

## Research Article

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**A new approach for solving Bratu's problem**<https://doi.org/10.1515/dema-2019-0023>

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**Abstract:** A numerical technique for one-dimensional Bratu's problem is displayed in this work. The technique depends on Bernstein polynomial approximation. Numerical examples are exhibited to verify the efficiency and accuracy of the proposed technique. In this sequel, the obtained error was shown between the proposed technique, Chebyshev wavelets, and Legendre wavelets. The results display that this technique is accurate.

**Keywords:** Bernstein polynomials and series, numerical solution, Bezier method

**MSC:** 34B10, 34B15

**1 Introduction**

In this sequel, Liouville-Bratu-Gelfand equation is considered (see [1])

$$\begin{aligned} \Delta u + \lambda e^u &= 0, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where  $\lambda > 0$  is a physical parameter and  $\Omega \in R^N$  is a bounded domain.

Now, one may focus on the boundary value problem and initial value problem of Bratu's problem. Here, the Bratu's boundary value problem in one-dimensional planner coordinates is as follows:

$$\begin{aligned} u''(x) + \lambda e^{u(x)} &= 0, \quad 0 < x < 1, \\ u(0) = u(1) &= 0, \quad \lambda > 0, \end{aligned} \tag{2}$$

where the exact solution of this problem is (see [2])

$$u(x) = -2 \ln \left[ \frac{\cosh(0.5\theta(x-0.5))}{\cosh(0.25\theta)} \right], \tag{3}$$

$$\theta = \sqrt{2\lambda} \sinh(0.25\theta). \tag{4}$$

The problem has zeros when  $\lambda > \lambda_c$ ,  $\lambda = \lambda_c$  and  $\lambda < \lambda_c$ , separately, where the critical value  $\lambda_c$  satisfies the following equation

$$1 = \frac{1}{4} \sqrt{2\lambda_c} \cosh\left(\frac{1}{4}\theta\right).$$

The critical value  $\lambda_c$  was calculated in [2–4] and found to be  $\lambda_c = 3.513830719$ .

This solution displays a bifurcation pattern, which only characterizes nonlinear differential equations. In fact, the following one-dimensional Bratu's problem is utilized in various types of applications such as

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the fuel ignition model of the thermal combustion theory, radioactive heat transfer and nanotechnology (see [5–20]). In this paper, we have the following Bratu's boundary value problem:

$$\begin{aligned} u''(x) + \lambda e^{u(x)} &= 0, \quad 0 < x < 1, \\ u(0) = u'(0) &= 0. \end{aligned} \quad (5)$$

The way to solve a boundary or initial value problem consists of computing first the general solution to the differential equation and secondly of finding the arbitrary parameters by applying the boundary or initial conditions (see [21]). Boyd [3, 22] utilized Chebyshev polynomial expansions and the Gegenbauer polynomial expansion as base functions. Syam and Hamdan [6] exhibited the Laplace decomposition method for solving Bratu's problem. Additionally, Aksoy and Pakdemirli [23] improved a perturbation solution to Bratu's problem. Wazwaz [8] exhibited the Adomian decomposition method for solving Bratu's problem. Also, the uses of spline strategy, wavelet technique and Sinc-Galerkin method for solving the Bratu's problem have been utilized in [24, 25]. By utilizing the authors solved the Bratu's problem [26]. Additionally there are many papers and books that manage the Bezier curves. Harada, and Nakamae [27], Nürnberger and Zeilfelder [28] utilized the Bezier control points in approximating functions. Zheng et al. [29] proposed the utilization of control points of the Bezier curve for solving differential equations numerically. Also, to solve delay differential equations, the Bezier control points strategy is utilized in [30]. Some other uses of the Bezier functions are found in [31]. In the present work, we recommend a system like the one which was utilized in [31] for solving Bratu's problem. The use of Bezier curves for solving Bratu's problem is a novel idea. The approach used in this article reduces the CPU time.

