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Reproducing Kernel Hilbert Space and Coalescence Hidden-variable Fractal Interpolation Functions

<https://doi.org/10.1515/dema-2019-0027>

Received February 15, 2019; accepted July 22, 2019

Abstract: Reproducing Kernel Hilbert Spaces (RKHS) and their kernel are important tools which have been found to be incredibly useful in many areas like machine learning, complex analysis, probability theory, group representation theory and the theory of integral operator. In the present paper, the space of Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) is demonstrated to be an RKHS and its associated kernel is derived. This extends the possibility of using this new kernel function, which is partly self-affine and partly non-self-affine, in diverse fields wherein the structure is not always self-affine.

Keywords: fractal, interpolation, coalescence, attractor, reproducing kernel, Hilbert space

MSC: 42C40, 41A15, 65T60, 42C10, 28A80

1 Introduction

The notion of Fractal Interpolation Function (FIF) and its construction was introduced by Barnsley [1] using the theory of Iterated Function System (IFS) and Read-Bajraktarevic operator. Since then, FIFs have been an amazing asset for interpolation of experimental data by a non-smooth curve and has extensive applications in engineering [2], biological sciences [3], planetary science [4] and arts [5]. After the introduction of FIF, different other kinds of FIFs namely Hidden-variable FIFs, Hermite FIFs, Spline FIFs and Super FIFs have also been constructed [6–8] and properties such as smoothness [9], approximation property [10, 11] and regularity [12] have been discussed. The idea of constructing Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) for simulation of curves that exhibit partly self-affine and partly non-self-affine nature was introduced by Chand and Kapoor [13]. The author had studied the effect of insertion of a new point in the interpolation data on the related IFS and the CHFIF [14] and Riemann-Liouville fractional calculus of CHFIF [15].

Reproducing Kernel Hilbert Spaces (RKHS) and their kernel are significant device which have been found valuable in numerous regions e.g. machine learning, complex analysis, probability theory, group representation theory and the theory of integral operator. The theory of RKHS was introduced [16–18] and have been used in the statistics literature for the past twenty years. Different kinds of kernels along with their respective RKHS (e.g. Gaussian Kernel) have been in use for a long time. However, most of these kernel spaces consisted of smooth functions. Bouboulis and Mavroforakis [19] showed that the space of any family of FIFs or Recurrent FIFs is an RKHS with a specific associated kernel function. This introduced the self-affine fractal functions to the RKHS universe. Multiresolution analysis arising from CHFIFs which exhibit partly self-affine and partly non-self-affine was developed [20] and as a natural follow-up, they have also been applied to construct orthonormal wavelets [21]. In this paper, the space of CHFIFs is shown to constitute an RKHS and its associated

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kernel is obtained. This increases the chance of utilizing this new kernel function which is partly self-affine and partly non-self-affine to fields where the structure is not always self-affine.

The organization of the paper is as follows: Section 2 summarizes the construction of a CHFIF. Section 3 discusses about the vector space of CHFIFs and its dimension. Section 4 begins with a brief introductory note on RKHS. Then, the space of CHFIFs is shown to be an RKHS and subsequently, its associated kernel is also derived.

2 Construction of a CHFIF

In this section, the basics of the construction of a CHFIF is discussed. Given interpolation data on \mathbb{R}^2 , a CHFIF is constructed as the first component of the attractor of a suitably defined IFS in \mathbb{R}^3 with the introduction of generalized interpolation data.

Let the given interpolation data be $\Lambda = \{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$ where $-\infty < x_0 < x_1 < \dots < x_N < \infty$. Denote the largest interval $[x_0, x_N]$ by I and the sub intervals $[x_{n-1}, x_n]$ for $n = 1, 2, \dots, N$ by I_n . A set of real parameters $\{z_i\}$ for $i = 0, 1, \dots, N$ is introduced to form the generalized interpolation data $\Delta = \{(x_i, y_i, z_i) : i = 0, 1, \dots, N\}$. The IFS required to construct a CHFIF is defined as

$$\{I \times \mathbb{R}^2; \omega_n, n = 1, 2, \dots, N\},$$

where

$$\omega_n(x, y, z) = (L_n(x), F_n(x, y, z)); \quad (1)$$

$$L_n(x) = x_{n-1} + \frac{x_n - x_{n-1}}{x_N - x_0} (x - x_0);$$

and

$$F_n(x, y, z) = (\alpha_n y + \beta_n z + p_n(x), \gamma_n z + q_n(x)).$$

Here, α_n and γ_n are free variables chosen such that $|\alpha_n| < 1$ and $|\gamma_n| < 1$. However, β_n are said to be constrained variables as they are chosen such that $|\beta_n| + |\gamma_n| < 1$. The functions p_n and q_n are continuous chosen such that the functions F_n satisfy

$$\begin{aligned} F_n(x_0, y_0, z_0) &= (y_{n-1}, z_{n-1}) \\ \text{and } F_n(x_N, y_N, z_N) &= (y_n, z_n). \end{aligned} \quad (2)$$

