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Further results on the neutrix composition of distributions involving the delta function and the function $\cosh_+^{-1}(x^{1/r} + 1)$

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Abstract: The neutrix composition $F(f(x))$ of a distribution $F(x)$ and a locally summable function $f(x)$ is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The function $\cosh_+^{-1}(x + 1)$ is defined by

$$\cosh_+^{-1}(x + 1) = H(x) \cosh^{-1}(|x| + 1),$$

where $H(x)$ denotes Heaviside's function. It is then proved that the neutrix composition $\delta^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)]$ exists and

$$\delta^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] = \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} [(j-2i+1)^s - (j-2i-1)^s] \delta^{(k)}(x),$$

for $r, s = 1, 2, \dots$. Further results are also proved.

Our results improve, extend and generalize the main theorem of [Fisher B., Al-Sirehy F., Some results on the neutrix composition of distributions involving the delta function and the function $\cosh_+^{-1}(x + 1)$, Appl. Math. Sci. (Ruse), 2014, 8(153), 7629–7640].

Keywords: distribution, Dirac-delta function, composition of distributions, neutrix, neutrix limit

MSC: Primary 46F10; Secondary 33B10

1 Introduction

In the following, we let \mathcal{D} be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$. We let \mathcal{D}' be the space of distributions defined on \mathcal{D} and let $\mathcal{D}'[a, b]$ be the space of distributions defined on $\mathcal{D}[a, b]$.

A sequence of functions $\{f_n\}$ is called regular [1] if

- (i) $\{f_n\}$ is infinitely differentiable for all n ;
- (ii) the sequence $\{\langle f_n, \varphi \rangle\}$ converges to a limit $\langle f, \varphi \rangle$ for every $\varphi \in \mathcal{D}$;
- (iii) $\langle f, \varphi \rangle$ is continuous in φ in the sense that $\lim_{n \rightarrow \infty} \langle f, \varphi_n \rangle = 0$ for every sequence $\varphi_n \rightarrow 0$ in \mathcal{D} .

There are several methods for constructing a sequence of regular functions which converges to $\delta(x)$. For example, let $\rho(x)$ be a function in \mathcal{D} having the following properties:

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- (i) $\rho(x) = 0$ for $|x| \geq 1$;
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in \mathcal{D}' and $F_n(x) = \langle F(x-t), \delta_n(x) \rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

Antosik [2] defined the composition $g(f(x))$ as the limit of the sequence $\{g_n(f_n)\}$ providing the limit exists. By this definition he defined the compositions $\sqrt{\delta} = 0$, $\sqrt{\delta^2 + 1} = 1 + \delta$, $\sin \delta = 0$, $\cos \delta = 1$ etc. Using the definition of Antosik, it is not possible to define the compositions for many pairs of distributions. Fisher gave a general principle, by using the neutrix calculus developed by Van der Corput [3], for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions, [4]. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from divergent integral is referred to as the Hadamard finite part, see [5]. In fact his method can be regarded as a particular application of the neutrix calculus.

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|} \left[\frac{1}{|f'(x)|} \frac{d}{dx} \right]^r \delta(x - x_0),$$

for $r = 0, 1, 2, \dots$, see [6].

In order to give a more general definition for the composition of distributions, the following definition for the neutrix composition of distributions was given in [4] and was originally called the composition of distributions.

Definition 1. Let F be a distribution in \mathcal{D}' and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\text{N-lim}_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and N is the neutrix, see [3], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n : \quad \lambda > 0, r = 1, 2, \dots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

It was proved in [7] that if the composition $F(f(x))$ exists by Gel'fand and Shilov's definition, then it exists by Definition 1 and the two are equivalent.

The following theorems were proved in [8], [9] and [10] respectively.

Theorem 1. The neutrix composition $\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)]^{1/r}$ exists and

$$\begin{aligned} & \delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)]^{1/r} \\ &= \sum_{k=0}^{s-1} \sum_{j=0}^k \sum_{i=0}^j \frac{(-1)^{rs+r-j-1} r}{2^{j+2}} \binom{k}{j} \binom{j}{i} \times \frac{[(j-2i+1)^s - (j-2i-1)^s](rs+r-1)!}{k!s!} \delta^{(k)}(x), \end{aligned}$$

for $r, s = 1, 2, \dots$.