This study is arranged as follows: In Section 2, the statement of the problem is presented. Error estimation is given in Section 3. Some numerical examples are provided in Section 4. Finally, Section 5 will give a brief conclusion.

## 2 Statement of the problem

The goal is to obtain an approximate solution of Eq.(2), presented in the following form:

$$u(x) \cong u_{n+1}(x) = \sum_{i=0}^n c_i B_{i,n}(x), \quad 0 \leq x \leq 1, \quad n \geq 1, \quad (6)$$

where  $\{B_{i,n}(x)\}_{i=0}^n$  denotes the Bernstein polynomials (B-polynomials) of  $n$ -th degree, as introduced by

$$\begin{aligned} B_{i,n}\left(\frac{x-x_0}{h}\right) &= \binom{n}{i} \frac{1}{h^n} (x_f - x)^{n-i} (x - x_0)^i, \\ 0 = x_0 \leq x \leq x_f &= 1, \quad i = 0, 1, \dots, n, \quad h = x_f - x_0, \end{aligned}$$

where the unknown control points are  $c_i$ ,  $i = 0, 1, \dots, n$ .

In this paper, the Bezier curve is utilized for solving the Bratu's problem. This technique is applied in [32, 33] for solving optimal control of switched systems and some linear optimal control systems with pantograph delays. The convergence of the proposed method was proven when  $n$  tends to infinity [32, 33].

By substituting  $u(x)$  in Eq. (2), we may characterize  $f_{\text{objective}}$  for  $x \in [x_0, x_f]$  as follows:

$$f_{\text{objective}} = \sum_{i=0}^n c_i^2. \quad (7)$$

Our goal is to solve the following problem to find the values  $c_r$ , for  $r = 0, 1, \dots, n$ .

$$\begin{aligned} \text{Min} \quad & f_{\text{objective}} \\ \text{such that} \quad & u''(x) + \lambda e^u = 0, \quad 0 < x < 1, \end{aligned}$$

$$u(0) = u'(0) = 0, \quad (8)$$

where the first derivative and the second derivative of the Bezier curve are as follows:

$$\begin{aligned} u'(x) &\cong n \sum_{i=0}^{n-1} B_{i,n}(x)(c_{i+1} - c_i), \\ u''(x) &\cong n(n-1) \sum_{i=0}^{n-2} B_{i,n-2}(x)(c_{i+2} - 2c_{i+1} + c_i). \end{aligned} \quad (9)$$

Now, one may utilize lagrange multipliers technique for solving (8).

### 3 Error estimation

We may define

$$(Du)(x) = u''(x) + \lambda e^{u(x)} = 0, \quad (10)$$

$$e^{u(x)} = 1 + u(x) + \frac{(u(x))^2}{2!} + \frac{(u(x))^3}{3!} + \dots, \quad (11)$$

then

$$(Du_n)(x) = u_n''(x) + \lambda e^{u_n(x)} = 0, \quad (12)$$

and by utilizing Eqs. (6), (9), we have

$$\begin{aligned} (Du_n)(x) &= n(n-1) \sum_{i=0}^{n-2} B_{i,n-2}(x)(c_{i+2} - 2c_{i+1} + c_i) \\ &+ \lambda \left( 1 + \sum_{i=0}^n c_i B_{i,n}(x) + \frac{(\sum_{i=0}^n c_i B_{i,n}(x))^2}{2!} + \frac{(\sum_{i=0}^n c_i B_{i,n}(x))^3}{3!} + \dots \right) = 0. \end{aligned}$$

Now, the following function is defined as

$$\min J = J(C) = J(c_0, c_1, \dots, c_n) = f_{\text{objective}} = \sum_{i=0}^n c_i^2 \quad (13)$$

$$\text{such that} \quad (Du_n)(x) = u_n''(x) + \lambda e^{u_n(x)} = 0, \quad u_n(0) = u_n'(0) = 0.$$

We therefore conclude that an error estimation of the proposed method is represented.

