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Elliptic operators and their symbols

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Abstract: We consider special elliptic operators in functional spaces on manifolds with a boundary which has some singular points. Such an operator can be represented by a sum of operators, and for a Fredholm property of an initial operator one needs a Fredholm property for each operator from this sum.

Keywords: elliptic operator, local representative, enveloping operator

MSC: 47A05; 58J05

1 Introduction

This paper is devoted to describing the structure of a special class of linear bounded operators on a manifold with non-smooth boundary. Our description is based on Simonenko's theory of envelopes [1] and explains why we obtain distinct theories for pseudo-differential equations and boundary value problems and distinct index theorems for such operators.

1.1 Operators of a local type

In this section we will give some preliminary ideas and definitions from [1].

Let B_1, B_2 be Banach spaces consisting of functions defined on compact m-dimensional manifold M, $A : B_1 \to B_2$ be a linear bounded operator, $W \subset M$, and P_W be a projector on W, i.e.

$$(P_W u)(x) = \begin{cases} u(x), & \text{if } x \in W; \\ 0, & \text{if } x \notin \overline{W}. \end{cases}$$

Definition 1. *An operator A is called an operator of local type if the operator*

$$P_U A P_V$$

is a compact operator for arbitrary non-intersecting compact sets $U, V \subset M$.

1.2 Simple examples

These are two of the simplest examples for illustration.

Example 1. *If A is a differential operator of the type*

$$(Au)(x) = \sum_{|k|=0}^{n} a_k(x) D^k u(x), \quad D^k u = \frac{\partial^k u}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}},$$

then A is an operator of local type.

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Example 2. *If A is a Calderon–Zygmund operator with variable kernel $K(x,y) \in C^1(\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ of the following type*

$$(Au)(x) = v.p. \int_{\mathbb{R}^m} K(x, x-y)u(y)dy,$$

then A is an operator of a local type.

Everywhere below we say "an operator" instead of "an operator of local type".

1.3 Functional spaces on a manifold

1.3.1 Spaces $H^s(\mathbb{R}^m)$, $L_p(\mathbb{R}^m)$, $C^\alpha(\mathbb{R}^m)$

It is possible to work with distinct functional spaces [2, 3].

Definition 2. [4] *The space $H^s(\mathbb{R}^m)$, $s \in \mathbb{R}$, is a Hilbert space of functions with the finite norm*

$$||u||_s = \left(\int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 (1+|\xi|)^{2s} d\xi \right)^{1/2},$$

where the sign $\sim$ over a function means its Fourier transform.

Definition 3. [2] *The space $L_p(\mathbb{R}^m)$, $1 < p < +\infty$, is a Banach space of measurable functions with the finite norm*

$$||u||_p = \left(\int_{\mathbb{R}^m} |u(x)|^p dx \right)^{1/p}.$$

Definition 4. [2] *The space $C^\alpha(\mathbb{R}^m)$, $0 < \alpha \leq 1$, is a space of continuous functions u on $\mathbb{R}^m$ satisfying the Hölder condition*

$$|u(x) - u(y)| \leq c|x-y|^\alpha, \quad \forall x, y \in \mathbb{R}^m,$$

with the finite norm

$$||u||_\alpha = \inf\{c\},$$

where infimum is taken over all constants c from the above inequality.

1.3.2 Partition of unity and spaces $H^s(M)$, $L_p(M)$, $C^\alpha(M)$

If M is a compact manifold then there is *a partition of unity* [5]. It means the following. For every finite open covering $\{U_j\}_{j=1}^k$ of the manifold M there exists a system of functions $\{\varphi_j(x)\}_{j=1}^k$, $\varphi_j(x) \in C^\infty(M)$, such that

- $0 \leq \varphi_j(x) \leq 1$,
- $supp\ \varphi_j \subset U_j$,
- $\sum_{j=1}^k \varphi_j(x) = 1$.

So we have

$$f(x) = \sum_{j=1}^k \varphi_j(x) f(x)$$

for an arbitrary function f defined on M.