In particular,

$$\delta^{(2r-1)}[\cosh_+^{-1}(x+1)]^{1/r} = \frac{(2r-1)!}{2} \delta(x),$$

for $r = 1, 2, \dots$.

Theorem 2. The neutrix composition $\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}]$ exists and

$$\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}] = \sum_{k=0}^{M-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{i+k} r c_{s,k,i}}{2^{k+1} k!} \delta^{(k)}(x)$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$, where M is the smallest positive integer greater than $(s - r^2 + 1)/r$ and

$$c_{s,k,i} = \begin{cases} \frac{[(k-2i+1)^p + (k-2i-1)^p](-1)^s s!}{2p!}, & p = \frac{s-r+1}{r} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the neutrix composition $\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}]$ exists and

$$\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}] = 0$$

for $s = 0, 1, 2, \dots, r-1$ and $r = 2, 3, \dots$.

Theorem 3. The neutrix composition $\delta^{(s)}[\cosh^{-1}(x_+^{1/r} + 1)]$ exists and

$$\delta^{(s)}[\cosh^{-1}(x_+^{1/r} + 1)] = \sum_{k=0}^{M-1} \sum_{i=0}^{kr+r} \binom{k}{i} \frac{(-1)^k r c_{r,s,k}}{(kr+r)k!} \delta^{(k)}(x)$$

for $s = M-1, M, M+1, \dots$ and $r = 1, 2, \dots$, where

$$c_{r,s,k} = \sum_{j=0}^i \binom{i}{j} \frac{(-1)^{kr+r+s-i} (2j-i)^{s+1}}{2^{i+1}},$$

M is the smallest integer for which $s - 2r + 1 < 2Mr$ and $r \leq s/(2M+2)$.

We now need the following lemma, which can be easily proved by induction:

Lemma 1.

$$\int_{-1}^1 t^i \rho^{(s)}(t) dt = \begin{cases} 0, & 0 \leq i < s, \\ (-1)^s s!, & i = s \end{cases}$$

and

$$\int_0^1 t^s \rho^{(s)}(t) dt = \frac{1}{2} (-1)^s s!$$

for $s = 0, 1, 2, \dots$.

In the following we define the functions $\cosh_+^{-1}(x+1)$ and $\cosh_-^{-1}(|x|+1)$ by

$$\begin{aligned} \cosh_+^{-1}(x+1) &= H(x) \cosh^{-1}(|x|+1), \\ \cosh_-^{-1}(x+1) &= H(-x) \cosh^{-1}(|x|+1). \end{aligned}$$

It follows that

$$\cosh^{-1}(|x|+1) = \cosh_+^{-1}(x+1) + \cosh_-^{-1}(x+1).$$

2 Main results

We now prove the following improvement of Theorem 3:

Theorem 4. *The neutrix composition $\delta^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)]$ exists and*

$$\begin{aligned} \delta^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] &= \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} \\ &\quad \times [(j-2i+1)^s - (j-2i-1)^s] \delta^{(k)}(x), \end{aligned} \quad (2.1)$$

for $r, s = 1, 2, \dots$.

In particular,

$$\delta[\cosh_+^{-1}(x^{1/r} + 1)] = 0 \quad (2.2)$$

and

$$\delta'[\cosh_+^{-1}(x^{1/r} + 1)] = \sum_{j=0}^{r-1} \sum_{i=0}^j \frac{(-1)^{r-j} r}{2^{j+1}} \binom{r-1}{j} \binom{j}{i} \delta(x), \quad (2.3)$$

for $r = 1, 2, \dots$.

Proof. It is clear that $\delta^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] = 0$ on any interval not containing the origin and so we only need prove equation (2.1) on the interval $[-1, 1]$. To do this, we will first of all need to evaluate

$$\begin{aligned} \int_{-1}^1 x^k \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] dx &= \int_0^1 x^k \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] dx + \int_{-1}^0 x^k \delta_n^{(s)}(0) dx \\ &= n^{s+1} \int_0^1 x^k \rho^{(s)}[n(\cosh_+^{-1}(x^{1/r} + 1))] dx + n^{s+1} \int_{-1}^0 x^k \rho^{(s)}(0) dx \\ &= I_1 + I_2. \end{aligned} \quad (2.4)$$