The above conditions are called join-up conditions. It is proved [13] that the above IFS is hyperbolic with respect to a metric d^* on \mathbb{R}^3 , equivalent to the Euclidean metric. For a hyperbolic IFS, it is known that there exists a unique non-empty compact set $A \subseteq \mathbb{R}^3$ such that $A = \bigcup_{n=1}^N \omega_n(A)$. This set A is called the attractor of IFS for the given interpolation data and shown to be the graph of a continuous function $f : I \rightarrow \mathbb{R}^2$ such that $f(x) = (f_1(x), f_2(x))$ and $f(x_i) = (y_i, z_i)$ for $i = 0, 1, \dots, N$. Now, a CHFIF is defined as follows:

Definition 2.1. The **Coalescence Hidden-variable Fractal Interpolation Function (CHFIF)** for the given interpolation data $\{(x_i, y_i) : i = 0, 1, \dots, N\}$ is defined as the continuous function $f_1 : I \rightarrow \mathbb{R}$ where f_1 is the first component of the vector valued function $f = (f_1, f_2)$ which is graph of an attractor.

Remark 2.2. If the functions $q_n(x)$ are linear polynomials, then the function $f_2(x)$ for the same interpolation data is called an Affine Fractal Interpolation Function (AFIF).

3 Space of CHFIFs

In this section, the vector space of CHFIFs is introduced and its dimension is found.

Definition 3.1. Let the set S_0 consist of functions $f : I \rightarrow \mathbb{R}^2$ such that $S_0 = \{f : f = (f_1, f_2), f_1$ is a CHFIF passing through $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$ and f_2 is an AFIF passing through $\{(x_i, z_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}\}$. For $f, g \in S_0$ and $a \in \mathbb{R}$, define $af + g = (af_1 + g_1, af_2 + g_2)$. Then, S_0 is a vector space, with usual point-wise addition and scalar multiplication. The set S_0 together with the maximum metric $d^*(f, g) = \max_{x \in I} (|f_1(x) - g_1(x)|, |f_2(x) - g_2(x)|)$ is a normed space.

Definition 3.2. Let S_0^1 be the set of functions $f_1 : I \rightarrow \mathbb{R}$ that are first components of functions $f \in S_0$. Then, S_0^1 is also a vector space with point-wise addition and scalar multiplication.

The following proposition, proved in [20], gives the dimension of S_0 and S_0^1 :

Proposition 3.1. *The dimension of vector space S_0 consisting of functions $f = (f_1, f_2)$, where f_1 is a CHFIF passing through $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$ and f_2 is an AFIF passing through $\{(x_i, z_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$ is $2(N+1)$.*

The dimension of vector space of CHFIFs S_0^1 is $2N$.

Proof. Let $V = \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \equiv \mathbb{R}^{2(N+1)}$. Then, V is a vector space with usual point-wise addition and scalar multiplication. Let $\mathbb{B}(I, \mathbb{R}^2)$ denote the set of bounded functions from I to \mathbb{R}^2 and $\mathbb{C}(I, \mathbb{R}^2)$ denote the set of continuous functions from I to \mathbb{R}^2 . Define the maximum metric on $\mathbb{B}(I, \mathbb{R}^2)$ and $\mathbb{C}(I, \mathbb{R}^2)$ as $d^*(f, g) = \max_{x \in I} \{|f_1(x) - g_1(x)|, |f_2(x) - g_2(x)|\}$. For every $t = (y, z) \in V$, define the operator Φ_t on $\mathbb{B}(I, \mathbb{R}^2)$ by

$$\begin{aligned} (\Phi_t f)(x) &= F_n(L_n^{-1}(x), f(L_n^{-1}(x))) \\ &= F_n(L_n^{-1}(x), f_1(L_n^{-1}(x)), f_2(L_n^{-1}(x))) \\ &= (\alpha_n f_1(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)), \gamma_n f_2(L_n^{-1}(x)) + q_n(L_n^{-1}(x))) \end{aligned}$$

for $x \in I_n$, $n = 1, 2, \dots, N$, where p_n and q_n are linear polynomials that satisfy the join up conditions: $p_n(x_0) = y_{n-1} - \alpha_n y_0 - \beta_n z_0$, $p_n(x_N) = y_n - \alpha_n y_N - \beta_n z_N$, $q_n(x_0) = z_{n-1} - \gamma_n z_0$ and $q_n(x_N) = z_n - \gamma_n z_N$. Then Φ_t is a contraction map on $\mathbb{B}(I, \mathbb{R}^2)$ and hence by the Banach contraction mapping theorem, Φ_t has a unique fixed point $f_t \in \mathbb{B}(I, \mathbb{R}^2)$. By join-up conditions (2), it follows that f_t is continuous.