Since every set U_j is diffeomorphic to an open set $D_j \subset \mathbb{R}^m$ we have corresponding diffeomorphisms $\omega_j : U_j \to D_j$. Further, for a function f defined on M we compose mappings $f_j = f \cdot \varphi_j$ and as long as *supp* $f_j \subset U_j$ we put $\hat{f}_j = f_j \circ \omega_j^{-1}$ so that $\hat{f}_j : D_j \to \mathbb{R}$ is a function defined on a domain of m-dimensional space $\mathbb{R}^m$. We can consider, for example, the following functional spaces [2–4].

Definition 5. *A function $f \in H^s(M)$ if the following norm*

$$||f||_{H^s(M)} = \sum_{j=1}^{k} ||\hat{f}_j||_s$$

is finite.

A function $f \in L_p(M)$ if the following norm

$$||f||_{L_p(M)} = \sum_{j=1}^{k} ||\hat{f}_j||_p$$

is finite.

A function $f \in C^\alpha(M)$ if the following norm

$$||f||_{C^\alpha(M)} = \sum_{j=1}^{k} ||\hat{f}_j||_\alpha$$

is finite.

2 Operators on a compact manifold

On the manifold M we fix a finite open covering and a partition of unity corresponding to this covering $\{U_j, f_j\}_{j=1}^n$. We then choose smooth functions $\{g_j\}_{j=1}^n$ so that *supp* $g_j \subset V_j$, $\overline{U_j} \subset V_j$, and $g_j(x) \equiv 1$ for $x \in$ *supp* f_j, *supp* $f_j \cap (1 - g_j) = \emptyset$.

Proposition 1. *The operator A on the manifold M can be represented in the form*

$$A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,$$

where $T : B_1 \to B_2$ is a compact operator.

Proof. The proof is straightforward. Since

$$\sum_{j=1}^{n} f_j(x) \equiv 1, \quad \forall x \in M,$$

then we have

$$A = \sum_{j=1}^{n} f_j \cdot A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + \sum_{j=1}^{n} f_j \cdot A \cdot (1 - g_j),$$

and the proof is completed. □

Remark 1. *Obviously such an operator is defined uniquely up to a compact operators which have no influence on an index.*

By definition, for an arbitrary operator $A : B_1 \to B_2$

$$|||A||| \equiv \inf ||A + T||,$$

where *infimum* is taken over all compact operators $T : B_1 \to B_2$.

Let B_1', B_2' be Banach spaces consisting of functions defined on $\mathbb{R}^m$, and let $\widetilde{A} : B_1' \to B_2'$ be a linear bounded operator.

Since M is a compact manifold, then for every point $x \in M$ there exists a neighborhood $U \ni x$ and a diffeomorphism $\omega : U \to D \subset \mathbb{R}^m$, $\omega(x) \equiv y$. We denote by S_ω the following operator acting from B_k to B'_k, $k = 1, 2$. For every function $u \in B_k$ vanishing out of U

$$(S_\omega u)(y) = u(\omega^{-1}(y)), \quad y \in D, \quad (S_\omega u)(y) = 0, \quad y \notin D.$$

Definition 6. *A local representative of the operator $A : B_1 \to B_2$ at the point $x \in M$ is called the operator $\widetilde{A} : B'_1 \to B'_2$ such that for all $\varepsilon > 0$ there exists the neighborhood U_j of the point $x \in U_j \subset M$ with the property*

$$|||g_j A f_j - S_{\omega_j^{-1}} \hat{g}_j \widetilde{A} \hat{f}_j S_{\omega_j}||| < \varepsilon.$$

3 Algebra of symbols

Definition 7. *Symbol of an operator A is called the family of its local representatives $\{A_x\}$ at each point $x \in \overline{M}$.*

One can show like [1] this definition of an operator symbol conserves all properties of a symbolic calculus. Namely, up to compact summands we have the following:

- the product and the sum of two operators corresponds to the product and the sum of their local representatives;
- the adjoint operator corresponds to its adjoint local representative;
- a Fredholm property of an operator corresponds to a Fredholm property of its local representative.

4 Operators with symbols. Examples of operators

It seems not every operator has a symbol, and we give some examples for operators with symbols.

