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On approximation in generalized Zygmund class

Abstract: Here, we estimate the degree of approximation of a conjugate function $\tilde{g}$ and a derived conjugate function $\tilde{g}'$, of a 2π-periodic function $g \in Z_r^\lambda, r \geq 1$, using Hausdorff means of CFS (conjugate Fourier series) and CDFS (conjugate derived Fourier series) respectively. Our main theorems generalize four previously known results. Some important corollaries are also deduced from our main theorems. We also partially review the earlier work of the authors in respect of order of the Euler-Hausdorff product method.

Keywords: generalized Zygmund ($Z_r^\lambda, r \geq 1$) class of function, error approximation, CFS (conjugate Fourier series), CDFS (conjugate derived Fourier series), generalized Minkowski's inequality, Hausdorff means

MSC: Primary: 42A10, 41A10; Secondary: 42B05, 42B08

1 Introduction

The study of error estimation of a conjugate function $\tilde{g}$ of a 2π-periodic function g belonging to $Lip(\alpha)$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$ using single summability operators such as Euler, Cesàro, Nörlund, generalized Nörlund, Hölder, Karamata, Riesz, matrix and almost matrix, has been a centre of creative study for the several researchers like Kushwaha [1], Nigam and Sharma [2], Qureshi [3–6], Lal and Nigam [7], Lal [8], Rhoades [9], Mittal et al. [10], Mishra [11] and Kranz et al. [12] in past few decades.

The study of error estimates of a conjugate function $\tilde{g}$ of a 2π-periodic function g belonging to different Lipschitz classes using the products of Cesàro, Hölder and Euler has also been of great interest among the researchers like Lal and Singh [13, 14], Dhakal [15], Mishra et al. [16], Nigam and Sharma [17–20] and Padhy et al. [21] in the recent past.

It can be noted that the matrices involved in Cesàro, Hölder, Euler and their product are Hausdorff matrices[1]. Thus, considering this view point, Singh and Srivastava [25] studied error estimates of a conjugate function $\tilde{g}$ of a 2π-periodic function $g \in W(L_r, \xi(t))$ using Hausdorff means.

Lal and Mishra [26] have obtained results on approximations of a conjugate function $\tilde{g}$ of a 2π-periodic function $g \in Lip\alpha$ and $g \in Lip(\xi(t), r)$ using Euler-Hausdorff means.

The review of the above research clearly suggests that the studies of error estimation of a conjugate function $\tilde{g}$ of 2π-periodic function $g \in Z_r^\lambda, r \geq 1$, using Hausdorff means of CFS[2] have not been initiated so far.

Note 1: The CFS is not necessarily a FS[3], for example, the series $\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$ conjugate to the Fourier series $\sum_{n=2}^{\infty} \frac{\cos(nx)}{\log n}$ is not a Fourier series (Zygmund [27, p. 186]).

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[1] The product of two Hausdorff matrices is again a Hausdorff matrix [9, 22–24].

[2] CFS denotes conjugate Fourier series (Zygmund [27]) and this abbreviation will be used throughout the paper.

[3] FS denotes Fourier series (Zygmund [27]) and this abbreviation will be used throughout the paper.

In view of the above example, a separate study in case of CFS in the present direction of work is so required.

The study of DFS[4] by single and product summability means has been of great interest among researchers.

A study of CDFS[5] was initiated for the first time by Moursund [28] in 1935. Moursund [28] established the results on Nörlund and Cesàro means of CDFS.

Thus, in the same direction, Chandra and Dikshit [29] have studied $|B|$ and $|E, q|$ summabilities of DFS and CDFS. Lal and Nigam [30] have studied K^λ-summability of DFS and Lal and Yadav [31] have studied $(N, p, q)(C, 1)$ product summability of DFS.

Very recently, Mursaleen and Alotaibi [32] have studied generalized matrix summability of CDFS.

Since the studies of error estimates of a conjugate derived function $\tilde{g}'$ of a 2π-periodic function g either in the Lipschitz space or in the Zygmund space have not been initiated so far. Therefore, in this paper, we, for the very first time, also study the error estimates of a conjugate derived function $\tilde{g}'$ of a 2π-periodic function $g \in Z_r^\lambda$, using Hausdorff means of CDFS.

A separate study in case of CDFS in the present direction is justified due to its importance in applications to science and engineering.

As the trigonometric series FS, CFS, DFS and CDFS are well known, we will not present them here. The detailed work on FS, CFS and DFS can be found in [27] and that on CDFS in [28].

We denote the jth partial sum of C F S as $\tilde{s}_j(g; y)$, which is given by

$$\tilde{s}_j(g; y) - \tilde{g}(y) = \frac{1}{2\pi} \int_0^\pi \gamma_{(y)}(l) \frac{\cos(j + 1/2)l}{\sin(l/2)} dl,$$ (1)

where $\tilde{g}$ is the conjugate function of 2π-periodic function g, and is expressed as

$$\tilde{g}(y) = -\frac{1}{2\pi} \int_0^\pi \gamma_{(y)}(l) \cot(l/2) \, dl \quad (\text{Zygmund } [27, \text{p. } 50]).$$ (2)

Now, we denote the jth partial sum of C D F S as $\tilde{s}'_j(g; y)$, which is given by

$$\tilde{s}'_j(g; y) - \tilde{g}'(y) = -\frac{2}{\pi} \int_0^\pi \frac{\rho_{(y)}(l)}{4\sin(l/2)} \left(j + \frac{1}{2}\right) \sin\left(j + \frac{1}{2}\right) l \, dl - \frac{1}{\pi} \int_0^\pi \frac{\rho_{(y)}(l)}{4\sin(l/2)} \frac{\cos(j + 1/2)l}{\tan(l/2)} dl,$$ (3)

where $\tilde{g}'$ is the conjugate derived function of 2π-periodic function g, which is expressed as

$$\tilde{g}'(y) = \frac{1}{4\pi} \int_0^\pi \text{cosec}^2 \frac{l}{2} \rho_{(y)}(l) dl \quad (\text{Moursund } [28]).$$ (4)

Note 2: For better understanding, formulas (3) and (4) are derived in the proof of Theorem 2. Here,

$$\gamma_{(y)}(l) = g(y + l) - g(y - l)$$

and

$$\rho_{(y)}(l) = g(y + l) + g(y - l) \text{ (Zygmund } [27]).$$

We mention here some standard inequalities which are used in the paper:

$$\frac{1}{\sin(l/2)} \le \frac{\pi}{l}, \qquad 0 < l \le \pi;$$ (5)

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[4] DFS denotes derived Fourier series (Zygmund [27]) and this abbreviation will be used throughout the paper.
[5] CDFC denotes conjugate derived Fourier series (Moursund [28]) and this abbreviation will be used throughout the paper.