Now, the approximate solution obtained by the proposed approach can be represented in the form of

$$u_n(x) = \sum_{i=0}^n c_i B_{i,n}(x).$$

Also, an upper bound for the error estimation of the proposed numerical scheme can be established in the following theorem.

**Theorem 1.** Suppose that  $u(x) \in C^{n+1}[0, 1]$  is the exact solution of this problem,  $u_n(x) = \sum_{i=0}^n c_i B_{i,n}(x)$  is the truncated series solution with degree  $n$ . It holds that

$$\begin{aligned} \|u(x) - u_n(x)\|_\infty &\leq \frac{M}{(n+1)!} \max_{i=0,1,\dots,n} |\tilde{c}_i|, \\ M &= \max_{0 \leq x \leq 1} |u^{n+1}(x)|, \quad \tilde{c}_k = \frac{u^{(k)}(0)}{k!} - c_i. \end{aligned}$$

*Proof.* See [34]. □

Now, the following problem is considered:

$$\begin{aligned}
 L(u(x), \frac{du(x)}{dx}) &= \frac{d^2u(x)}{dx^2} + \lambda(1 + u(x) + \frac{u^2(x)}{2!} + \dots) \\
 &= F(x) \\
 u(0) &= a, \\
 u'(0) &= b.
 \end{aligned} \tag{14}$$

**Lemma 1.** *For a polynomial in Bezier form*

$$u(x) = \sum_{i=0}^{n_1} c_{i,n_1} B_{i,n_1}(x),$$

we have

$$\frac{\sum_{i=0}^{n_1} c_{i,n_1}^2}{n_1 + 1} \geq \frac{\sum_{i=0}^{n_1+1} c_{i,n_1+1}^2}{n_1 + 2} \geq \dots \geq \frac{\sum_{i=0}^{n_1+m_1} c_{i,n_1+m_1}^2}{n_1 + m_1 + 1},$$

where  $c_{i,n_1+m_1}$  are the Bezier coefficients of  $u(x)$  after it is degree-elevated to degree  $n_1 + m_1$ .

*Proof.* See [29]. □

**Theorem 2.** *If problem (14) has a unique  $C^2[0, 1]$  continuous solution  $\bar{u}$ , then the approximate solution obtained by the control-point-based method converges to the exact solution  $\bar{u}$  as the degree of the approximate solution tends to infinity.*

*Proof.* Given an arbitrary small positive number  $\epsilon > 0$ , by the Weierstrass Theorem, one can easily find polynomials  $Q_{1,N_1}(x)$  of degree  $N_1$  such that

$$\|Q_{1,N_1}(x) - \bar{u}(x)\|_\infty \leq \frac{\epsilon}{16 \times 2^n |\lambda| (\|Q_{1,N_1}(x)\|_\infty + \|\bar{u}(x)\|_\infty)},$$

and

$$\left\| \frac{d^i Q_{1,N_1}(x)}{dx^i} - \frac{d^i \bar{u}(x)}{dx^i} \right\|_\infty \leq \frac{\epsilon}{16 \times 2^n |\lambda|}, \quad i = 1, 2,$$

where  $\|\cdot\|_\infty$  stands for the  $L_\infty$ -norm over  $[0, 1]$ . In particular, we have

$$\|a - Q_{1,N_1}(0)\|_\infty \leq \frac{\epsilon}{16 \times 2^n |\lambda|}. \tag{15}$$

In general,  $Q_{1,N_1}(x)$  does not satisfies the boundary conditions. After a small perturbation with linear and constant polynomials  $\alpha x + \beta$  for  $Q_{1,N_1}(x)$ , we can obtain polynomials  $P_{1,N_1}(x) = Q_{1,N_1}(x) + (\alpha x + \beta)$  such that  $P_{1,N_1}(x)$  satisfy the following boundary conditions

$$P_{1,N_1}(0) = a, \quad \frac{dP_{1,N_1}(x)}{dx} \Big|_{x=0} = b.$$

Therefore, we have

$$Q_{1,N_1}(0) + \beta = a,$$

and

$$\frac{dQ_{1,N_1}(x)}{dx} \Big|_{x=0} + \alpha = b.$$

By using (15), we have

$$\begin{aligned}\|b - \frac{dQ_{1,N_1}(x)}{dx}|_{x=0}\|_\infty &= \|\alpha\|_\infty \leq \frac{\epsilon}{16 \times 2^n |\lambda|}, \\ \|\alpha - Q_{1,N_1}(0)\|_\infty &= \|\beta\|_\infty \leq \frac{\epsilon}{16 \times 2^n |\lambda|}.\end{aligned}$$