Example 3. *Let A be the differential operator from Example 1, and functions $a_k(x)$ be continuous functions on $\mathbb{R}^m$. Then its symbol is an operator family consisting of multiplication operators on the function*

$$\sum_{|k|=0}^{n} a_k(x)\xi^k,$$

where $\xi^k = \xi_1^{k_1} \cdots \xi_m^{k_m}$.

Example 4. *Let A be the Calderon–Zygmund operator from Example 2 and $\sigma(x, \xi)$ be its symbol in the sense of [2], then its symbol is an operator family consisting of multiplication operators on the function $\sigma(x, \xi)$.*

The more important point is that the symbol of an operator is simpler than general operator, and it permits to verify its Fredholm properties. For the two above examples a Fredholm property of an operator symbol is equivalent to its invertibility.

5 Stratification of manifolds and operators

5.1 Sub-manifolds

The above definition of an operator on a manifold supposes that all neighborhoods $\{U_j\}$ have the same type. But even if a manifold has a smooth boundary then there are two types of neighborhoods related to a placement of neighborhood, namely inner neighborhoods and boundary ones. For an inner neighborhood U such that $\overline{U} \subset \mathring{M}$ we have the diffeomorphism $\omega : U \to D$, where $D \in \mathbb{R}^m$ is an open set. For a boundary neighborhood such that $U \cap \partial M \neq \emptyset$ we have another diffeomorphism $\omega_1 : U \to D \cap \mathbb{R}^m_+$, where

$$\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > 0\}.$$

Maybe this boundary ∂M has some singularities like conical points and wedges. The conical point at the boundary is such a point, for which its neighborhood is diffeomorphic to the cone

$$C_+^a = \{x \in \mathbb{R}^m : x_m > a|x'|, \ x' = (x_1, ..., x_{m-1}), \ a > 0\}.$$

The wedge point of codimension k, $1 \le k \le m-1$, is such a point for which its neighborhood is diffeomorphic to the set $\{x \in \mathbb{R}^m : x = (x', x''), x'' \in \mathbb{R}^{m-k}, x' = (x_1, \cdots, x_{m-k-1}), x_{m-k-1} > a|x'''|, x''' = (x_1, \cdots, x_{m-k-2}), a > 0\}$. So if the manifold M has such singularities we suppose that we can extract certain k-dimensional submanifolds, namely an $(m-1)$-dimensional boundary ∂M, and k-dimensional wedges M_k, $k = 0, \cdots, m-2$; M_0 are a collection of conical points.

5.2 Enveloping operators

If the family $\{A_x\}_{x \in M}$ is continuous in the operator topology, then according to Simonenko's theory there is an enveloping operator, i.e. such an operator A for which every operator A_x is the local representative for the operator A in the point $x \in M$.

Example 5. *If $\{A_x\}_{x \in M}$ consists of Calderon–Zygmund operators in $\mathbb{R}^m$ [2] with symbols $\sigma_x(\xi)$ parametrized by points $x \in M$ and this family smoothly depends on $x \in M$ then the Calderon–Zygmubd operator with variable kernel and symbol $\sigma(x, \xi)$ will be an enveloping operator for this family.*

Example 6. *If $\{A_x\}_{x \in M}$ consists of null operators then an enveloping operator is a compact operator [1].*

Theorem 1. *The operator A has a Fredholm property if and only if its all local representatives $\{A_x\}_{x \in M}$ have the same property.*

This property was proved in [1], but we will give the proof (see Lemma 2) including some new constructions because it will be used below for a decomposition of the operator.

5.3 Hierarchy of operators

We will remind the reader here of the following definition and Fredholm criteria for operators [6].

Definition 8. *Let B_1, B_2 be Banach spaces, and $A : B_1 \to B_2$ be a linear bounded operator. The operator $R : B_2 \to B_1$ is called a regularizer for the operator A if the following properties*

$$RA = I_1 + T_1, \quad AR = I_2 + T_2$$

hold, where $I_k : B_k \to B_k$ is an identity operator, $T_k; B_k \to B_k$ is a compact operator, $k = 1, 2$.