Making the substitution $t = n[\cosh^{-1}(x^{1/r} + 1)]$ or $x = [\cosh(t/n) - 1]^r$, we have

$$\begin{aligned} I_1 &= rn^s \int_0^1 [\cosh(t/n) - 1]^{kr+r-1} \sinh(t/n) \rho^{(s)}(t) dt \\ &= rn^s \sum_{j=0}^{kr+r-1} (-1)^{kr+r-j-1} \binom{kr+r-1}{j} \cosh^j(t/n) \sinh(t/n) \rho^{(s)}(t) dt \\ &= rn^s \sum_{j=0}^{kr+r-1} \frac{(-1)^{kr+r-j-1}}{2^{j+1}} \binom{kr+r-1}{j} \int_0^1 (e^{t/n} + e^{-t/n})^j (e^{t/n} - e^{-t/n}) \rho^{(s)}(t) dt \\ &= rn^s \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r-j-1}}{2^{j+1}} \binom{kr+r-1}{j} \binom{j}{i} \int_0^1 (e^{(j-2i+1)t/n} - e^{(j-2i-1)t/n}) \rho^{(s)}(t) dt \end{aligned}$$

and it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} I_1 &= \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r-j-1} r}{2^{j+1}} \binom{kr+r-1}{j} \binom{j}{i} \frac{(j-2i+1)^s - (j-2i-1)^s}{s!} \int_0^1 t^s \rho^{(s)}(t) dt \\ &= \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} [(j-2i+1)^s - (j-2i-1)^s]. \end{aligned} \quad (2.5)$$

It is obvious that

$$\lim_{n \rightarrow \infty} I_2 = 0 \quad (2.6)$$

and it now follows from equations (2.4), (2.5) and (2.6) that

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 x^k \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] dx &= \\ &= \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} [(j-2i+1)^s - (j-2i-1)^s] \end{aligned} \quad (2.7)$$

for $k = 0, 1, 2, \dots$.

Next, when $k = s$, we note that

$$[\cosh(t/n) - 1]^{rs+r-1} \sinh(t/n) = O(n^{-2rs-2r+1})$$

and it follows that

$$\begin{aligned} |I_1| &\leq rn^s \int_0^1 |[\cosh(t/n) - 1]^{rs+r-1} \sinh(t/n) \rho^{(s)}(t)| dt \\ &= O(n^{-2rs-2r+s+1}). \end{aligned}$$

Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n \rightarrow \infty} \int_0^1 x^s \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] \psi(x) dx = 0, \quad (2.8)$$

for $s = 0, 1, 2, \dots$.

Further,

$$\int_{-1}^0 x^s \delta_n^{(s)}(0) \psi(x) dx = n^{s+1} \int_{-1}^0 x^s \rho^{(s)}(0) \psi(x) dx$$

and it follows that

$$\text{N-lim}_{n \rightarrow \infty} \int_{-1}^0 x^s \delta_n^{(s)}(0) \psi(x) dx = 0. \quad (2.9)$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-,]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{s-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^s}{s!} \varphi^{(s)}(\xi x),$$

where $0 < \xi < 1$. Then with $s \geq 1$, we have

$$\begin{aligned} \langle \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)], \varphi(x) \rangle &= \int_{-1}^1 \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] \varphi(x) dx \\ &= \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 x^k \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] dx \\ &\quad + \int_0^1 \frac{x^s}{s!} \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)] \varphi^{(s)}(\xi x) dx + \int_{-1}^0 \frac{x^s}{s!} \delta_n^{(rs+r-1)}(0) \varphi^{(s)}(\xi x) dx. \end{aligned}$$

It now follows from equations (2.7), (2.8) and (2.9) that

$$\text{N-lim}_{n \rightarrow \infty} \langle \delta_n^{(s)}[\cosh_+^{-1}(x^{1/r} + 1)], \varphi(x) \rangle =$$

$$\begin{aligned}
&= \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-k-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} \\
&\quad \times [(j-2i+1)^s - (j-2i-1)^s] \varphi^{(k)}(0) \\
&= \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} [(j-2i+1)^s - (j-2i-1)^s] \langle \delta(x), \varphi(x) \rangle
\end{aligned}$$

and equation (2.1) follows.

Equations (2.2) and (2.3) follow immediately from equation (2.1).