Now, we have

$$\begin{aligned}\|P_{1,N_1}(x) - \bar{u}(x)\|_\infty &= \|Q_{1,N_1}(x) + \alpha x + \beta - \bar{u}(x)\|_\infty \\ &\leq \|Q_{1,N_1}(x) - \bar{u}(x)\|_\infty + \|\alpha + \beta\|_\infty \leq \frac{3\epsilon}{16 \times 2^n |\lambda|} < \frac{\epsilon}{5 \times 2^n |\lambda|},\end{aligned}$$

$$\left\| \frac{d^2 P_{1,N_1}(x)}{dx^2} - \frac{d^2 \bar{u}(x)}{dx^2} \right\|_\infty = \left\| \frac{d^2 Q_{1,N_1}(x)}{dx^2} - \frac{d^2 \bar{u}(x)}{dx^2} \right\|_\infty \leq \frac{\epsilon}{16 \times 2^n |\lambda|} < \frac{\epsilon}{5 \times 2^n |\lambda|}.$$

Since  $C^2[0, 1]$  is a Banach algebra, we may have

$$\begin{aligned}\|P_{1,N_1}^2(x) - \bar{u}^2(x)\|_\infty &\leq \|P_{1,N_1}(x) - \bar{u}(x)\|_\infty \|P_{1,N_1}(x) + \bar{u}(x)\|_\infty \\ &\leq \frac{\epsilon}{16 \times 2^n |\lambda| (\|Q_{1,N_1}(x)\|_\infty + \|\bar{u}(x)\|_\infty)} \left( \|Q_{1,N_1}(x)\|_\infty + \|\bar{u}(x)\|_\infty \right) \\ &< \frac{\epsilon}{5 \times 2^n |\lambda|}.\end{aligned}$$

Now, we define

$$\begin{aligned}LP_N(x) &= L(P_{1,N_1}(x), \frac{d^2 P_{1,N_1}(x)}{dx^2}) \\ &= \frac{d^2 P_{1,N_1}(x)}{dx^2} + \lambda \left( 1 + P_{1,N_1}(x) + \frac{P_{1,N_1}^2(x)}{2!} + \frac{P_{1,N_1}^3(x)}{3!} + \dots \right) = F(x)\end{aligned}$$

for every  $x \in [0, 1]$ . Thus for  $N \geq N_1$ , we find an upper bound for the following residual:

$$\begin{aligned}\|LP_N(x) - F(x)\|_\infty &= \|L\left(P_{1,N_1}(x), \frac{d^2 P_{1,N_1}(x)}{dx^2}\right) - F(x)\|_\infty \\ &\leq \left\| \frac{d^2 P_{1,N_1}(x)}{dx^2} - \frac{d\bar{u}^2(x)}{dx^2} \right\|_\infty + |\lambda| \left( \|P_{1,N_1}(x) - \bar{u}(x)\|_\infty + \frac{1}{2} \|P_{1,N_1}^2(x) - \bar{u}^2(x)\|_\infty + \dots \right) \\ &\leq \epsilon.\end{aligned}$$

Since the residual  $R(P_N) := LP_N(x) - F(x)$  is a polynomial, we can represent it by a Bezier form. Thus we have

$$R(P_N) := \sum_{i=0}^m d_{i,m} B_{i,m}(x).$$

Then from Lemma 1 in [29], there exists an integer  $M (\geq N)$  such that when  $m > M$ , we have

$$\left| \frac{1}{m+1} \sum_{i=0}^m d_{i,m}^2 - \int_0^1 (R(P_N))^2 dx \right| < \epsilon,$$

which gives

$$\frac{1}{m+1} \sum_{i=0}^m d_{i,m}^2 < \epsilon + \int_0^1 (R(P_N))^2 dt \leq \epsilon. \quad (16)$$