$$\sin l \le l, \qquad l \ge 0; \tag{6}$$

$$|\sin l| \le 1, \quad |\cos l| \le 1, \qquad \text{for all } l. \tag{7}$$

In 1921, Hausdorff [33] gave the following definition:

A Hausdorff matrix $H \equiv (\Theta_{j,a})$ is an infinite lower triangular matrix with nonzero entries

$$\Theta_{j,a} = \begin{cases} \begin{pmatrix} j \\ a \end{pmatrix} \Delta^{j-a}\mu_a & 0 \le a \le j \\ 0 & a > j \end{cases};$$

where Δ is a forward difference operator defined by $\Delta\mu_j = \mu_j - \mu_{j+1}$ and $\Delta^{a+1}\mu_j = \Delta^a(\Delta\mu_j)$. If H is regular, then μ_j is known as a moment sequence. μ_j has the representation $\mu_j = \int_0^1 v^j \, d\zeta(v)$, where $\zeta(v)$ is known as the mass function. $\zeta(v)$ is continuous at $v = 0$ and it belongs to $BV[0, 1]$ such that $\zeta(0) = 0$, $\zeta(1) = 1$; and for $0 < v < 1$, $\zeta(v) = [\zeta(v + 0) + \zeta(v - 0)]/2$.

The Hausdorff means H of CFS is given by

$$\tilde{M}_j^H(g; y) := \sum_{a=0}^{j} \Theta_{j,a}\tilde{s}_a(g; y), \quad j = 0, 1, 2, \dots$$

If $\tilde{M}_j^H(g; y) \to s$ as $j \to \infty$, then the CFS is said to be summable to s by the H method [34].

Note 3: A detailed study of Hausdorff matrices can be found in [23, 35].

Particular cases of Hausdorff means:

Hausdorff means have following particular cases:

(i) Cesàro means (C, m) is a Hausdorff method with the mass function
$\zeta(v) = m \int_0^v (1 - l)^{m-1} dl$.

(ii) Hölder means (H, m) is a Hausdorff method with the mass function
$\zeta(v) = \int_0^v \frac{1}{(m-1)!} \left(\log\left(\frac{1}{l}\right)\right)^{m-1} dl$ [24, pp. 39-40].

(iii) For the mass function

$$\zeta(v) = \begin{cases} 0, & \text{if } v \in [0, b] \\ 1, & \text{if } v \in [b, 1], \end{cases}$$

where $b = 1/(1 + q)$, $q > 0$, the Hausdorff method $H \equiv (\Theta_{j,a})$ is the Euler means (E, q), $q > 0$.

Remark 1. As per the above discussion of particular cases of Hausdorff means, our main results also hold for Cesàro, Euler and Hölder means, as well as their product methods.

In view of Remark 1, the products of Euler and Cesàro are again Hausdorff matrices [9, 22–24]. Note that we need to check the order of the product of Euler and Cesàro means. Now a question arises: Is the (E, q) method generated from the Hausdorff moment sequence $\zeta_j = \frac{1}{(q+1)^j}$ a right or left shifting sequence of moments. Similarly, is the $(C, 1)$ method generated from the Hausdorff moment sequence $\zeta_j = \frac{1}{j+1}$ a right or left shifting sequence of moments.

Lal and Mishra [26] have considered the (E, q)-Hausdorff product transform and have reduced the Hausdorff matrix to a (C,1) matrix as a right shifting sequence (see corollary 8.3 of [26, p. 12]), which appears false because a $(C, 1)$ matrix is a left shifting sequence as defined in Keska [36, p. 68]. Infact, Keska [36] gave the following:

Definition 1. [36, p. 67] A moment sequence ζ_j is known as a right-shifting sequence of moments if there exists $\zeta_0^* \in \mathbb{R}$ such that the sequence

$$\zeta_0^*, \ \zeta_1^* = \zeta_0, \dots, \zeta_{k+1}^* = \zeta_k^*, \dots$$

is a moment sequence

$$\zeta_j = \int_0^1 x^j d\chi(x),$$

where χ is a real, bounded variation function defined on the interval $[0, 1]$ satisfying the conditions

$$\chi(0+) = \chi(0) = 0 \text{ and } \chi(1) = 1.$$

Example 1. [36, p. 68] The Hausdorff moment sequence $\zeta_j = \frac{1}{(q+1)^j}$, which generates the Euler method (E, q), is a right-shifting sequence of moments because $\Delta^j \zeta_0^\star = (1 + q)\Delta^j \zeta_0 \geq 0$ for $\zeta_0^\star = (1 + q)$.

Example 2. [36, p. 68] The Haudsorff moment sequence $\zeta_j = \frac{1}{j+1}$, which generates the Cesàro method C^1, is not a right-shifting sequence of moments.

Remark 2. In view of above facts, Corollary 8.3 of [26, p. 12] cannot be reduced to the method $(E, 1)(C, 1)$ unless $(C, 1)$ is considered as a left shifting sequence. Also, Corollary 8.4-8.6 of [26, p. 13] cannot be reduced to the methods $(E, q)(C, m), (E, q)(C, 1), (E, 1)(C, m)$ or $(E, 1)(C, 1)$ unless (C, m) or $(C, 1)$ is considered as a left shifting sequence.

Let

$$L^r[0, 2\pi] := \left\{ g : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} |g(y)|^r dy < \infty \right\}, \quad r \geq 1, \tag{8}$$

be the space of 2π-periodic and integrable functions.

We define $\| \cdot \|_r$ by

$$\|g\|_r := \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(y)|^r dy \right\}^{1/r} & \text{for } 1 \leq r < \infty; \\ ess \sup_{y \in (0, 2\pi)} |g(y)| & \text{for } r = \infty. \end{cases}$$

Now, we define

$$Z_r^{(\lambda)} := \left\{ g \in L^r[0, 2\pi] : r \geq 1, \sup_{l \neq 0} \frac{\|g(\cdot + l) - g(\cdot - l)\|_r}{\lambda(l)} < \infty \right\}$$

and a norm $\|g\|_r^{(\lambda)}$ on $Z_r^{(\lambda)}$ is defined by

$$\|g\|_r^{(\lambda)} := \|g\|_r + \sup_{l \neq 0} \frac{\|g(\cdot + l) - g(\cdot - l)\|_r}{\lambda(l)}, \quad r \geq 1,$$

where $\lambda : [0, 2\pi] \to \mathbb{R}$ is an arbitrary function with $\lambda(l) > 0$ for $0 < l \leq 2\pi$ and $\lim_{l \to 0+} \lambda(l) = \lambda(0) = 0$. (Zygmund [27])

The completeness of the space $Z_r^{(\lambda)}$ can be discussed by considering the completeness of $L^r, r \geq 1$. Hence the space Z_r^λ is a Banach space under the norm $\| \cdot \|_r^{(\lambda)}$.