Let $\Theta : V \rightarrow S_0 \subset \mathbb{C}(I, \mathbb{R}^2)$ be defined by $\Theta(t) = f_t$. Consider $y = (y_0, y_1, \dots, y_N)$, $z = (z_0, z_1, \dots, z_N)$, $\bar{y} = (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_N)$ and $\bar{z} = (\bar{z}_0, \bar{z}_1, \dots, \bar{z}_N)$. For every $t = (y, z)$ and $\bar{t} = (\bar{y}, \bar{z}) \in V$,

$$\begin{aligned} (\Phi_t(f_t))(x) &= F_n(L_n^{-1}(x), f_t(L_n^{-1}(x))) \\ &= F_n(L_n^{-1}(x), f_{t_1}(L_n^{-1}(x)), f_{t_2}(L_n^{-1}(x))) \\ &= (\alpha_n f_{t_1}(L_n^{-1}(x)) + \beta_n f_{t_2}(L_n^{-1}(x)) + p_n(L_n^{-1}(x)), \gamma_n f_{t_2}(L_n^{-1}(x)) + q_n(L_n^{-1}(x))), \end{aligned}$$

where $p_n(x_0) = y_{n-1} - \alpha_n y_0 - \beta_n z_0$, $p_n(x_N) = y_n - \alpha_n y_N - \beta_n z_N$, $q_n(x_0) = z_{n-1} - \gamma_n z_0$, $q_n(x_N) = z_n - \gamma_n z_N$ and

$$\begin{aligned} (\Phi_{\bar{t}}(f_{\bar{t}}))(x) &= F_n(L_n^{-1}(x), f_{\bar{t}}(L_n^{-1}(x))) \\ &= F_n(L_n^{-1}(x), f_{\bar{t}_1}(L_n^{-1}(x)), f_{\bar{t}_2}(L_n^{-1}(x))) \\ &= (\alpha_n f_{\bar{t}_1}(L_n^{-1}(x)) + \beta_n f_{\bar{t}_2}(L_n^{-1}(x)) + \bar{p}_n(L_n^{-1}(x)), \gamma_n f_{\bar{t}_2}(L_n^{-1}(x)) + \bar{q}_n(L_n^{-1}(x))), \end{aligned}$$

where $\bar{p}_n(x_0) = \bar{y}_{n-1} - \alpha_n \bar{y}_0 - \beta_n \bar{z}_0$, $\bar{p}_n(x_N) = \bar{y}_n - \alpha_n \bar{y}_N - \beta_n \bar{z}_N$, $\bar{q}_n(x_0) = \bar{z}_{n-1} - \gamma_n \bar{z}_0$, $\bar{q}_n(x_N) = \bar{z}_n - \gamma_n \bar{z}_N$. Therefore,

$$(\Phi_{at+\bar{t}}(af_t + f_{\bar{t}}))(x) = F_n(L_n^{-1}(x), (af_t + f_{\bar{t}})(L_n^{-1}(x)))$$

$$\begin{aligned}
&= F_n(L_n^{-1}(x), (af_{t_1} + f_{\bar{t}_1})(L_n^{-1}(x)), (af_{t_2} + f_{\bar{t}_2})(L_n^{-1}(x))) \\
&= \left(\alpha_n(af_{t_1}(L_n^{-1}(x)) + f_{\bar{t}_1}(L_n^{-1}(x))) + \beta_n(af_{t_2}(L_n^{-1}(x)) + f_{\bar{t}_2}(L_n^{-1}(x))) + (a p_n(L_n^{-1}(x)) + \bar{p}_n(L_n^{-1}(x))), \right. \\
&\quad \left. \gamma_n(af_{t_2}(L_n^{-1}(x)) + f_{\bar{t}_2}(L_n^{-1}(x))) + (a q_n(L_n^{-1}(x)) + \bar{q}_n(L_n^{-1}(x))) \right).
\end{aligned}$$

The above equation gives the following on simplification:

$$\begin{aligned}
((\Phi_{at+\bar{t}}(af_t + f_{\bar{t}}))(x)) &= a \left(\alpha_n f_{t_1}(L_n^{-1}(x)) + \beta_n f_{t_2}(L_n^{-1}(x)) + p_n(L_n^{-1}(x)), \gamma_n f_{t_2}(L_n^{-1}(x)) + q_n(L_n^{-1}(x)) \right) \\
&\quad + \left(\alpha_n f_{\bar{t}_1}(L_n^{-1}(x)) + \beta_n f_{\bar{t}_2}(L_n^{-1}(x)) + \bar{p}_n(L_n^{-1}(x)), \gamma_n f_{\bar{t}_2}(L_n^{-1}(x)) + \bar{q}_n(L_n^{-1}(x)) \right) \\
&= (af_t + f_{\bar{t}})(x).
\end{aligned}$$

Therefore, $af_t + f_{\bar{t}}$ is a fixed point of $\Phi_{at+\bar{t}}$ for all $a \in \mathbb{R}$ and $t, \bar{t} \in V$. By uniqueness of fixed point of $\Phi_{at+\bar{t}}$, it follows that $af_t + f_{\bar{t}} = f_{at+\bar{t}}$. So, $\Theta(at + \bar{t}) = f_{at+\bar{t}} = af_t + f_{\bar{t}}$.