Suppose  $u(x)$  is a approximated solution of (14) obtained by the control-point-based method of degree  $k$  ( $k \geq m \geq M$ ). Let

$$R \left( u(x), \frac{du(x)}{dx} \right) = L \left( u(x), \frac{du(x)}{dx} \right) - F(x) = \sum_{i=0}^k c_{i,k} B_{i,k}(x), \quad k \geq m \geq M, \quad x \in [0, 1].$$

Define the following norm for difference approximated solutions  $u(x)$  and exact solutions  $\bar{u}(x)$ :

$$\|u(x) - \bar{u}(x)\| := \int_0^1 \sum_{j=0}^2 \left| \frac{d^j u(x)}{dx^j} - \frac{d^j \bar{u}(x)}{dx^j} \right|^2 dx. \quad (17)$$

It is easy to show that:

$$\begin{aligned} \|u(x) - \bar{u}(x)\| &\leq C(|u(0) - \bar{u}(0)| + \left| \frac{du}{dx} \Big|_{x=0} - \frac{d\bar{u}}{dx} \Big|_{x=0} \right| + \|R(u(x), \frac{du(x)}{dx})\|_2^2) \\ &= C \int_0^1 \sum_{i=0}^k (c_{i,k} B_{i,k}(t))^2 dx \\ &\leq \frac{C}{k+1} \sum_{i=0}^k c_{i,k}^2. \end{aligned} \quad (18)$$

The last inequality in (18) is obtained from Lemma 1 in [29] in (18) which  $C$  is a constant positive number. Now, from Lemma 1 and (16), one can easily show that:

$$\|u(x) - \bar{u}(x)\| \leq \frac{C}{k+1} \sum_{i=0}^k c_{i,k}^2 \leq \frac{C}{k+1} \sum_{i=0}^k d_{i,k}^2 \leq \frac{C}{m+1} \sum_{i=0}^m d_{i,m}^2 \leq C(\epsilon) = \epsilon_1, \quad m \geq M, \quad (19)$$

where the last inequality in (19) comes from (16).

This completes the proof. □

## 4 Numerical applications

To illustrate the ability of this method, some examples are given. The findings reveal that the method is very successful for obtaining a solution of the Bratu's problem.

*Example 1.* Consider the following Bratu's problem:

$$u'' - 2e^u = 0, \quad 0 < x < 1, \quad (20)$$

$$u(0) = u(1) = 0. \quad (21)$$

The exact solution is  $u(x) = -2\ln(\cos(x))$ . We use the proposed method with  $n = 8$  for solving this problem. The graphs of approximate and exact solutions  $u(x)$  are plotted in Figure 1 (with  $n = 8$ ). Table 1 demonstrates the comparison between the absolute error of the proposed technique, Chebyshev wavelets (with  $k = 1$  and  $M = 6$ ) [35], and Legendre wavelets [35]. Table 2 gives the comparison of the error with different  $n$  (for the convergence of the present method). The computation takes 3 seconds of CPU time when it is performed using Maple 12 on an AMD Athelon X4 PC with 2 GB of RAM. The NLPsolve command solves (8), which involves computing the minimum of a nonlinear objective function possibly subject to linear constraints. For  $n = 3$ , we have

$$u_{bezier} = (1-x)^3 c_0 + 3x(1-x)^2 c_1 + 3x^2(1-x) c_2 + t^3 c_3.$$

In Maple with Eq. (20), we have

```

f := 1 + x + x^2/2 + x^3/6;
R2 := subs(t = u_bezier, f);
eqq1 := diff(u_bezier, [x$2]) - 2 * R2;
h := 10; a := 0; b := 1;
for i from 0 by 1 to h do
x[i] := a + ((b - a)/h) * i;
eq[i] := subs(x = x[i], eqq1);
enddo;
obj := 0; n := 3;
for i from 1 by 1 to n - 1 do
obj := obj + c[i]^2;
enddo:
cnsts := [eq[0], eq[1], eq[2], eq[3], eq[4], eq[5], eq[6], eq[7], eq[8], eq[9], eq[10]];
NLPsolve(obj, cnsts);