Remark 3. Throughout the paper, λ and ν denote moduli of continuity of order two such that $\frac{\lambda(l)}{\nu(l)}$ is positive and non-decreasing in l, then

$$\|g\|_r^{(\nu)} \leq \max\left(1, \frac{\lambda(2\pi)}{\nu(2\pi)}\right) \|g\|_r^{(\lambda)} < \infty.$$

Thus,

$$Z_r^{(\lambda)} \subset Z_r^{(\nu)} \subset L^r, r \geq 1.$$

Remark 4. Any modulus of continuity of first order $\lambda = \lambda(g, \cdot)$ satisfies the following conditions:

(i) $\lambda(0) = 0$;
(ii) the function λ is continuous on $[0, +\infty)$;
(iii) the function λ is non-decreasing on $[0, +\infty)$;
(iv) the function λ is semiadditive; i. e. the inequality $\lambda(l_1 + l_2) \leq \lambda(l_1) + \lambda(l_2)$ holds for any $l_1 \geq 0$ and $l_2 \geq 0$.

Conversely, if a function λ satisfies conditions (i)-(iv), then it is the first order modulus of continuity of the function $g(y) = \lambda(|y|)$. Moreover, it can be easily seen that λ is the second order modulus of continuity of the function $g(y) = \frac{\lambda(|y|)}{2}$. If a function λ satisfies conditions (i)-(iii) and the function $\frac{\lambda(l)}{l}$ is non-increasing on $(0, +\infty)$, then the semiadditivity condition (iv) also holds, and so λ is the modulus of continuity of the first and second order for some continuous functions.

The modulus of continuity of second order satisfies conditions (i)-(iii) and a further condition, given as follows:

(v) the inequality $\lambda(nl) \leq n^2 \lambda(l)$ holds for any $l \geq 0$ and $n \in N$.

Geit [37] constructed a wide class of the functions that are second-order moduli of continuity of 2π-periodic functions. It can be easily shown that condition (v) for non-negative functions follows from the following condition:

(vi) the function $\frac{\lambda(l)}{l^2}$ is non-increasing on $(0, +\infty)$.

Note 4: Readers may refer to the paper of Konyagin [38] in support of Remark 4. Readers may also refer to the paper of Weiss and Zygmund [39] which dealt with conditions on the second order modulus of smoothness sufficient to force absolute continuity of a function. The technique employed in [39] is nearly identical to that of [40].

Remark 5.

(i) If we take $r \to \infty$ in $Z_r^{(\lambda)}$, then $Z_r^{(\lambda)}$ reduces to $Z^{(\lambda)}$.
(ii) If we take $\lambda(l) = l^\alpha$ in $Z^{(\lambda)}$, then $Z^{(\lambda)}$ reduces to Z_α.
(iii) If we take $\lambda(l) = l^\alpha$ in $Z_r^{(\lambda)}$, then $Z_r^{(\lambda)}$ reduces to $Z_{\alpha,r}$.
(iv) If we take $r \to \infty$ in $Z_{\alpha,r}$, then $Z_{\alpha,r}$ reduces to Z_α.
(v) Let $0 \leq \alpha_1 < \alpha_2 < 1$. If $\lambda(l) = l^{\alpha_2}$ and $v(l) = l^{\alpha_1}$, then $\lambda(l)/v(l)$ is increasing, but $\lambda(l)/lv(l)$ is a decreasing function of l.

We write

$$f(v, l) = Re\left[\sum_{a=0}^{j} \binom{j}{a} v^a (1-v)^{j-a} e^{i(a+1/2)l}\right];$$

$$f_1(v, l) = Im\left[\sum_{a=0}^{j} \binom{j}{a} v^a (1-v)^{j-a} e^{i(a+1/2)l}\right];$$

$$f_2(v, l) = Re\left[\sum_{a=0}^{j} \binom{j}{a} v^a (1-v)^{j-a} e^{ial}\right];$$

$$\tilde{Y}_j^H(l) = \frac{1}{2\pi \sin(l/2)} \int_0^1 f(v, l) d\zeta(v);$$

$$\tilde{Y}_{1j}'^H(l) = -\frac{a}{2\pi \sin(l/2)} \int_0^1 f_1(v, l) d\zeta(v);$$

$$\tilde{Y}_{2j}'^H(l) = -\frac{1}{4\pi \sin^2(l/2)} \int_0^1 f_2(v, l) d\zeta(v).$$

2 The Main Theorems

Theorem 1. Error approximation of a conjugate function $\tilde{g}$ of 2π-periodic function $g \in Z_r^{(\lambda)}, r \geq 1$, using $H \equiv (\Theta_{j,a})$ of CFS is given by

$$\|\tilde{M}_j^H(g, \cdot) - \tilde{g}(\cdot)\|_r^{(v)} = O\left(\frac{1}{j+1}\int_{\frac{1}{j+1}}^{\pi}\frac{\lambda(l)}{l^2\,v(l)}dl\right),$$

where $\lambda(l)$ and $v(l)$ are as defined in Remark 3, provided

$$\int_0^\eta \frac{\lambda(l)}{l v(l)}dl = O\left(\frac{\lambda(\eta)}{v(\eta)}\right), \quad 0 < \eta < \pi. \tag{9}$$

Theorem 2. Error approximation of a conjugate derived conjugate function $\tilde{g}'$ of a 2π-periodic function $g \in Z_r^{(\lambda)}, r \geq 1$, using $H \equiv (\Theta_{j,a})$ of CDFS is given by

$$\|\tilde{M'}_j^H(g, \cdot) - \tilde{g}'(\cdot)\|_r^v = O\left(\frac{1}{j+1}\int_{\frac{1}{j+1}}^{\pi}\frac{(l+1)\,\lambda(l)}{l^3\,v(l)}dl\right),$$

where $\lambda(l)$ and $v(l)$ are as defined in Remark 3, provided

$$\int_0^\eta \frac{\lambda(l)}{l^2 v(l)}dl = O\left(\frac{1}{\eta}\frac{\lambda(\eta)}{v(\eta)}\right), \quad 0 < \eta < \pi. \tag{10}$$

3 Lemmas

We use the following lemmas:

Lemma 1. For $l \in (0, \frac{1}{j+1}], v \in [0, 1]$,

$$\left|\int_0^1 f(v, l)d\zeta(v)\right| = O(1).$$

Lemma 2. For $l \in [\frac{1}{j+1}, \pi], v \in [0, 1]$,

$$\left|\int_0^1 f(v, l)d\zeta(v)\right| = O\left(\frac{1}{l(j+1)}\right).$$

Lemmas 1 and 2 are proved using similar arguments to those used in the proofs of Singh and Srivastava ([25], Lemma 1).

Lemma 3.

$$\left|\tilde{Y}_j^H(l)\right| = \begin{cases} O(\frac{1}{l}), & l \in (0, 1/(j+1)]; \\ O(1/l^2(j+1)), & l \in [1/(j+1), \pi]. \end{cases}$$

Proof. Using (5) and above Lemmas 1 and 2, Lemma 3 can be obtained directly.