Let $f = (f_1, f_2) \in \mathcal{S}_0 \subset \mathbb{C}(I, \mathbb{R}^2)$. Then f_1 is a CHFIF passing through $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$ and f_2 is an AFIF passing through $\{(x_i, z_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$. Define $y(f) = (y_0, y_1, \dots, y_N)$ and $z(f) = (z_0, z_1, \dots, z_N)$. Then $t(f) = (y(f), z(f)) \in V$ whenever $f \in \mathcal{S}_0$. Also $f_{t(f)} = f$ by uniqueness. So, $\Theta(t) = f_t$ is an onto map.

Let $f_t(x) = (0, 0)$ for all values of $x \in I$. Then, $f_t(x) = (0, 0) \Leftrightarrow \Phi_t(f_t)(x) = (0, 0) \Leftrightarrow F_n(L_n^{-1}(x), f_t(L_n^{-1}(x))) = (0, 0) \Leftrightarrow (p_n(L_n^{-1}(x)), q_n(L_n^{-1}(x))) = (0, 0)$ for every $n \Leftrightarrow t = (y, z) = ((0, \dots, 0), (0, \dots, 0))$. This gives that $\Theta(t) = f_t$ is injective.

Therefore, $\Theta : V \rightarrow \mathcal{S}_0$ defined by $\Theta(t) = f_t$ is a linear isomorphism. Hence,

$$\text{dimension of } \mathcal{S}_0 = 2(N + 1). \quad (3)$$

Now, consider the projection map $P : \mathcal{S}_0 \rightarrow \mathcal{S}_0^1$. Then, Kernel of $P \equiv \{f \in \mathcal{S}_0 \text{ such that } P(f) = 0\}$ is a proper subset of \mathcal{S}_0 and consists of elements of the form $(0, 0)$ and $(0, f_2)$. For the element $(0, f_2) \in \text{Ker } P$, it is observed that $\beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)) = 0$ for $x \in I_n$. Hence, for all $x \in I$, it is seen that $f_2(x) = \frac{1}{\beta_n} p_n(x)$. Since p_n is a linear polynomial, denote $p_n(x) = c_n x + d_n$. With $x = x_0$ and $\beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)) = 0$ for $x \in I_n$, it follows that $c_i = \frac{\beta_i}{\beta_1} c_1$ and $d_i = \frac{\beta_i}{\beta_1} d_1$, $i = 2, \dots, N$. Consequently, if $(0, f_2) \in \text{Ker } P$ then f_2 is a linear polynomial. So, dimension of $\text{Ker } P = 2$. Therefore, by the Rank-Nullity Theorem, dimension of $\mathcal{S}_0^1 = \text{dimension of } \mathcal{S}_0 - \text{dimension of } \text{Ker } P = 2(N + 1) - 2 = 2N$. \square

Remark 3.3. Suppose the operator Φ_t on $\mathbb{B}(I, \mathbb{R}^2)$ is written component-wise as $\Phi_t f = (\Phi_{t,1} f_1, \Phi_{t,2} f_2)$, where

$$\Phi_{t,1} f_1(x) = \alpha_n f_1(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)) \quad (4)$$

and

$$\Phi_{t,2} f_2(x) = \gamma_n f_2(L_n^{-1}(x)) + q_n(L_n^{-1}(x)) \quad (5)$$

for $x \in [x_{n-1}, x_n]$. The linear isomorphism between V and the vector space \mathcal{S}_0 together with equations (4) and (5) gives that for $f = (f_1, f_2) \in \mathcal{S}_0$, the function f_1 is the unique CHFIF passing through (x_i, y_i) , while the function f_2 is the unique AFIF passing through (x_i, z_i) . By the linear isomorphism between V and the vector space \mathcal{S}_0 , it follows using (5) that f_2 is completely determined by $f_2(x_i)$ for $i = 0, 1, \dots, N$. Further, it follows by (4) that f_1 depends on f_2 . If f_2 is a polynomial of degree of at most 1, then f_1 is affine FIF.

4 Reproducing kernel Hilbert space of CHFIFs

In this section, a brief introductory note on RKHS is given. Then, the space of \mathcal{S}_0 is shown to be an RKHS and its associated kernel function is derived. Subsequently, the space of CHFIFs is also proven to be an RKHS and its associated kernel function is also obtained.