```

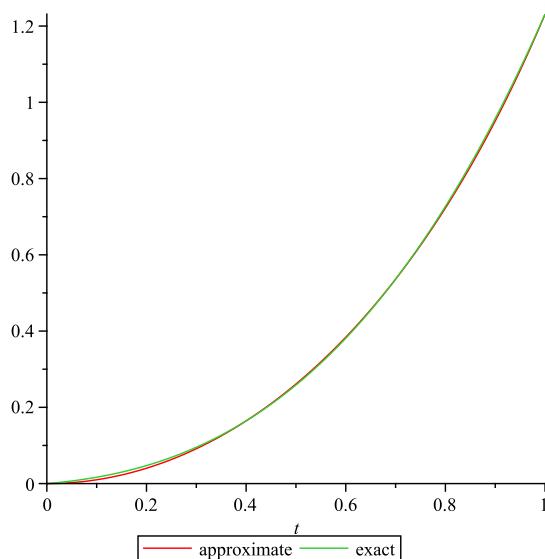
Now, by substituting into Eq. (8), we obtain the following results:

$$\begin{aligned}
c_0 &= 0, \quad c_1 = -2.71050543121376 \times 10^{-20}, \\
c_2 &= -2.71050543121376 \times 10^{-20}, \quad c_3 = -2 \ln(\cos(1)), \\
u_{bezier} &= -8.131516294 \times 10^{-20} + 8.131516294 \times 10^{-20}x^2 + 1.231252941x^3.
\end{aligned}$$

If we select  $\lambda = 3.51$  with  $n = 8$ , the following results are achieved

$$\begin{aligned}
u_{bezier} &= 4.19840927x^8 - 4.293691442x^2 + 15.95911927x^3 - 59.2204564x^4 \\
&\quad + 62.12047483x^5 - 15.87307814x^6 - 9.093378061x^7 + 6.202600671x.
\end{aligned}$$

Table 3 gives the error with  $n = 3.51$ .



**Figure 1:** The graphs of approximated and exact solutions  $u(x)$  for Example 1.

**Table 1:** Comparison of the error for proposed method, Chebyshev, and Legendre wavelets method for Example 1.

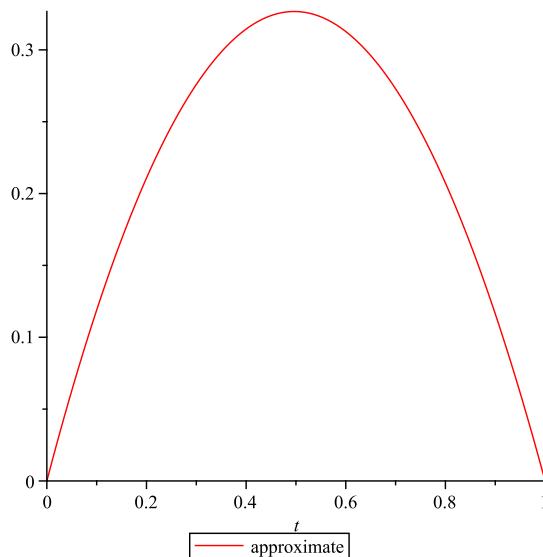
$x$	error of $u$ for the proposed method	error of Chebyshev wavelets	error of Legendre wavelets
0.1	$2.529410000 \times 10^{-12}$	0.0	$9 \times 10^{-6}$
0.2	$2.472000000 \times 10^{-12}$	$4 \times 10^{-7}$	$15 \times 10^{-6}$
0.3	0.0001918049054	$15 \times 10^{-7}$	$0.614 \times 10^{-4}$
0.4	0.0001658529272	$6 \times 10^{-7}$	$0.888 \times 10^{-4}$
0.5	$1.000000000 \times 10^{-12}$	$0.737 \times 10^{-4}$	$0.5669 \times 10^{-3}$
0.6	0.00005651941000	$0.53673 \times 10^{-2}$	$0.25577 \times 10^{-2}$
0.7	$7.000000000 \times 10^{-11}$	$4 \times 10^{-5}$	$0.92461 \times 10^{-2}$
0.8	$1.000000000 \times 10^{-10}$	$0.6398 \times 10^{-3}$	$0.286198 \times 10^{-1}$
0.9	$2.000000000 \times 10^{-10}$	$0.1528 \times 10^{-3}$	$0.791248 \times 10^{-1}$