Lemma 4. [41, p. 3]: For $0 \le v \le 1$ and $0 \le l \le \pi$,

$$\left| \int_0^1 f(v, l) d\zeta(v) \right| = \begin{cases} O((j+1)l), & l \in [0, \frac{1}{j+1}]; \\ O((j+1)^{-1}l^{-1}), & l \in [\frac{1}{j+1}, \pi]. \end{cases}$$

Lemma 5.

$$\left| \tilde{Y}_{1j}'^H(l) \right| = \begin{cases} O(j+1), & l \in (0, \frac{1}{j+1}]; \\ O((j+1)^{-1}l^{-2}), & l \in [\frac{1}{j+1}, \pi]. \end{cases}$$

Proof. Using (5) and above Lemma 4, Lemma 5 can be obtained directly.

Lemma 6. For $l \in (0, \frac{1}{j+1}]$, $v \in [0, 1]$,

$$\left| \int_0^1 f_2(v, l) d\zeta(v) \right| = O(1).$$

Proof. We can write

$$f_2(v, l) = (1-v)^j Re\left[\sum_{a=0}^j {}_jC_a \left(\frac{ve^{il}}{1-v} \right)^a \right] = Re\left[(1-v+ve^{il})^j \right], \quad \text{continuous for } v \in [0, 1].$$

If $K = \sup_{v \in [0,1]} \frac{d\zeta(v)}{dv}$, then

$$\int_0^1 f_2(v, l) d\zeta(v) \le K \int_0^1 f_2(v, l) dv$$

$$= O(1) \int_0^1 Re\left[(1-v+ve^{il})^j \right] dv$$

$$= O(1) Re\left[\frac{(1-v+ve^{il})^{j+1}}{(e^{il}-1)(j+1)} \right]_{v=0}^1$$

$$= O(1) Re\left[\frac{e^{il(j+1)} - 1}{(e^{il}-1)(j+1)} \right]$$

$$= O(1) Re\left[\frac{\cos(j+1)l + i\sin(j+1)l - 1}{(\cos l + i\sin l - 1)(j+1)} \right]$$

$$= O\left(\frac{1}{j+1} \right) Re\left[\frac{\cos(j+1)l - 1}{\cos l - 1 - i\sin l\{(\cos l - 1)^2 + \sin^2 l\}} + \frac{i\sin(j+1)l\{(\cos l - 1) - i\sin l\}}{\{(\cos l - 1)^2 + \sin^2 l\}} \right]$$

$$= O\left(\frac{1}{j+1} \right) \left[\frac{\cos(j+1)l\cos l - \cos(j+1)l - \cos l + 1}{2(1-\cos l)} + \frac{\sin(j+1)l\sin l}{2(1-\cos l)} \right].$$

Using trigonometric properties, we get

$$\int_0^1 f_2(v, l) d\zeta(v) = O\left(\frac{1}{j+1} \right) \left[\frac{\cos jl - \cos(j+1)l - \cos l + 1}{2(1-\cos l)} \right]$$

$$= O\left(\frac{1}{j+1} \right) \left[\frac{2\sin(j+1/2)l\sin(l/2) + 2\sin^2(l/2)}{4\sin^2(l/2)} \right]$$

$$= O\left[\frac{\sin(j+1/2)l + \sin(l/2)}{2(j+1)\sin(l/2)} \right]$$

$$= O\left[\frac{\sin((j+1)l/2)\cos(jl/2)}{(j+1)\sin(l/2)}\right]. \tag{11}$$

Using (5), (6) and (7), we get Lemma 6.

Lemma 7. For $l \in [\frac{1}{j+1}, \pi]$, $v \in [0, 1]$,

$$\left|\int_0^1 f_2(v, l)d\zeta(v)\right| = O\left(\frac{1}{l(j+1)}\right).$$

Proof. Using (5) and (7), we get Lemma 7.

Lemma 8.

$$\left|\tilde{Y}'^H_{2j}(l)\right| = \begin{cases} O(l^{-2}), & l \in (0, \frac{1}{j+1}]; \\ O((j+1)^{-1}l^{-3}), & l \in [\frac{1}{j+1}, \pi]. \end{cases}$$

Proof. Using (5), Lemmas 6 and 7, we get Lemma 8.

4 Proof of the Main Theorems

4.1 Proof of Theorem 1

Using (1),

$$\tilde{M}^H_j(g, y) - \tilde{g}(y) = \sum_{a=0}^j \Theta_{j,a}\{\tilde{s}_a(g; y) - \tilde{g}(y)\} = \frac{1}{2\pi}\int_0^\pi \left(\frac{\gamma_{(y)}(l)}{\sin(l/2)}\sum_{a=0}^j \binom{j}{a}\Delta^{j-a}\mu_a \cos(a+1/2)l\right) dl$$

$$= \frac{1}{2\pi}\int_0^\pi \left[\left(\frac{\gamma_{(y)}(l)}{\sin(l/2)}\sum_{a=0}^j \binom{j}{a}\right)\left(\int_0^1 v^a(1-v)^{j-a}d\zeta(v)Re\{e^{i(a+1/2)l}\}\right)\right] dl$$

$$= \frac{1}{2\pi}\int_0^\pi \left[\left(\frac{\gamma_{(y)}(l)}{\sin(l/2)}\right)\left(\int_0^1 Re\left[\sum_{a=0}^j \binom{j}{a}v^a(1-v)^{j-a}e^{i(a+1/2)l}\right] d\zeta(v)\right)\right] dl$$

$$= \frac{1}{2\pi}\int_0^\pi \left(\frac{\gamma_{(y)}(l)}{\sin(l/2)}\int_0^1 f(v, l) d\zeta(v)\right) dl$$

$$= \int_0^\pi \gamma_{(y)}(l)\tilde{Y}^H_j(l)dl.$$

Let

$$k_j(y) := \tilde{M}^H_j(g, y) - \tilde{g}(y) = \int_0^\pi \gamma_{(y)}(l)\tilde{Y}^H_j(l)dl. \tag{12}$$

Then

$$k_j(y + z) - k_j(y - z) = \int_0^\pi [\gamma_{(y+z)}(l) - \gamma_{(y-z)}(l)]\tilde{Y}^H_j(l)dl.$$

Using the generalized Minkowski inequality (Chui [42]),

$$\left\|k_j(\cdot + z) - k_j(\cdot - z)\right\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|k_j(y + z) - k_j(y - z)\right|^r dy\right\}^{1/r}$$

$$\leq \int_0^\pi \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|\{\gamma_{(y+z)}(l) - \gamma_{(y-z)}(l)\}\right|^r dy \right\}^{\frac{1}{r}} |\tilde{Y}_j^H(l)| dl$$

$$= \int_0^{1/(j+1)} \left\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\right\|_r |\tilde{Y}_j^H(l)| dl + \int_{1/(j+1)}^\pi \left\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\right\|_r |\tilde{Y}_j^H(l)| dl$$

$$:= I_1 + I_2 \quad \text{(say)}. \tag{13}$$

Further,

$$|\gamma_{(y+z)}(l) - \gamma_{(y-z)}(l)| \leq |g(y+z+l) - g(y+z-l)| + |g(y-z+l) - g(y-z-l)|.$$