Let \mathcal{H} denote a linear class of real-valued functions f defined on a set X and define an inner product $\langle \cdot, \cdot \rangle$ with corresponding norm $\|\cdot\|$ on \mathcal{H} such that \mathcal{H} is complete with respect to that norm. Then \mathcal{H} is a Hilbert space. The space \mathcal{H} is said to be a Reproducing Kernel Hilbert Space (RKHS) if there exists a function $\kappa : X \times X \rightarrow \mathbb{R}$ satisfying the following two properties: 1. For every $x \in X$, $\kappa(x, \cdot) \in \mathcal{H}$ and 2. For all $f \in \mathcal{H}$, $f(x) = \langle f, \kappa(x, \cdot) \rangle$. In particular, $\kappa(x, y) = \langle \kappa(x, \cdot), \kappa(y, \cdot) \rangle$. Then, κ is called the reproducing kernel of \mathcal{H} .

The definition of reproducing kernel says that it depends on the inner product and the Hilbert space. There could be several inner products defined in the same Hilbert space. Hence, the reproducing kernel of a Hilbert space varies if the inner product is changed. The following result is useful in describing the Kernel of a Hilbert space:

Proposition 4.1. [18] Suppose \mathcal{H} is finite dimensional of dimension N and let $e_j; j = 1, \dots, N$ be a basis of \mathcal{H} . If $f, g \in \mathcal{H}$, then $\langle f, g \rangle = \sum_{i,j=1}^N A_{i,j} c_i d_j$, where $f = \sum_{i=1}^N c_i e_i$, $g = \sum_{j=1}^N d_j e_j$ and $A_{i,j} = \langle e_i, e_j \rangle$. Let A be a $N \times N$ matrix such that $A_{i,j}$ is the entry at (i, j) position. A function $\kappa : X \times X \rightarrow \mathbb{R}$ is reproducing kernel of \mathcal{H} if and only if it is of the form $\kappa(x, y) = \sum_{i,j=1}^N B_{i,j} e_i(x) e_j(y)$, where B is a positive definite matrix with entry $B_{i,j}$ at (i, j) position and B is inverse of A .

Let $\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. Using the above proposition, we shall now derive a kernel for the space \mathcal{S}_0 .

Theorem 4.1. For $i = 1, \dots, N+1$ and $j = 0, \dots, N$, let $y_{i,j} = \delta_{i,j+1}$ and $z_{i,j} = 0$ and for $i = N+2, \dots, 2N+2$ and $j = 0, 1, \dots, N$, let $z_{i,j} = \delta_{i-N-1,j+1}$ and $y_{i,j} = 0$. Then, there exists a kernel for the space \mathcal{S}_0 given by $\kappa(x, y) = \sum_{i=1}^{2N+2} f_i(x) f_i(y)$, where $f_i = (f_{i_1}, f_{i_2})$, f_{i_1} are CHFIFs passing through $\{(x_j, y_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$ and f_{i_2} are AFIFs passing through $\{(x_j, z_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$ for $i = 1, 2, \dots, 2N+2$ and hence \mathcal{S}_0 is an RKHS.

Proof. For every $t = (y, z)$ and $\bar{t} = (\bar{y}, \bar{z}) \in V$, define $\langle\langle t, \bar{t} \rangle\rangle_V = \langle y, \bar{y} \rangle + \langle z, \bar{z} \rangle$, where $\langle y, \bar{y} \rangle = \sum_{j=1}^{N+1} y_j \bar{y}_j$ and $\langle z, \bar{z} \rangle = \sum_{j=1}^{N+1} z_j \bar{z}_j$. Then, $(V, \langle\langle \cdot, \cdot \rangle\rangle_V)$ is a Hilbert space and its dimension is $2(N+1)$. For $i = 1, \dots, 2N+2$, let $t_i = (y_i, z_i)$, where y_i and z_i are $N+1$ tuples whose j^{th} coordinate is given as $y_{i,j-1}$ and $z_{i,j-1}$ respectively. It is easily seen that for any $i \neq k$ and $i, k \in \{1, \dots, 2N+2\}$, $\langle\langle t_i, t_k \rangle\rangle_V = 0$ and for each $i \in \{1, \dots, 2N+2\}$, $\langle\langle t_i, t_i \rangle\rangle_V = 1$. Therefore, the collection $\{t_i; i = 1, \dots, 2N+2\}$ is a linearly independent set and hence a basis of V .

The linear isomorphism $\Theta(t) = f_t$ between V and \mathcal{S}_0 implies that the set of functions $\{f_i = (f_{i_1}, f_{i_2}); i = 1, \dots, 2N+2\}$, where f_{i_1} are CHFIFs passing through $\{(x_j, y_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$ and f_{i_2} are AFIFs passing through $\{(x_j, z_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$ is a basis for the set \mathcal{S}_0 . Suppose we choose the inner product on \mathcal{S}_0 as $\langle\langle f, \bar{f} \rangle\rangle = \langle\langle \Theta^{-1}(f), \Theta^{-1}(\bar{f}) \rangle\rangle_V$. Then,

$$\begin{aligned} A_{i,k} &= \langle\langle f_i, f_k \rangle\rangle = \langle\langle \Theta^{-1}(f_i), \Theta^{-1}(f_k) \rangle\rangle_V \\ &= \sum_{j=1}^{N+1} y_{i,j} y_{k,j} + \sum_{j=1}^{N+1} z_{i,j} z_{k,j} = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases} \end{aligned}$$

