q, h), \tag{5.3}$$

$k = 0, \dots, n$, $l = 0, \dots, m$. Moreover,

$$P_{k,l}^{LL} = \sum_{i=0}^{k} \sum_{j=0}^{l} P_{i,j} B_{i,j}^{k,l}(x, y; [a, b] \times [c, d]; q, h). \tag{5.4}$$

(Left-Upper (q, h)-subdivision) A control polygon for $P(t, s)$ over the rectangle $[a, x] \times [y, d]$ is generated by selecting $\sigma_1(k) = k$, $k = 1, \dots, n$, and $\sigma_2(l) = m + 1 - l$, $l = 1, \dots, m$, in Proposition 3.4. Then

$$P(t, s) = \sum_{k=0}^{n} \sum_{l=0}^{m} P_{k,l}^{LU} B_{k,l}^{n,m}(t, s; [a, x] \times [y, d]; q, h), \tag{5.5}$$

where

$$P_{k,l}^{LU} = p(g^{[k]}(a), \dots, g^{[n-1]}(a), x, \dots, g^{[k-1]}(x); g^{[l]}(y), \dots, g^{[m-1]}(y), d, \dots, g^{[l-1]}(d); q, h), \tag{5.6}$$

$k = 0, \dots, n$, $l = 0, \dots, m$. Moreover,

$$P_{k,l}^{LU} = \sum_{i=0}^{k} \sum_{j=l}^{m} P_{i,j} B_{i,j-l}^{n,m-l}(x, g^{[l]}(y); [a, b] \times [g^{[l]}(c), g^{[l]}(d)]; q, h). \tag{5.7}$$

(Right-Lower (q, h)-subdivision) A control polygon for $P(t, s)$ over the rectangle $[x, b] \times [c, y]$ is generated by selecting $\sigma_1(k) = n + 1 - k$, $k = 1, \dots, n$, and $\sigma_2(l) = l$, $l = 1, \dots, m$, in Proposition 3.4. Then

$$P(t, s) = \sum_{k=0}^{n} \sum_{l=0}^{m} P_{k,l}^{RL} B_{k,l}^{n,m}(t, s; [x, b] \times [c, y]; q, h), \tag{5.8}$$

where

$$P_{k,l}^{RL} = p(g^{[k]}(x), \dots, g^{[n-1]}(x), b, \dots, g^{[k-1]}(b); g^{[l]}(c), \dots, g^{[m-1]}(c), y, \dots, g^{[l-1]}(y); q, h), \tag{5.9}$$

$k = 0, \dots, n$, $l = 0, \dots, m$. Moreover,

$$P^{RL}_{k,l} = \sum_{i=k}^{n} \sum_{j=0}^{l} P_{i,j} B^{n-k,l}_{i-k,j}(g^{[k]}(x), y; [g^{[k]}(a), g^{[k]}(b)] \times [c, d]; q, h). \tag{5.10}$$

(Right-Upper (q, h)-subdivision) A control polygon for $P(t, s)$ over the rectangle $[x, b] \times [y, d]$ is generated by selecting $\sigma_1(k) = n + 1 - k$, $k = 1, \dots, n$, and $\sigma_2(l) = m + 1 - l$, $l = 1, \dots, m$, in Proposition 3.4. Then

$$P(t, s) = \sum_{k=0}^{n} \sum_{l=0}^{m} P^{RU}_{k,l} B^{n,m}_{k,l}(t, s; [x, b] \times [y, d]; q, h), \tag{5.11}$$

where

$$\begin{aligned} P^{RU}_{k,l} = p(&g^{[k]}(x), \dots, g^{[n-1]}(x), b, \dots, g^{[k-1]}(b); \\ &g^{[l]}(y), \dots, g^{[m-1]}(y), d, \dots, g^{[l-1]}(d); q, h), \end{aligned} \tag{5.12}$$

$k = 0, \dots, n$, $l = 0, \dots, m$. Moreover,

$$P^{RU}_{k,l} = \sum_{i=k}^{n} \sum_{j=l}^{m} P_{j,m} B^{n-k,m-l}_{i-k,j-l}(g^{[k]}(x), g^{[l]}(y); [g^{[k]}(a), g^{[k]}(b)] \times [g^{[l]}(c), g^{[l]}(d)]; q, h). \tag{5.13}$$