Applying Minkowski's inequality (Zygmund [27]),

$$\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\|_r \leq \|g(\cdot+z+l) - g(\cdot+z-l)\|_r + \|g(\cdot-z+l) - g(\cdot-z-l)\|_r = O(\lambda(l)). \tag{14}$$

Also

$$\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\|_r \leq \|g(\cdot+l+z) - g(\cdot+l-z)\|_r + \|g(\cdot-l+z) - g(\cdot-l-z)\|_r = O(\lambda(z)). \tag{15}$$

Since $v(l)$ is positive and non-decreasing and for $l \leq z$, we have

$$\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\|_r = O(\lambda(l)) = O\left(v(l)\left(\frac{\lambda(l)}{v(l)}\right)\right) = O\left(v(z)\left(\frac{\lambda(l)}{v(l)}\right)\right).$$

Since $\lambda(l)/v(l)$ is positive and increasing, and if $l \geq z$, then $\lambda(l)/v(l) \geq \lambda(z)/v(z)$ and we have

$$\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\|_r = O(\lambda(z)) = O\left(v(z)\left(\frac{\lambda(l)}{v(l)}\right)\right). \tag{16}$$

Using (9), (16), Lemma 3 and $\lambda(l)/v(l)$ being monotonic with l, we have

$$I_1 = \int_0^{1/(j+1)} \left\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\right\|_r |\tilde{Y}_j^H(l)| dl = O\left(\int_0^{1/(j+1)} v(z)\frac{\lambda(l)}{v(l)}\frac{dl}{l}\right) = O\left(v(z)\frac{\lambda(1/(j+1))}{v(1/(j+1))}\right). \tag{17}$$

Also, using Lemma 3 and (16),

$$I_2 = \int_{1/(j+1)}^\pi \left\|\gamma_{(\cdot+z)}(l) - \gamma_{(\cdot-z)}(l)\right\|_r |\tilde{Y}_j^H(l)| dl = O\left(\int_{1/(j+1)}^\pi v(z)\frac{\lambda(l)}{v(l)}\frac{dl}{l^2(j+1)}\right) = O\left(\frac{v(z)}{j+1}\int_{1/(j+1)}^\pi \frac{\lambda(l)}{l^2 v(l)} dl\right). \tag{18}$$

By (12), (17) and (18),

$$\left\|k_j(\cdot+z) - k_j(\cdot-z)\right\|_r = O\left(v(z)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})}\right) + O\left(\frac{v(z)}{j+1}\int_{1/(j+1)}^\pi \frac{\lambda(l)}{l^2 v(l)} dl\right).$$

Thus,

$$\sup_{z \neq 0}\frac{\left\|k_j(\cdot+z) - k_j(\cdot-z)\right\|_r}{v(z)} = O\left(\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})}\right) + O\left(\frac{1}{j+1}\int_{1/(j+1)}^\pi \frac{\lambda(l)}{l^2 v(l)} dl\right). \tag{19}$$

Since $g \in Z_r^{(\lambda)}$, then

$$\|\gamma_{(\cdot)}(l)\|_r = \|g(\cdot+l) - g(\cdot-l)\|_r = O(\lambda(l)). \tag{20}$$

From (9), when $v(l) = 1$, we get

$$\int_0^\eta \frac{\lambda(l)}{l} dl = O(\lambda(\eta)) \quad 0 < \eta < \pi. \tag{21}$$

Using (20), (21) and Lemma 3,

$$\|k_j(\cdot)\|_r \le \left(\int_0^{1/(j+1)} + \int_{1/(j+1)}^{\pi} \right) \|\gamma_{(\cdot)}(l)\|_r |\tilde{Y}_j^H(l)| dl$$

$$= \int_0^{1/(j+1)} \|\gamma_{(\cdot)}(l)\|_r |\tilde{Y}_j^H(l)| dl + \int_{1/(j+1)}^{\pi} \|\gamma_{(\cdot)}(l)\|_r |\tilde{Y}_j^H(l)| dl$$

$$= O\left(\int_0^{1/(j+1)} \frac{\lambda(l)}{l} dl \right) + O\left(\int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2(j+1)} dl \right)$$

$$= O\left(\lambda(1/(j+1)) \right) + O\left(\int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2(j+1)} dl \right)$$

$$= O\left(\lambda(1/(j+1)) \right) + O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2} dl \right). \tag{22}$$

Now, by (19) and (22), we have

$$\|k_j(\cdot)\|_r^{(v)} = \|k_j(\cdot)\|_r + \sup_{z \neq 0} \frac{\left\| k_j(\cdot + z) - k_j(\cdot - z) \right\|_r}{v(z)}$$

$$= O\left(\lambda((j+1)^{-1}) \right) + O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2} dl \right) + O\left(\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right) + O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right)$$

$$= \sum_{i=1}^{4} O(S_i) \text{ (say)}.$$

Now, we write S_1, S_2 and S_3 in terms of S_4. Since, we have $\lambda(l) = \frac{\lambda(l)}{v(l)} \cdot v(l) \le \frac{\lambda(l)}{v(l)} v(\pi)$, $0 < l \le \pi$, then

$$S_1 = O(S_3) \tag{23}$$

and

$$S_2 = \left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} v(l) dl \right) \le \left(\frac{v(\pi)}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right) = O(S_4). \tag{24}$$

Since $\lambda(l)/v(l)$ is positive and increasing where $\lambda(l)$ and $v(l)$ are as in Remark 3, then

$$S_4 = \frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \ge \frac{\lambda(1/(j+1))}{v(1/(j+1))} \frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{dl}{l^2}$$

$$= \frac{\lambda(1/(j+1))}{v(1/(j+1))} \frac{1}{(j+1)} \left[-\frac{1}{l} \right]_{1/(j+1)}^{\pi} = \frac{\lambda(1/(j+1))}{v(1/(j+1))} \frac{1}{(j+1)} \left[j+1 - \frac{1}{\pi} \right] \ge \frac{\lambda(1/(j+1))}{v(1/(j+1))} \frac{j}{(j+1)} \ge \frac{\lambda(1/(j+1))}{2v(1/(j+1))}.$$

Then,

$$S_3 = O(S_4). \tag{25}$$

From (23), (24) and (25),

$$\|k_j(\cdot)\|_r^{(v)} = \sum_{i=1}^{4} O(S_i) = O(S_4) = O\left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right).$$