**Table 2:** Comparison of the error for the proposed method with different  $n$  for Example 1.

$x$	error with $n = 3$	error with $n = 5$	error with $n = 7$	error with $n = 15$
0.1	0.008785458259	0.01710549764	0.01770104387	$1.231252941 \times 10^{-15}$
0.2	0.03041952266	$1.1 \times 10^{-12}$	$2.360000000 \times 10^{-12}$	$3.456963600 \times 10^{-13}$
0.3	0.05813948250	0.02678752885	0.008769247952	0.002757317303
0.4	0.08565785000	0.04294036881	$1.800000000 \times 10^{-11}$	$5.222200000 \times 10^{-11}$
0.5	0.1072618632	0.03517577600	0.01101646320	0.004087227458
0.6	0.1179797036	$2 \times 10^{-11}$	$2.000000000 \times 10^{-11}$	$5.5 \times 10^{-11}$
0.7	0.1138517564	0.05364174050	0.04855446610	0.006610826229
0.8	0.09237998820	0.1018934012	0.1210180696	$1.000000000 \times 10^{-11}$
0.9	0.05330149330	0.1038404221	0.1520070074	0.0

**Table 3:** The error for the proposed method with  $n = 8$  and  $\lambda = 3.51$  for Example 1.

$x$	error
0.1	$1.110223025 \times 10^{-16}$
0.2	$2.220446049 \times 10^{-16}$
0.3	0.003880930546
0.4	0.003442827953
0.5	$2.220446049 \times 10^{-16}$
0.6	0.0009925212416
0.7	$2.220446049 \times 10^{-16}$
0.8	0.0
0.9	$2.220446049 \times 10^{-16}$



**Figure 2:** The graph of approximated  $u(x)$  for Example 2.

*Example 2.* Presently, the following problem is considered as follows (see [36]):

$$\begin{aligned} u'' + 2e^u &= 0, \quad 0 < x < 1, \\ u(0) &= u(1) = 0, \end{aligned}$$

These results are obtained by utilizing the Bezier curve technique with  $n = 8$ . The graph of approximated solutions  $u(x)$  is plotted in Figure 2. Table 4 demonstrates the comparison between the absolute error of exact and approximated solutions.

**Table 4:** The exact value and absolute errors of the this method for Example 2.

$x$	exact of $u$	error of $u$
0.125	0.13960278219	$1.493350000 \times 10^{-11}$
0.250	0.24333656779	0.0001214358601
0.375	0.30731941062	$4.960000000 \times 10^{-11}$
0.500	0.32895242134	0.00001053263900
0.625	0.30731941062	$2.000000000 \times 10^{-11}$
0.750	0.24333656779	$1.000000000 \times 10^{-10}$
0.875	0.13960278219	0.0002111432700

## 5 Conclusions

The aim of this sequel is to improve an effective and accurate technique for solving Bratu's problem. The Bezier curve strategy is utilized to obtain the approximate solution of this problem. The method used in this article reduces the CPU time. Illustrative examples are included to show the validity of this technique. Comparing with other methods, the results of numerical examples demonstrate that this method is more accurate than some existing methods.

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