Proof. Equations (5.2)-(5.3), (5.5)-(5.6), (5.8)-(5.9), and (5.11)-(5.12) follow from Theorem 3.5. Equation (5.4) follows by applying [1, Theorem 6.1] to both x and y, equation (5.7) follows by applying [1, Theorem 6.1] to x and [1, Theorem 6.2] to y, equation (5.10) follows by applying [1, Theorem 6.2] to x and [1, Theorem 6.1] to y, and equation (5.13) follows by applying [1, Theorem 6.2] to both x and y. □

Theorem 5.2. *Let $P(t, s) \in \mathbb{P}_{n,m}[t, s]$ be a (q, h)-Bézier surface on $R = [a, b] \times [c, d]$. Then the control polygons generated by the (q, h)-Bézier midpoint subdivision converge to the (q, h)-Bézier surface $P(t, s)$ uniformly on R at the rate of $(2/3)^N$, where N is the number of subdivision iterations.*

Proof. We begin by selecting $x \in (a, b)$ and $y \in (c, d)$. In the first iteration of the subdivision algorithm, we subdivide the surface $P(t, s)$ into four segments, over $[a, x] \times [c, y]$, $[a, x] \times [y, d]$, $[x, b] \times [c, y]$, and $[x, b] \times [y, d]$. Then we apply the same procedure to each new segment: At the N-th iteration of subdivision, we subdivide each segment generated after the $(N - 1)$-th iteration.

Next, we estimate the sizes of the corresponding control polygons. To estimate the area of the surface polygon $\mathcal{P}^{LL}_{k,l}$ with vertices $P^{LL}_{k,l}$, $P^{LL}_{k+1,l}$, $P^{LL}_{k+1,l+1}$ and $P^{LL}_{k,l+1}$, we estimate the areas of triangles $\Delta^1_{k,l}$ with vertices $P^{LL}_{k,l}$, $P^{LL}_{k+1,l}$ and $P^{LL}_{k,l+1}$ and $\Delta^2_{k,l}$ with vertices $P^{LL}_{k+1,l}$, $P^{LL}_{k+1,l+1}$ and $P^{LL}_{k,l+1}$. We start by estimating the length of the segment $P^{LL}_{k,l} P^{LL}_{k+1,l}$. Let $p(\tilde{u}_1, \dots, \tilde{u}_n; \tilde{v}_1, \dots, \tilde{v}_m; q, h)$ be the homogeneous (q, h)-blossom of $P(t, s)$ (see Remark 2.4). By (5.3), the multilinear property of the homogeneous (q, h)-blossom, and (2.3), we have

$$\begin{aligned} P^{LL}_{k+1,l} - P^{LL}_{k,l} = q^k (x - a) p(&g^{[k+1]}(a), \dots, g^{[n-1]}(a), x, \dots, g^{[k-1]}(x), \delta; \\ &g^{[l]}(c), \dots, g^{[m-1]}(c), y, \dots, g^{[l-1]}(y); q, h), \end{aligned}$$

where $\delta = (1, 0)$. Let

$$\begin{aligned} M = \max_{r=0,1} \max_{\substack{0 \le k \le n \\ 0 \le l \le m}} \Big\{ &|p(x, \dots, g^{[k-1]}(x), g^{[k+1-r]}(z), \dots, g^{[n-1]}(z), \delta; \\ &y, \dots, g^{[l-1]}(y), g^{[l+r]}(w), \dots, g^{[m-1]}(w); q, h)|, x, z \in [a, b], y, w \in [c, d] \Big\}, \end{aligned}$$

and

$$Q = \max\{1, |q|^{n-1}\}.$$

Then

$$|P^{LL}_{k+1,l} - P^{LL}_{k,l}| \le QM|x - a|.$$

Similarly,

$$|P^{LL}_{k,l+1} - P^{LL}_{k,l}| \le QM|y - c|.$$

Therefore, the maximum distance between two points in $\mathcal{P}^{LL}_{k,l}$ is estimated by

$$d^{LL}_{k,l} \le QM(|x - a| + |y - c|).$$

Similarly, the maximum distances between points in the polygons arising from the left-upper, right-lower, and right-upper (q,h)-subdivisions are estimated by

$$d^{LU}_{k,l} \le QM(|x - a| + |d - y|), \qquad d^{RL}_{k,l} \le QM(|b - x| + |y - c|),$$

$$d^{RU}_{k,l} \le QM(|b - x| + |d - y|),$$

respectively.