Hence

$$\|\tilde{M}_j^H(g, \cdot) - \tilde{g}(\cdot)\|_r = O\left(\frac{\int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl}{j+1} \right). \tag{26}$$

4.2 Proof of Theorem 2

The jth partial sum of CDFS is given by

$$\tilde{s}'_j(g;y) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(u) \frac{\partial}{\partial y} \left(\sum_{m=1}^{a} \sin m(u-y) \right) du$$

$$= -\frac{1}{\pi} \int_{0}^{\pi} \frac{d}{dl} \left(\frac{\cos(l/2) - \cos(a+1/2)l}{2\sin(l/2)} \right) \{ g(y+l) + g(y-l) \} dl$$

$$= -\frac{1}{\pi} \int_{0}^{\pi} \frac{d}{dl} \left(\frac{1}{2} \cos \frac{l}{2} \right) \rho_{(y)}(l) dl + \frac{1}{\pi} \int_{0}^{\pi} \frac{d}{dl} \left(\frac{\cos(a+1/2)l}{2\sin(l/2)} \right) \rho_{(y)}(l) dl$$

$$= \frac{1}{4\pi} \int_{0}^{\pi} \operatorname{cosec}^2 \frac{l}{2} \rho_{(y)}(l) dl$$

$$- \frac{2}{\pi} \int_{0}^{\pi} \frac{\rho_{(y)}(l)}{4\sin(l/2)} \left(a + \frac{1}{2} \right) \sin \left(a + \frac{1}{2} \right) l \, dl - \frac{1}{\pi} \int_{0}^{\pi} \frac{\rho_{(y)}(l)}{4\sin(l/2)} \frac{\cos(a+1/2)l}{\tan(l/2)} dl.$$

Writing $\varphi_{(y)}(l) = \frac{\rho_{(y)}(l)}{4\sin(l/2)}$, we have

$$\tilde{s}'_j(g;y) - \frac{1}{4\pi} \int_{0}^{\pi} \operatorname{cosec}^2 \frac{l}{2} \rho_{(y)}(l) dl \tilde{g}'(y) = -\frac{2}{\pi} \left(a + \frac{1}{2} \right) \int_{0}^{\pi} \varphi_{(y)}(l) \sin \left(a + \frac{1}{2} \right) l \, dl$$

$$- \frac{1}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \frac{\cos(a+1/2)l \cos(l/2)}{\sin(l/2)} dl$$

$$\tilde{s}'_j(g;y) - \tilde{g}'(y) = \left[-\frac{2a}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \sin \left(a + \frac{1}{2} \right) l \, dl - \frac{1}{\pi} \int_{0}^{\pi} \left(\varphi_{(y)}(l) \frac{\sin(a+1/2)l \sin(l/2) + \cos(a+1/2)l \cos(l/2)}{\sin(l/2)} \right) dl \right]$$

$$= -\frac{2a}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \sin \left(a + \frac{1}{2} \right) l \, dl - \frac{1}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \frac{\cos al}{\sin(l/2)} dl.$$

Further,

$$\tilde{M}'^{H}_{j}(y) - \tilde{g}'(y) = \sum_{a=0}^{j} \Theta_{j,a} \left\{ \tilde{s}'_a(g;y) - \tilde{g}'(y) \right\}$$

$$= -\frac{2a}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \sum_{a=0}^{j} \Theta_{j,a} \sin(a+\frac{1}{2})l \, dl - \frac{1}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \sum_{a=0}^{j} \Theta_{j,a} \frac{\cos al}{\sin(l/2)} dl$$

$$= -\frac{2a}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \sum_{a=0}^{j} \binom{j}{a} \Delta^{j-a} \mu_a \sin(a+\frac{1}{2})l \, dl - \frac{1}{\pi} \int_{0}^{\pi} \varphi_{(y)}(l) \sum_{a=0}^{j} \binom{j}{a} \Delta^{j-a} \mu_a \frac{\cos al}{\sin(l/2)} dl$$

$$= -\frac{2a}{\pi} \int_{0}^{\pi} \left(\varphi_{(y)}(l) \sum_{a=0}^{j} \binom{j}{a} \right) \left(\int_{0}^{1} v^a (1-v)^{j-a} d\zeta(v) Im[e^{i(a+1/2)l}] \right) dl$$

$$- \frac{1}{\pi} \int_{0}^{\pi} \left(\frac{\varphi_{(y)}(l)}{\sin(l/2)} \sum_{a=0}^{j} \binom{j}{a} \right) \left(\int_{0}^{1} v^a (1-v)^{j-a} d\zeta(v) Re[e^{ial}] \right) dl$$

$$= -\frac{2a}{\pi} \int_0^\pi \left(\frac{\rho_{(y)}(l)}{4\sin(l/2)} \right) \left(\int_0^1 Im \left[\sum_{a=0}^j \binom{j}{a} v^a(1-v)^{j-a} e^{i(a+1/2)l} \right] d\zeta(v) \right) dl$$

$$- \frac{1}{\pi} \int_0^\pi \left(\frac{\rho_{(y)}(l)}{4\sin^2(l/2)} \right) \left(\int_0^1 Re \left[\sum_{a=0}^j \binom{j}{a} v^a(1-v)^{j-a} e^{ial} \right] d\zeta(v) \right) dl$$

$$= -\frac{a}{2\pi} \int_0^\pi \left(\frac{\rho_{(y)}(l)}{\sin(l/2)} \int_0^1 f_1(v,l) d\zeta(v) \right) dl - \frac{1}{4\pi} \int_0^\pi \left(\frac{\rho_{(y)}(l)}{\sin^2(l/2)} \int_0^1 f_2(v,l) d\zeta(v) \right) dl$$

$$= \int_0^\pi \rho_{(y)}(l) \left[\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l) \right] dl$$

Let

$$k'_j(y) := \tilde{M}'^H_j(y) - \tilde{g}'(y) = \int_0^\pi \rho_{(y)}(l) \left[\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l) \right] dl. \tag{27}$$

Then

$$k'_j(y+z) + k'_j(y-z) = \int_0^\pi [\rho_{(y+z)}(l) + \rho_{(y-z)}(l)] \left[\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l) \right] dl.$$

Using the generalized Minkowski inequality (Chui [42]),

$$\left\| k'_j(\cdot + z) + k'_j(\cdot - z) \right\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| k'_j(y+z) + k'_j(y-z) \right|^r dy \right\}^{1/r}$$

$$\le \int_0^\pi \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \{\rho_{(y+z)}(l) + \rho_{(y-z)}(l)\} \right|^r dy \right\}^{1/r} |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl$$

$$= \int_0^{1/(j+1)} \left\| \rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l) \right\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl$$

$$+ \int_{1/(j+1)}^\pi \left\| \rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l) \right\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl$$

$$:= I'_1 + I'_2 \text{ (say).} \tag{28}$$

Further,

$$|\rho_{(y+z)}(l) + \rho_{(y-z)}(l)| \le |g(y+z+l) + g(y+z-l)| + |g(y-z+l) + g(y-z-l)|$$