Proposition 2. *The operator $A : B_1 \to B_2$ has a Fredholm property if and only if there exists a linear bounded regularizer $R : B_2 \to B_1$.*

Lemma 1. *Let f be a smooth function on the manifold M, $U \subset M$ be an open set, and supp $f \subset U$. Then the operator $f \cdot A - A \cdot f$ is a compact operator.*

Proof. Let g be a smooth function on M, *supp* $g \subset V \subset M$, moreover $\overline{U} \subset V$, $g(x) \equiv 1$ for $x \in$ *supp* f. Then we have

$$f \cdot A = f \cdot A \cdot g + f \cdot A \cdot (1-g) = f \cdot A \cdot g + T_1,$$
$$A \cdot f = g \cdot A \cdot f + (1-g) \cdot A \cdot f = g \cdot A \cdot f + T_2,$$

where T_1, T_2 are compact operators. Let us denote $g \cdot A \cdot g \equiv h$ and write

$$f \cdot A \cdot g = f \cdot g \cdot A \cdot g = f \cdot h, \quad g \cdot A \cdot f = g \cdot A \cdot g \cdot f = h \cdot f,$$

and we obtain the required property. □

Definition 9. *The operator A is called an elliptic operator if its operator symbol $\{A_x\}_{x\in M}$ consists of Fredholm operators.*

Now we will show that each elliptic operator really has a Fredholm property. Our proof in general follows the book [1], but our constructions are more stratified and we need such constructions below.

Lemma 2. *Let A be an elliptic operator. Then the operator A has a Fredholm property.*

Proof. To obtain the proof we will construct the regularizer for the operator A. For this purpose we choose two coverings like Proposition 1 and write the operator A in the form

$$A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T, \tag{1}$$

where T is a compact operator. Without loss of generality we can assume that there are n points $x_k \in U_k \subset V_k$, $k = 1, 2, \dots n$. Moreover, we can construct such coverings by balls in the following way. Let $\varepsilon > 0$ be a small enough number. First, for every point $x \in M_0$ we take two balls U_x, V_x with the center at x of radius ε and 2ε and construct two open coverings for M_0 namely $\mathfrak{U}_0 = \cup_{x\in M_0} U_x$ and $\mathfrak{V}_0 = \cup_{x\in M_0} V_x$. Second, we consider the set $L_1 = \overline{M}\setminus\mathfrak{V}_0$ and construct two coverings $\mathfrak{U}_1 = \cup_{x\in L_1\cap M_1} U_x$ and $\mathfrak{V}_1 = \cup_{x\in L_1\cap M_1} V_x$. Further, we introduce the set $L_2 = \overline{M}\setminus(\mathfrak{V}_0\cup\mathfrak{V}_1)$ and two coverings $\mathfrak{U}_2 = \cup_{x\in L_2\cap M_2} U_x$ and $\mathfrak{V}_2 = \cup_{x\in L_2\cap M_2} V_x$. Continuing these actions we will come to the set $L_{m-1} = \overline{M}\setminus(\cup_{k=0}^{m-2}\mathfrak{U}_k)$ which consists of smoothness points of ∂M and inner points of M. We then construct two covering $\mathfrak{U}_{m-1} = \cup_{x\in L_{m-1}\cap\partial M} U_x$ and $\mathfrak{V}_{m-1} = \cup_{x\in L_{m-1}\cap\partial M} V_x$. Finally, the set $L_m = \overline{M}\setminus(\cup_{k=0}^{m-1}\mathfrak{U}_k)$ consists of inner points of the manifold M only. We finish this process by choosing the covering $\mathfrak{U}_m$ for the latter set L_m. So, the covering $\cup_{k=0}^{m}\mathfrak{U}_k$ will be a covering for the whole manifold M. According to the compactness property we can take into account that this covering is finite, and the centers of balls which cover M_k are placed at M_k.