In particular, if we subdivide each time using the midpoint of the intervals, then the diameter of the control polygon of each segment of the (q,h)-Bézier surface generated at the N-th iteration of the subdivision algorithm is bounded by a constant times 2^{-N}. In case this is not possible (if (x,y) does no satisfy the condition of Proposition 5.1), we can subdivide at $(x,y) \in [a,b] \times [c,d]$ such that $\frac{x-a}{b-a}, \frac{y-c}{d-c} \in (\frac{1}{3}, \frac{2}{3})$, and replace the rate 2^{-N} by $(2/3)^N$. This is so, because the equation $\prod_{j=0}^{n-1} g^{[j]}(x) = A$, with fixed $g(x) = qx + h$, has finitely many (n) real solutions for any A.

Let $\tilde{P}(t,s)$ be a segment of the original (q,h)-Bézier surface $P(t,s)$ constructed after N iterations of midpoint subdivisions and let $\mathcal{P}(t,s)$ denote the corresponding control polygon. Then $\tilde{P}(t,s)$ is the restriction of $P(t,s)$ over some $\tilde{R} = [t_0, t_1] \times [s_0, s_1] \subset [a,b] \times [c,d]$ of area at most $(4/9)^N(b-a)(d-c)$ and by Corollary 3.8, $P(t,s)$ and $\mathcal{P}(t,s)$ coincide at the vertices of $\tilde{R}$. Hence, for every $(t,s) \in \tilde{R}$ we have

$$\begin{aligned}
|\tilde{P}(t,s) - \mathcal{P}(t,s)| &= |P(t,s) - \mathcal{P}(t,s)| \\
&\le |P(t,s) - P(t_0,s_0)| + |P(t_0,s_0) - \mathcal{P}(t,s)| \\
&\le \max_{(\tau,\sigma)\in\tilde{R}} \left|\frac{\partial P}{\partial t}(\tau,\sigma)\right| |t - t_0| + \max_{(\tau,\sigma)\in\tilde{R}} \left|\frac{\partial P}{\partial s}(\tau,\sigma)\right| |s - s_0| + QM(|t_1 - t_0| + |s_1 - s_0|) \\
&\le C(2/3)^N,
\end{aligned}$$

where

$$C = \left(\max_{(\tau,\sigma)\in R} \left|\frac{\partial P}{\partial t}(\tau,\sigma)\right| + \max_{(\tau,\sigma)\in R} \left|\frac{\partial P}{\partial s}(\tau,\sigma)\right| + 2QM\right) \max\{b - a, d - c\}.$$

□

Example 5.3. We perform the recursive midpoint subdivision algorithm on a quadratic (q,h)-Bézier surface with $(q,h) = (0.4, 0.05)$ on $R = [0,2] \times [0,2]$. The control points are $P_{0,0}(0,0,0)$, $P_{0,1}(1,0,0)$, $P_{0,2}(2,0,0)$, $P_{1,0}(0,1,0)$, $P_{1,1}(0,2,0)$, $P_{1,2}(1,1,1)$, $P_{2,0}(1,2,0)$, $P_{2,1}(2,1,0)$, and $P_{2,2}(2,2,0)$. We plot this (q,h)-Bézier surface and the control points obtained after each of four iterations of this algorithm in Figures 1–2.

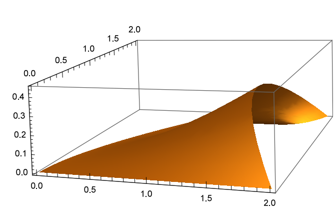

Figure 1: The (q,h)-Bézier surface.

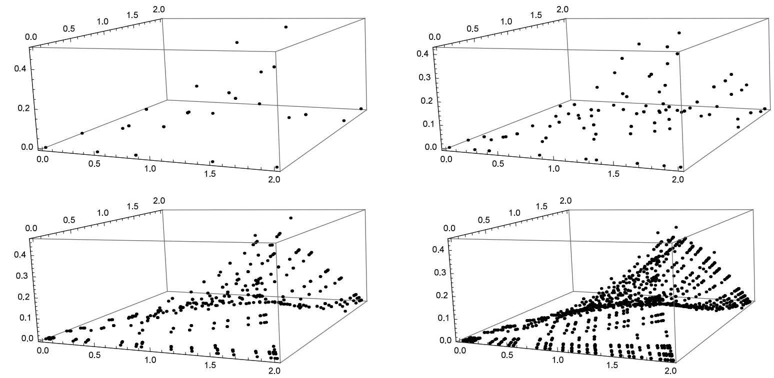

Figure 2: The control points from the first four iterations of subdivision.