Applying Minkowski's inequality (Zygmund [27]),

$$\|\rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l)\|_r \le \|g(\cdot+z+l) + g(\cdot+z-l)\|_r \|g(\cdot-z+l) + g(\cdot-z-l)\|_r = O(\lambda(l)). \tag{29}$$

Also

$$\|\rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l)\|_r \le \|g(\cdot+l+z) + g(\cdot+l-z)\|_r + \|g(\cdot-l+z) + g(\cdot-l-z)\|_r = O(\lambda(z)). \tag{30}$$

Since $v(l)$ is positive and increasing and for $l \le z$, we have

$$\|\rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l)\|_r = O(\lambda(l)) = O\left(v(l) \left(\frac{\lambda(l)}{v(l)} \right) \right) = O\left(v(z) \left(\frac{\lambda(l)}{v(l)} \right) \right).$$

Since $\lambda(l)/v(l)$ is positive and increasing, and if $l \geq z$, then $\lambda(l)/v(l) \geq \lambda(z)/v(z)$ and we have

$$\left\| \rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l) \right\|_r = O(\lambda(z)) = O\left(v(z) \left(\frac{\lambda(l)}{v(l)} \right) \right). \tag{31}$$

Using (10), (31), Lemma 5, Lemma 8 and $\lambda(l)/v(l)$ being monotonic with l, we have

$$I'_1 = \int_0^{1/(j+1)} \left\| \rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l) \right\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl$$

$$= O\left(\int_0^{1/(j+1)} v(z)\frac{\lambda(l)}{v(l)}(j+1)dl + \int_0^{1/(j+1)} v(z)\frac{\lambda(l)}{v(l)} \frac{dl}{l^2} \right)$$

$$= O\left((j+1)v(z)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \int_0^{1/(j+1)} dl \right) + O\left(v(z)(j+1)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right)$$

$$= O\left(v(z)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} + v(z)(j+1)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right). \tag{32}$$

Also, using (31), Lemma 5 and Lemma 8, we have

$$I'_2 = \int_{1/(j+1)}^{\pi} \left\| \rho_{(\cdot+z)}(l) + \rho_{(\cdot-z)}(l) \right\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl$$

$$= O\left(\int_{1/(j+1)}^{\pi} v(z)\frac{\lambda(l)}{v(l)} \frac{dl}{l^2(j+1)} \right) + O\left(\int_{1/(j+1)}^{\pi} v(z)\frac{\lambda(l)}{v(l)} \frac{dl}{l^3(j+1)} \right)$$

$$= O\left(\frac{v(z)}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl + \frac{v(z)}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3 v(l)} dl \right). \tag{33}$$

By (29), (33) and (34),

$$\left\| k'_j(\cdot+z) + k'_j(\cdot-z) \right\|_r = O\left(v(z)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} + v(z)(j+1)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right)$$

$$+ O\left(\frac{v(z)}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl + \frac{v(z)}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3 v(l)} dl \right).$$

Thus,

$$\sup_{z \neq 0} \frac{\left\| k'_j(\cdot+z) + k'_j(\cdot-z) \right\|_r}{v(z)} = O\left(\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} + (j+1)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right)$$

$$+ O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right) + O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3 v(l)} dl \right). \tag{34}$$

Since $g \in Z_r^{(\lambda)}$, then

$$\left\| \rho_{(\cdot)}(l) \right\|_r = \left\| g(\cdot+l) + g(\cdot-l) \right\|_r = O(\lambda(l)). \tag{35}$$

Also from (10), when $v(l) = 1$, we get

$$\int_0^{\eta} \frac{\lambda(l)}{l^2} dl = O\left(\frac{\lambda(\eta)}{\eta} \right), \quad 0 < \eta < \pi. \tag{36}$$

Using (35), (36), Lemma 5 and Lemma 8, we have

$$\|k'_j(\cdot)\|_r \le \left(\int_0^{1/(j+1)} + \int_{1/(j+1)}^{\pi} \right) \|\rho_{(\cdot)}(l)\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl \tag{37}$$

$$= \int_0^{1/(j+1)} \|\rho_{(\cdot)}(l)\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl + \int_{1/(j+1)}^{\pi} \|\rho_{(\cdot)}(l)\|_r |\tilde{Y}'^H_{1j}(l) + \tilde{Y}'^H_{2j}(l)| dl$$

$$= O\left((j+1) \int_0^{1/(j+1)} \lambda(l) dl + \int_0^{1/(j+1)} \frac{\lambda(l)}{l^2} dl \right) + O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2} dl + \frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3} dl \right)$$

$$= O\left(\lambda((j+1)^{-1}) + (j+1)\lambda((j+1)^{-1}) \right) + O\left(\frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2} dl + \frac{1}{j+1} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3} dl \right). \tag{38}$$

Now, we have

$$\|k'_j(\cdot)\|_r^{(v)} = \|k'_j(\cdot)\|_r + \sup_{z \ne 0} \frac{\|k'_j(\cdot + z) + k'_j(\cdot - z)\|_r}{v(z)}.$$

Using (33) and (36),

$$\|k'_j(\cdot)\|_r^{(v)} = O\left(\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right) + O\left((j+1)\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \right) + O\left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right)$$

$$+ O\left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3 v(l)} dl \right) + O\left(\lambda((j+1)^{-1}) \right) + O\left((j+1)\lambda((j+1)^{-1}) \right)$$

$$+ O\left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2} dl \right) + O\left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3} dl \right)$$

$$= \sum_{i=1}^{8} O(S'_i) \quad \text{(say)}.$$

Now, we write S'_i, $1 \le i \le 6$ in terms of S'_7 and S'_8. Since we have $\lambda(l) = \frac{\lambda(l)}{v(l)} \cdot v(l) \le v(\pi)\frac{\lambda(l)}{v(l)}$, $l \in (0, \pi]$, then

$$S'_1 = O(S'_5) \quad \text{and} \quad S'_2 = S'_6 \tag{39}$$

and

$$S'_3 = \left(\frac{1}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} v(l) dl \right) \le \left(\frac{v(\pi)}{(j+1)} \int_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right) = O(S'_7). \tag{40}$$