Using the above proposition, if we define $\kappa(x, y) = \sum_{i,k=1}^{2N+2} A_{i,k}^{-1} f_i(x) f_k(y)$, then $\kappa(x, y) = \sum_{i=1}^{2N+2} f_i(x) f_i(y)$ is a reproducing kernel of the space \mathcal{S}_0 and hence \mathcal{S}_0 is an RKHS. \square

The above theorem does not help in determining the kernel of space of CHFIFs. It neither induces a norm in the space \mathcal{S}_0 . Suppose the inner product on \mathcal{S}_0 is chosen as $\langle\langle f, \bar{f} \rangle\rangle = \langle f_1, \bar{f}_1 \rangle + \langle f_2, \bar{f}_2 \rangle$, where $f = (f_1, f_2)$, $\bar{f} =$

$(\bar{f}_1, \bar{f}_2) \in \mathcal{S}_0$, $\langle f_1, \bar{f}_1 \rangle = \int_{x_0}^{x_N} f_1(x) \bar{f}_1(x) dx$ and $\langle f_2, \bar{f}_2 \rangle = \int_{x_0}^{x_N} f_2(x) \bar{f}_2(x) dx$. Then, it induces a norm on \mathcal{S}_0 given by $\|f\| = \sqrt{\|f_1\|^2 + \|f_2\|^2}$.

In order to study the kernel of space of CHFIF, assume that $y_{i,j} = \delta_{i,j+1}$, $z_{i,j} = 0$ and $z_{N+1+i,j} = \delta_{i,j+1}$ for $i = 1, \dots, N+1$ and $j = 0, 1, \dots, N$. Also, assume that $y_{N+2,j} = y_{2N+2,j} = 0$ for $j = 0, 1, \dots, N$. To derive the kernel of space of CHFIFs, $y_{i,0}$ and $y_{i,N}$ for $i = N+3, \dots, 2N+1$ are chosen such that $y_{i,0} = y_{i,N} = 0$ and for $j = 2, \dots, N$, $y_{i,j-1}$ are real numbers chosen such that $\langle f_{i_1}, f_{j_1} \rangle = 0$. Let, for $i = N+3, \dots, 2N+1$,

$$\zeta_i = \langle f_{i_1}, f_{1_1} \rangle \quad \text{and} \quad \eta_i = \langle f_{i_1}, f_{(N+1)_1} \rangle. \quad (6)$$

The functions f_{i_1} for $i = N+2, \dots, 2N+2$ are CHFIFs passing through $\{(x_j, y_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$. The free variables α_n, γ_n and constrained variables β_n for $n = 1, 2, \dots, N$, in the construction of f_{i_1} for $i = N+3, \dots, 2N+1$ are chosen such that $\zeta_i = 0$ and $\eta_i = 0$. This is possible because there are $3N$ unknowns which is the total of free variables α_n, γ_n and constrained variables β_n for $n = 1, 2, \dots, N$ while $\zeta_i = \eta_i = 0$ for $i = N+3, \dots, 2N+1$ is a system of $2N-2$ equations. Suppose there exist no α_n, γ_n and β_n for $n = 1, \dots, N$, in $(-1, 1)$ such that $\zeta_i = \eta_i = 0$ for $i = N+3, \dots, 2N+1$, then the number of linearly independent functions in S_0^1 is less than $2N$, which is a contradiction as dimension of $S_0^1 = 2N$. Hence, there exists at least one set of α_n, γ_n and β_n for $n = 1, \dots, N$, in $(-1, 1)$ such that $\zeta_i = \eta_i = 0$ for $i = N+3, \dots, 2N+1$. With this choice of free variables and constrained variables, a kernel for the space of CHFIFs is described below:

Theorem 4.2. *The space \mathcal{S}_0^1 is an RKHS and the kernel for the space \mathcal{S}_0^1 is given by $\kappa(x, y) = \sum_{i,k=1}^{N+1} B_{i,k} f_{i_1}(x) f_{k_1}(y) + \sum_{i,k=N+3}^{2N+1} B_{i-1,k-1} f_{i_1}(x) f_{k_1}(y)$ where B is a positive definite matrix of order $2N$ with entry $B_{i,k}$ at (i, k) position, f_{i_1} are CHFIFs passing through $\{(x_j, y_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$ such that f_{i_1} are first components of functions $f_i \in \mathcal{S}_0$; for $i = 1, \dots, N+1$ and $j = 0, 1, \dots, N$, $y_{i,j} = \delta_{i,j+1}$, $z_{i,j} = 0$, $z_{N+1+i,j} = \delta_{i,j+1}$; for $i = N+3, \dots, 2N+1$, $y_{i,0} = y_{i,N} = 0$ and for $j = 2, \dots, N$, $y_{i,j-1}$ are real numbers such that $\langle f_{i_1}, f_{j_1} \rangle = 0$, and the free variables α_n, γ_n and constrained variables β_n for $n = 1, 2, \dots, N$ in the construction of CHFIFs are chosen such that $\zeta_i = \langle f_{i_1}, f_{1_1} \rangle = 0$ and $\eta_i = \langle f_{i_1}, f_{(N+1)_1} \rangle = 0$ for $i = N+3, \dots, 2N+1$.*