Now we will rewrite the formula (1) in the following way

$$A = \sum_{k=0}^{m}\left(\sum_{j=1}^{n_k} f_{jk}\cdot A\cdot g_{jk}\right) + T, \tag{2}$$

where the coverings and partitions of unity $\{f_{jk}\}$ and $\{g_{jk}\}$ are chosen as mentioned above. In other words the operator

$$\sum_{j=1}^{n_k} f_{jk}\cdot A\cdot g_{jk}$$

is related to some neighborhood of the sub-manifold M_k; this neighborhood is generated by covering the sub-manifold M_k by balls with centers at points $x_{jk}\in M_k$. Since $A_{x_{jk}}$ is a local representative for the operator A at point x_{jk} we can rewrite the formula (2) as follows

$$A = \sum_{k=0}^{m}\left(\sum_{j=1}^{n_k} f_{jk}\cdot A_{x_{jk}}\cdot g_{jk}\right) + T. \tag{3}$$

Let us denote $S_{\omega_j^{-1}}\hat{g}_j \equiv \tilde{g}_j and \hat{f}_j S_{\omega_j} \equiv \tilde{f}_j$. Further, we can assert that the operator

$$R = \sum_{k=0}^{m}\left(\sum_{j=1}^{n_k} g_{jk}\cdot A_{x_{jk}}^{-1} f_{jk}\right)$$

will be the regularizer for the operator A'; here $A_{x_{jk}}^{-1}$ is a regularizer for the operator $A_{x_{jk}}$.

Indeed,

$$RA = \left(\sum_{k=0}^{m}\left(\sum_{j=1}^{n_k} g_{jk}A_{x_{jk}}^{-1}f_{jk}\right)\right)\cdot A = \sum_{k=0}^{m}\sum_{j=1}^{n_k} g_{jk}\cdot A_{x_{jk}}^{-1}\cdot(A - A_{x_{jk}} + A_{x_{jk}})\cdot f_{jk} + T_1$$

$$= \sum_{k=0}^{m}\sum_{j=1}^{n_k} g_{jk}\cdot A_{x_{jk}}^{-1}\cdot (A - A_{x_{jk}})\cdot f_{jk} + \sum_{k=0}^{m}\sum_{j=1}^{n_k} f_{jk} + T_1 = I_1 + T_1 + \Theta_1,$$

$$\Theta_1 = \sum_{k=0}^{m}\sum_{j=1}^{n_k} g_{jk}\cdot A_{x_{jk}}^{-1}\cdot (A - A_{x_{jk}})\cdot f_{jk},$$

because $f_{jk}\cdot A_{x_{jk}} = A_{x_{jk}}\cdot f_{jk}$ + compact summand, and $f_{jk}\cdot g_{jk} = f_{jk}$, and

$$\sum_{k=0}^{m}\sum_{j=1}^{n_k} f_{jk} \equiv 1$$

as the partition of unity. The same property

$$AR = I_2 + T_2 + \Theta_2,$$

$$\Theta_2 = \sum_{k=0}^{m}\sum_{j=1}^{n_k} g_{jk}\cdot (A - A_{x_{jk}})\cdot A_{x_{jk}}^{-1}\cdot f_{jk},$$

is verified analogously. □

6 Piece-wise continuous operator families

Given an operator A with the symbol $\{A_x\}_{x\in\overline{M}}$ which generates a few operators in dependence on a quantity of singular manifolds; we consider this situation in the following way. We will assume additionally some smoothness properties for the symbol $\{A_x\}_{x\in\overline{M}}$.

Theorem 2. *If the symbol $\{A_x\}_{x\in\overline{M}}$ is a piece-wise continuous operator function then there are $m+1$ operators $A^{(k)}$, $k = 0, 1, \ldots, m$ such that the operator A and the operator*

$$A' = \sum_{k=0}^{m} A^{(k)} + T \tag{4}$$

have the same symbols, where the operator $A^{(k)}$ is an enveloping operator for the family $\{A_x\}_{x\in\overline{M}_k}$, $and T$ is a compact operator.

Proof. We will use the constructions from the proof of Lemma 2, namely the formula (3). We will extract the operator

$$\sum_{j=1}^{n_k} f_{jk}\cdot A_{x_{jk}}\cdot g_{jk}$$

which “serves” the sub-manifold M_k and consider it in detail. This operator is related to neighborhoods $\{U_{jk}\}$ and the partition of unity $\{f_{jk}\}$. Really, U_{jk} is the ball with the center at $x_{jk}\in M_k$ of radius $\varepsilon > 0$, and therefore f_{jk}, g_{jk}, n_k depend on ε.