Example 5.4. We consider the (q, h)-Bézier surface with the same control points, R, and h as in the previous example, but we take $q = 3.5$. We plot this (q, h)-Bézier surface and the control points obtained after each of four iterations of the recursive midpoint subdivision algorithm in Figures 3–4.

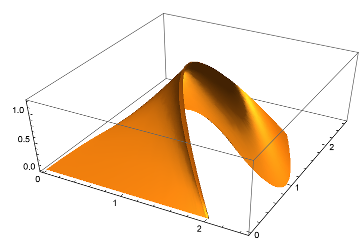

Figure 3: The (q, h)-Bézier surface.

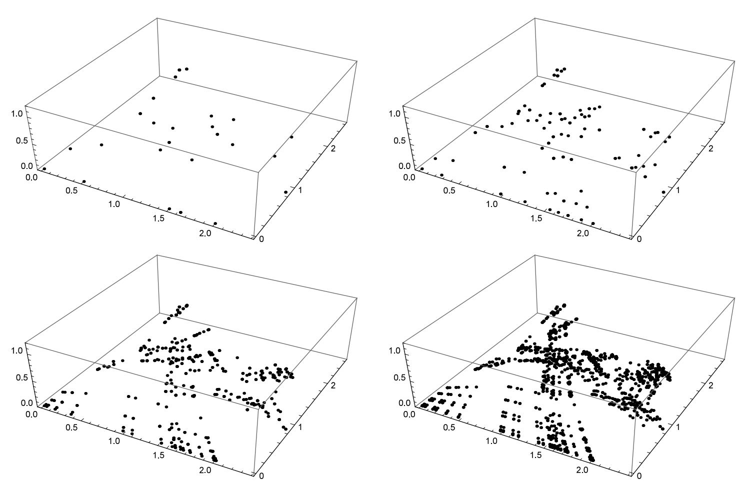

Figure 4: The control points from the first four iterations of subdivision.

6 Quantum partial derivatives of bivariate (q,h)-Bernstein bases and (q,h)-Bézier surfaces

Theorem 6.1. *Let $P(t,s) \in \mathbb{P}_{n.m}[t,s]$ with homogeneous (q,h)-blossom $p(\tilde u_1,\ldots,\tilde u_n;\tilde v_1,\ldots,_m;q,h)$, where $\tilde u_i = (u_i,w_i)$, $i = 1,\ldots,n$ and $\tilde v_j = (v_j,z_j)$, $j = 1,\ldots,m$. Let $g(t) = qt + h$. Set $\{Q^{0,0}_{i,j}\}$ as in (3.7), and define recursively*

$$\begin{aligned}
&Q^{r+1,l+1}_{i,j}(\tilde u_1,\ldots,\tilde u_{r+1};\tilde v_1,\ldots,\tilde v_{l+1};q,h)\\
&\quad=\frac{-u_{r+1}+g^{[i]}(b)w_{r+1}}{g^{[i]}(b)-g^{[i+r]}(a)}\,\frac{-v_{l+1}+g^{[j]}(d)z_{l+1}}{g^{[j]}(d)-g^{[j+l]}(c)}\,Q^{r,l}_{i,j}(\tilde u_1,\ldots,\tilde u_r;\tilde v_1,\ldots,\tilde v_l;q,h)\\
&\qquad+\frac{-u_{r+1}+g^{[i]}(b)w_{r+1}}{g^{[i]}(b)-g^{[i+r]}(a)}\,\frac{v_{l+1}-g^{[j+l]}(c)z_{l+1}}{g^{[j]}(d)-g^{[j+l]}(c)}\,Q^{r,l}_{i,j+1}(\tilde u_1,\ldots,\tilde u_r;\tilde v_1,\ldots,\tilde v_l;q,h)\\
&\qquad+\frac{u_{r+1}+g^{[i+r]}(a)w_{r+1}}{g^{[i]}(b)-g^{[i+r]}(a)}\,\frac{-v_{l+1}+g^{[j]}(d)z_{l+1}}{g^{[j]}(d)-g^{[j+l]}(c)}\,Q^{r,l}_{i+1,j}(\tilde u_1,\ldots,\tilde u_r;\tilde v_1,\ldots,\tilde v_l;q,h)\\
&\qquad+\frac{u_{r+1}+g^{[i+r]}(a)w_{r+1}}{g^{[i]}(b)-g^{[i+r]}(a)}\,\frac{v_{l+1}-g^{[j+l]}(c)z_{l+1}}{g^{[j]}(d)-g^{[j+l]}(c)}\,Q^{r,l}_{i+1,j+1}(\tilde u_1,\ldots,\tilde u_r;\tilde v_1,\ldots,\tilde v_l;q,h),
\end{aligned}$$