Proof. From Remark 3.3, it is clear that $f_{i,2}$ is completely determined by the points $(x_j, z_{i,j})$ for $i = 1, \dots, N+1$. As $z_{i,j} = 0$ for $i = 1, \dots, N+1$ and $j = 0, \dots, N$, it is clear that $f_{i,2}$ are zero functions. Since $y_{i,j} = \delta_{i,j+1}$ for $i = 1, \dots, N+1$ and $j = 0, 1, \dots, N$, it is easily seen that the functions f_{i_1} for $i = 1, \dots, N+1$ are AFIFs and are linearly independent. Also, since $z_{i,j} = \delta_{i-N-1,j+1}$ for $i = N+2, \dots, 2N+2$ and $j = 0, 1, \dots, N$, it is clear that the functions f_{i_2} for $i = N+3, \dots, 2N+1$, are not linear polynomials whereas f_{i_2} for $i = N+2$ and $i = 2N+2$ are linear polynomials. So, the functions f_{i_1} for $i = N+3, \dots, 2N+1$ are linearly independent as $\sum_{i=N+3}^{2N+1} a_i (f_{i_1}(x) - \alpha_n f_{i_1}(L_n^{-1}(x))) = 0$ if and only if $a_i = 0$.

Now, let $\sum_{i=1}^{N+1} a_i f_{i_1} + \sum_{i=N+3}^{2N+1} a_i f_{i_1} = 0$. Then, $\left\langle \sum_{i=1}^{N+1} a_i f_{i_1} + \sum_{i=N+3}^{2N+1} a_i f_{i_1}, f_{k_1} \right\rangle = 0$. For $i = N+3, \dots, 2N+1$, $y_{i,j-1}$ for $j = 2, \dots, N$ are real numbers chosen such that $\langle f_{i_1}, f_{j_1} \rangle = 0$. Also, the free variables α_n, γ_n and constrained variables β_n for $n = 1, 2, \dots, N$ are chosen such that $\zeta_i = \langle f_{i_1}, f_{1_1} \rangle = 0$ and $\eta_i = \langle f_{i_1}, f_{(N+1)_1} \rangle = 0$ for $i = N+3, \dots, 2N+1$. So, $\left\langle \sum_{i=1}^{N+1} a_i f_{i_1} + \sum_{i=N+3}^{2N+1} a_i f_{i_1}, f_{k_1} \right\rangle = 0$ gives $a_k = 0$. This is true for each $k = 1, \dots, N+1, N+3, \dots, 2N+1$. Hence, $\sum_{i=1}^{N+1} a_i f_{i_1} + \sum_{i=N+3}^{2N+1} a_i f_{i_1} = 0$ if and only if $a_i = 0$. Therefore, the functions f_{i_1} for $i = 1, \dots, N+1, N+3, \dots, 2N+1$ are linearly independent. As the dimension of \mathcal{S}_0^1 is $2N$, the linearly independent functions f_{i_1} for $i = 1, \dots, N+1, N+3, \dots, 2N+1$ form a basis of \mathcal{S}_0^1 .

Let $A_{i,k} = \langle f_{i_1}, f_{k_1} \rangle = \int_{x_0}^{x_N} f_{i_1}(x) f_{k_1}(x) dx$ for $i = 1, \dots, N+1$ and for $i = N+3, \dots, 2N+1$, let $A_{i-1,k-1} = \langle f_{i_1}, f_{k_1} \rangle = \int_{x_0}^{x_N} f_{i_1}(x) f_{k_1}(x) dx$. Then A is an invertible matrix of order $2N$ with entry $A_{i,k}$ at (i, k) position

and the inverse of A is B . Using the above proposition, it is clear that $\kappa(x, y) = \sum_{i,k=1}^{N+1} B_{i,k} f_{i_1}(x) f_{k_1}(y) + \sum_{i,k=N+3}^{2N+1} B_{i-1,k-1} f_{i_1}(x) f_{k_1}(y)$ is a reproducing kernel of the space \mathcal{S}_0^1 and hence \mathcal{S}_0^1 is an RKHS. \square

With the above choices of free variables and constrained variables and norm inducing inner product on the space \mathcal{S}_0 , another Kernel for the space is given as follows:

Theorem 4.3. *The space \mathcal{S}_0 is an RKHS and the kernel for the space \mathcal{S}_0 is given by $\kappa(x, y) = \sum_{i,k=1}^{2N+2} B_{i,k} f_i(x) f_k(y)$ where B is a positive definite matrix of order $2N+2$ with entry at (i, k) position as $B_{i,k} f_i = (f_{i_1}, f_{i_2}) \in \mathcal{S}_0$, f_{i_1} are CHFIFs passing through $\{(x_j, y_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$, f_{i_2} are AFIFs passing through $\{(x_j, z_{i,j}) \in \mathbb{R}^2 : j = 0, 1, \dots, N\}$; $y_{i,j} = \delta_{i,j+1}$, $z_{i,j} = 0$ and $z_{N+1+i,j} = \delta_{i,j+1}$ for $i = 1, \dots, N+1$ and $j = 0, 1, \dots, N$, and $y_{2N+2,j} = 0$ for $j = 0, 1, \dots, N$; for $i = N+3, \dots, 2N+1$, $y_{i,0} = y_{i,N} = 0$ and for $j = 2, \dots, N$, $y_{i,j-1}$ are real numbers such that $\langle f_{i_1}, f_{j_1} \rangle = 0$, and the free variables α_n, γ_n and constrained variables β_n for $n = 1, 2, \dots, N$ in the construction of CHFIFs are chosen such that $\zeta_i = \langle f_{(N+2+i)_1}, f_{0_1} \rangle = 0$ and $\eta_i = \langle f_{(N+2+i)_1}, f_{N+1_1} \rangle = 0$ for $i = 1, 2, \dots, N-1$.*

Proof. Although the functions f_{i_2} are zero functions for $i = 1, \dots, N+1$, the functions f_{i_1} for $i = 1, \dots, N+1$ are AFIFs and are linearly independent. So, the functions $f_i = (f_{i_1}, f_{i_2})$ for $i = 1, \dots, N+1$ are linearly independent. Again, although the functions f_{i_1} for $i = N+2$ and $i = 2N+2$ are zero functions, f_{i_2} for $i = N+2$ and $i = 2N+2$ are not zero functions and are linearly independent. Since $z_{i,j} = \delta_{i-N-1,j+1}$ for $i = N+2, \dots, 2N+2$ and $j = 0, 1, \dots, N$, it is clear that the functions f_{i_2} for $i = N+2, \dots, 2N+2$ are linearly independent. So, the functions $f_i = (f_{i_1}, f_{i_2})$ for $i = N+2, \dots, 2N+2$ are linearly independent. Since for $j = 2, \dots, N$, $y_{i,j-1}$ are real numbers such that $\langle f_{i_1}, f_{j_1} \rangle = 0$, and the free variables α_n, γ_n and constrained variables β_n for $n = 1, 2, \dots, N$ in the construction of CHFIFs are chosen such that $\zeta_i = \langle f_{(N+2+i)_1}, f_{0_1} \rangle = 0$ and $\eta_i = \langle f_{(N+2+i)_1}, f_{N+1_1} \rangle = 0$ for $i = 1, 2, \dots, N-1$, the functions $f_i = (f_{i_1}, f_{i_2})$ for $i = 1, \dots, 2N+2$ are linearly independent. Since dimension of \mathcal{S}_0 is $2N+2$, the linearly independent functions $f_i = (f_{i_1}, f_{i_2})$ for $i = 1, \dots, 2N+2$ form a basis of \mathcal{S}_0 .

Let $A_{i,k} = \langle f_i, f_k \rangle = \langle f_{i_1}, f_{k_1} \rangle + \langle f_{i_2}, f_{k_2} \rangle = \int_{x_0}^{x_N} f_{i_1}(x) f_{k_1}(x) dx + \int_{x_0}^{x_N} f_{i_2}(x) f_{k_2}(x) dx$ for $i = 1, \dots, 2N+2$. Then A is an invertible matrix of order $2N+2$ with entry $A_{i,k}$ at (i, k) position and the inverse of A is B . Using the above proposition, it is clear that $\kappa(x, y) = \sum_{i,k=1}^{2N+2} B_{i,k} f_i(x) f_k(y)$ is a reproducing kernel of the space \mathcal{S}_0 and hence \mathcal{S}_0 is an RKHS. \square

5 Conclusions

In this paper, it is shown that the space of CHFIFs is an RKHS and its associated kernel is obtained. This broadens the likelihood of using this new kernel function which is partly self-affine and partly non-self-affine to fields where the structure is not always self-affine. The Space \mathcal{S}_0 consisting of vector valued functions $f = (f_1, f_2)$, where the first component f_1 is a CHFIF and second component f_2 is an AFIF is also shown to be an RKHS with respect to two different inner products and corresponding to each inner product, its associated kernel is also derived.

Acknowledgment The author is thankful to the referees for their constructive evaluation of the paper and their valuable suggestions.

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