According to Simonenko's ideas [1] we will construct the component $A^{(k)}$ in the following way. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence such that $\varepsilon_n > 0$ for all $n\in\mathbb{N}$, $\lim_{n\to\infty}\varepsilon_n = 0$. Given ε_n we choose coverings $\{U_{jk}\}_{j=1}^{n_k}$ and $\{V_{jk}\}_{j=1}^{n_k}$ as above with partition of unity $\{f_{jk}\}$ and corresponding functions $\{g_{jk}\}$ such that

$$|||f_{jk}\cdot (A_x - A_{x_{jk}})\cdot g_{jk}||| < \varepsilon_n, \quad \forall x\in V_{jk};$$

we remind that U_{jk}, V_{jk} are balls with centers at $x_{jk}\in\overline{M_k}$ of radius ε and 2ε. This requirement is possible according to continuity of family $\{A_x\}$ on the sub-manifold $\overline{M_k}$. Now we will introduce such a constructed operator

$$A_n = \sum_{j=1}^{n_k} f_{jk}\cdot A_{x_{jk}}\cdot g_{jk}$$

and will show that the sequence $\{A_n\}$ is a Cauchy sequence with respect to a norm $|||\cdot|||$. We have

$$A_l = \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik},$$

where the operator A_l is constructed for a given ε_l with corresponding coverings $\{u_{ik}\}_{i=1}^{l_k}$ and $\{v_{ik}\}_{j=1}^{i_k}$ with partition of unity $\{F_{ik}\}$ and corresponding functions $\{G_{ik}\}$ so that

$$|||F_{ik} \cdot (A_x - A_{y_{ik}}) \cdot G_{ik}||| < \varepsilon_l, \quad \forall x \in v_{ik};$$

here u_{ik}, v_{ik} are balls with centers at $y_{ik} \in \overline{M_k}$ of radius τ and 2τ.

We can write

$$\begin{aligned} A_n &= \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} = \sum_{i=1}^{l_k} F_{ik} \cdot \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \\ &= \sum_{i=1}^{l_k} \sum_{j=1}^{n_k} F_{ik} \cdot f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} = \sum_{i=1}^{l_k} \sum_{j=1}^{n_k} F_{ik} \cdot f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \cdot G_k + T_1, \end{aligned}$$

and the same can be done for A_l

$$\begin{aligned} A_l &= \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} = \sum_{j=1}^{n_k} f_{jk} \cdot \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} - \sum_{j=1}^{n_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} \\ &= \sum_{j=1}^{n_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} \cdot g_{jk} + T_2. \end{aligned}$$

Let us consider the difference

$$|||A_n - A_l||| = |||\sum_{j=1}^{n_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk}|||. \tag{5}$$

Obviously, summands with non-vanishing supplements to the formula (5) are those for which $U_{jk} \cap u_{ik} \neq \emptyset$. A number of such neighborhoods are finite always for arbitrary finite coverings, hence we obtain

$$\begin{aligned} |||A_n - A_l||| &\leq \sum_{j=1}^{n_k} \sum_{i=1}^{l_k} |||f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk}||| \\ &\leq \sum_{x \in U_{jk} \cap u_{ik} \neq \emptyset} |||f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_x) \cdot G_{ik} \cdot g_{jk}||| + \sum_{x \in U_{jk} \cap u_{ik} \neq \emptyset} |||f_{jk} \cdot F_{ik} \cdot (A_x - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk}||| \\ &\leq 2K \max[\varepsilon_n, \varepsilon_l], \end{aligned}$$

where K is a universal constant.

Thus, we have proved that the sequence $\{A_n\}$ is a Cauchy sequence, hence there exists $\lim_{n\to\infty} A_n = A^{(k)}$. □

Corollary 1. *The operator A has a Fredholm property if and only if all operators $A^{(k)}$, $k = 0, 1, \ldots, m$ have the same property.*

Remark 2. *The constructed operator A' generally speaking does not coincide with the initial operator A because they act in different spaces. But for some cases they may be the same.*

7 Conclusion

This paper is a general concept of my vision to the theory of pseudo-differential equations and boundary value problems on manifolds with a non-smooth boundary. The second part will be devoted to applying these abstract results to index theory for such operator families and then to concrete classes of pseudo-differential equations.

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