$i = 0,\ldots,n-r-1$, $r = 0,\ldots,n-1$ and $j = 0,\ldots,m-l-1$, $l = 0,\ldots,m-1$. Then

$$\begin{aligned}
Q^{r,l}_{i,j}(\tilde u_1,\ldots,\tilde u_r;\tilde v_1,\ldots,\tilde v_l;q,h) = p(&g^{[i+r]}(a),\ldots,g^{[n-1]}(a),b,\ldots,g^{[i-1]}(b),\tilde u_1,\ldots,\tilde u_r;\\
&g^{[j+l]}(c),\ldots,g^{[m-1]}(c),d,\ldots,g^{[j-1]}(d),\tilde v_1,\ldots,\tilde v_l;q,h),
\end{aligned}$$

$i = 0,\ldots,n-r$, $r = 0,\ldots,n$ and $j = 0,\ldots,m-l$, $l = 0,\ldots,m$. In particular,

$$Q^{n,m}_{0,0}(\tilde u_1,\ldots,\tilde u_n;\tilde v_1,\ldots,\tilde v_m;q,h) = p(\tilde u_1,\ldots,\tilde u_n;\tilde v_1,\ldots,\tilde v_m;q,h).$$

Proof. This result follows by induction on r and l. □

Corollary 6.2. *Let $P(t,s) \in \mathbb{P}_{n,m}[t,s]$ with homogeneous (q,h)-blossom $p(\tilde u_1,\ldots,\tilde u_n;\tilde v_1,\ldots,\tilde v_m;q,h)$. Let $0 \le r \le n$ and $0 \le l \le m$. There are $\frac{n!m!}{r!l!}$ recursive evaluation algorithms for the quantum partial derivative*

$$\frac{\partial^r_{1/q,-h/q}}{\partial t^r}\,\frac{\partial^l_{1/q,-h/q}}{\partial s^l}\,P(t,s). \tag{6.1}$$

Proof. Choose $\{i_\nu\}_{\nu=1}^r \subseteq \{1,\ldots,n\}$ (we have $\binom{n}{r}$ choices) and $\{j_\mu\}_{\mu=1}^l \subseteq \{1,\ldots,m\}$ (we have $\binom{m}{l}$ choices). Then pick $\sigma_1 \in S_{n-r}$ (we have $(n-r)!$ choices) and $\sigma_2 \in S_{m-l}$ (we have $(m-l)!$ choices). Next, in the recursive evaluation algorithm of Theorem 6.1, set $\tilde u_{i_\nu} = \delta$, $\nu = 1,\ldots,r$, $\tilde v_{j_\mu} = \delta$, $\mu = 1,\ldots,l$. Let $i'_1 < i'_2 < \ldots < i'_{n-r}$ be the elements of $\{1,\ldots,n\} \setminus \{i_\nu\}_{\nu=1}^r$ and let $j'_1 < j'_2 < \ldots < j'_{m-l}$ be the elements of $\{1,\ldots,m\} \setminus \{j_\mu\}_{\mu=1}^l$. Complete the algorithm of Theorem 6.1 by setting $\tilde u_{i'_\nu} = (g^{[\sigma_1(\nu)-1]}(t),1)$, $\nu = 1,\ldots,n-r$, and $\tilde v_{j'_\mu} = (g^{[\sigma_2(\mu)-1]}(s),1)$, $\mu = 1,\ldots,m-l$. Finally, multiply the resulting expression by $\frac{[n]_q![m]_q!}{[n-r]_q![m-l]_q!}$. By Theorem 2.5 we get (6.1). Clearly, the number of such algorithms is

$$\binom{n}{r}\binom{m}{l}(n-r)!(m-l)! = \frac{n!m!}{r!l!}.$$

□

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