Similarly,

$$S'_4 = S'_8. \tag{41}$$

Again, since $\frac{\lambda(l)}{v(l)} \le (j+1)\frac{\lambda(l)}{v(l)}$, then

$$S'_5 = O(S'_6). \tag{42}$$

Since $\lambda(l)/v(l)$ is positive and increasing where $\lambda(l)$ and $v(l)$ are as in Remark 3, then

$$S'_8 = \frac{1}{(j+1)} \int\limits_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3 v(l)} dl \geq \frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \frac{1}{(j+1)} \int\limits_{1/(j+1)}^{\pi} \frac{dl}{l^3}$$

$$= \frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} \frac{1}{(j+1)} \left[-\frac{1}{2l^2} \right]_{1/(j+1)}^{\pi} = \frac{\lambda((j+1)^{-1})}{2v((j+1)^{-1})} \frac{1}{(j+1)} \left[(j+1)^2 - \frac{1}{\pi^2} \right]$$

$$= \frac{\lambda((j+1)^{-1})}{2v((j+1)^{-1})} (j+1) \left[\frac{(j+1)^2 - \frac{1}{\pi^2}}{(j+1)^2} \right] \geq \frac{1}{2} \left(1 - \frac{1}{4\pi^2} \right) (j+1) \frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})}.$$

Then,

$$S'_6 = O(S'_8). \tag{43}$$

From (37) to (42), we have

$$\|k'_j(\cdot)\|_r^{(v)} = \sum_{i=1}^{8} O(S'_i) = O(S'_7) + O(S'_8) = O \left(\frac{1}{j+1} \int\limits_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl + \frac{1}{j+1} \int\limits_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^3 v(l)} dl \right).$$

Hence,

$$\|\tilde{M}'^H_j(g, \cdot) - \tilde{g}'(\cdot)\|_r^v = O \left(\frac{1}{j+1} \int\limits_{1/(j+1)}^{\pi} \frac{(l+1)}{l^3} \frac{\lambda(l)}{v(l)} dl \right).$$

5 Corollaries

Corollary 1. Error approximation of a conjugate function $\tilde{g}$ of a 2π-periodic function $g \in Z_r^{(\lambda)}, where r \geq 1$, using $H \equiv (\Theta_{j,a})$ of CFS is given by

$$\|\tilde{M}^H_j(g, \cdot) - \tilde{g}(\cdot)\|_r^{(v)} = O \left[\frac{\lambda\left(\frac{1}{j+1}\right)}{v\left(\frac{1}{j+1}\right)} (\log(j+1)) \right],$$

where $\frac{\lambda(l)}{lv(l)}$ is non-increasing.
Proof. From Theorem 1, we have

$$\|\tilde{M}^H_j(g, \cdot) - \tilde{g}(\cdot)\|_r = O \left(\frac{1}{j+1} \int\limits_{1/(j+1)}^{\pi} \frac{\lambda(l)}{l^2 v(l)} dl \right).$$

Since $\lambda(l)/lv(l)$ is non-increasing function, then

$$\|\tilde{M}^H_j(g, \cdot) - \tilde{g}(\cdot)\|_r = O \left(\frac{1}{(j+1)} (j+1) \frac{\lambda(1/(j+1))}{v(1/(j+1))} \int\limits_{1/(j+1)}^{\pi} \frac{dl}{l} \right) = O \left(\frac{\lambda(1/(j+1))}{v(1/(j+1))} (\log(j+1)) \right).$$

Corollary 2. Error approximation of a conjugate derived function $\tilde{g}'$ of a 2π-periodic function $g \in Z_r^{(\lambda)}, where r \geq$ 1, using $H \equiv (\Theta_{j,a})$ of CDFS is given by

$$\|\tilde{M}'^H_j(g, \cdot) - \tilde{g}'(\cdot)\|_r^v = O \left(\frac{\lambda((j+1)^{-1})}{v((j+1)^{-1})} ((j+1) + \log(j+1)) \right),$$

where $\frac{\lambda(l)}{l\nu(l)}$ is non-increasing.

Proof. From Theorem 2, we have

$$\|\tilde{M}'^{H}_{j}(g,\cdot) - \tilde{g}'(\cdot)\|^{\nu}_{r} = O\left(\frac{1}{j+1}\int\limits_{1/(j+1)}^{\pi}\frac{(l+1)}{l^3}\frac{\lambda(l)}{\nu(l)}dl\right).$$

Since $\lambda(l)/l\nu(l)$ is non-increasing function, then

$$\|\tilde{M}'^{H}_{j}(g,\cdot) - \tilde{g}'(\cdot)\|^{\nu}_{r} = O\left(\frac{1}{(j+1)}(j+1)\frac{\lambda((j+1)^{-1})}{\nu((j+1)^{-1})}\int\limits_{1/(j+1)}^{\pi}\frac{dl}{l}\right) + O\left(\frac{1}{(j+1)}(j+1)\frac{\lambda((j+1)^{-1})}{\nu((j+1)^{-1})}\int\limits_{1/(j+1)}^{\pi}\frac{dl}{l^2}\right)$$

$$= O\left(\frac{\lambda((j+1)^{-1})}{\nu((j+1)^{-1})}((j+1) + \log(j+1))\right).$$

Corollary 3. Error approximation of a conjugate function $\tilde{g}$ of a 2π-periodic $g \in Z_{\alpha,r}$, *where* $r \geq 1$, using $H \equiv (\Theta_{j,a})$ of CFS is given by

$$\|\tilde{M}^{H}_{j}(g,\cdot) - \tilde{g}(\cdot)\|^{(\nu)}_{r} = \begin{cases} O\left((j+1)^{\alpha_1-\alpha_2}\right), & 0 \leq \alpha_1 < \alpha_2 < 1; \\ O\left((j+1)^{-1}\{\log(j+1)\}\right), & \alpha_1 = 0, \alpha_2 = 1. \end{cases}$$

Proof. Putting $\lambda(l) = l^{\alpha_2}$ and $\nu(l) = l^{\alpha_1}$ in Theorems 1 and Corollary 1, the result follows.

6 Particular cases

Some particular cases of our main results are as follows:

(i) If $\lambda(l) = \xi(l)$ and $\nu(l) = u(l)$ in Theorem 1, then Theorem 4 of Rhoades [43] becomes a particular case of our result.

(ii) If $\lambda(l) = l^{\alpha_2}$ and $\nu(l) = l^{\alpha_1}$ in Theorem 1 and also if $r \to \infty$ and $\alpha_1 = 0$, then Theorem 6 of Rhoades [43] becomes a particular case of our result.

(iii) Since the Euler matrix is an $H \equiv (\Theta_{j,a})$ matrix, and in view of remark 1, if $\lambda(l) = \xi(l)$ and $\nu(l) = u(l)$ in Theorem 1 and if the order of the product (E, q)-Hausdorff is taken as Hausdorff-(E, q) (Example 1 [36, p. 68]), Theorem 3.3 of Lal and Mishra [26] becomes a particular case of our Theorem 1.

(iv) If we take $\lambda(l) = l^{\alpha_2}$ and $\nu(l) = l^{\alpha_1}$ in Theorem 1, where $r \to \infty$ and $\alpha_1 = 0$ and if the order of the product (E, q)-Hausdorff is taken as Hausdorff-(E, q) (Example 1 [36, p. 68]), Theorem 3.1 of Lal and Mishra [26] becomes a particular case of our Theorem 1.

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