Replacing x by $-x$ in Theorem 4, we get

Corollary 4.1 *The neutrix composition $\delta^{(s)}[\cosh_{-}^{-1}(|x|^{1/r} + 1)]$ exists and*

$$\begin{aligned}
\delta^{(s)}[\cosh_{-}^{-1}(|x|^{1/r} + 1)] &= \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+k+r+s-j-1} r}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} \\
&\quad \times [(j-2i+1)^s - (j-2i-1)^s] \delta^{(k)}(x),
\end{aligned} \tag{2.10}$$

for $r, s = 1, 2, \dots$

In particular,

$$\delta[\cosh_{-}^{-1}(|x|^{1/r} + 1)] = 0 \tag{2.11}$$

and

$$\delta'[\cosh_{-}^{-1}(|x|^{1/r} + 1)] = \sum_{j=0}^{r-1} \sum_{i=0}^j \frac{(-1)^{r-j} r}{2^{j+1}} \binom{r-1}{j} \binom{j}{i} \delta(x), \tag{2.12}$$

for $r = 1, 2, \dots$

Corollary 4.2 *The neutrix composition $\delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)]$ exists and*

$$\begin{aligned}
\delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] &= \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^j \frac{(-1)^{kr+r+s-j-1} r[1 - (-1)^k]}{2^{j+2}} \\
&\quad \times \binom{kr+r-1}{j} \binom{j}{i} [(j-2i+1)^s - (j-2i-1)^s] \delta^{(k)}(x),
\end{aligned} \tag{2.13}$$

for $r, s = 1, 2, \dots$

In particular,

$$\delta[\cosh^{-1}(|x|^{1/r} + 1)] = 0 \tag{2.14}$$

and

$$\delta'[\cosh^{-1}(|x|^{1/r} + 1)] = \sum_{j=0}^{r-1} \sum_{i=0}^j \frac{(-1)^{r-j} r[1 - (-1)^k]}{2^{j+1}} \binom{r-1}{j} \binom{j}{i} \delta(x), \tag{2.15}$$

for $r = 1, 2, \dots$

Proof. Equation (2.13) follows on noting that

$$\delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] = \delta^{(s)}[\cosh_{+}^{-1}(x^{1/r} + 1)] + \delta^{(s)}[\cosh_{-}^{-1}(|x|^{1/r} + 1)].$$

Equations (2.14) and (2.15) follow immediately.

For further related results, see [11], [7], [12], [13] and [14].

References

- [1] Temple G., The theory of generalized functions, In: *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences*, 1955, 228(1173), 175–190
- [2] Antosik P., Composition of distributions, *Technical Report no.9*, University of Wisconsin, Milwaukee, 1988–1989, 1–30
- [3] Van der Corput J. G., Introduction to the neutrix calculus, *J. Analyse Math.*, 1959, 7, 291–398
- [4] Fisher B., On defining the change of variable in distributions, *Rostock. Math. Kolloq.*, 1985, 28, 75–86
- [5] Jones, D. S., Hadamard's Finite Part., *Mathematical Methods in the Applied Sciences*, 1996, 19(13), 1017–1052
- [6] Gel'fand I. M., Shilov G. E., *Generalized functions*, Vol 1, Academic Press, London and New York, 1964
- [7] Fisher B., The composition and neutrix composition of distributions, In: Tas K., Tenreiro Machado J. A., Baleanu D. (Eds.), *Mathematical Methods in Engineering*, Springer, The Netherlands, 2007, 59–69
- [8] Fisher B., Al-Sirehy F., Some results on the neutrix composition of distributions involving the delta function and the function $\cosh_+^{-1}(x + 1)$, *Appl. Math. Sci. (Ruse)*, 2014, 8(153), 7629–7640
- [9] Fisher B., Kılıçman A., On the composition and neutrix composition of the delta function and powers of the inverse hyperbolic sine function, *Integral Transforms Spec. Funct.*, 2010, 21(12), 935–944
- [10] Fisher B., Özçağ E., Al-Sirehy F., On the composition and neutrix composition of the delta function and the function $\cosh^{-1}(|x|^{1/r} + 1)$, *Internat. J. Anal. Appl.*, 2017, 13(2), 161–169
- [11] Fisher B., The delta function and the composition of distributions, *Dem. Math.*, 2002, 35(1), 117–123
- [12] Fisher B., Kılıçman A., On the composition and neutrix composition of the delta function with the hyperbolic tangent and its inverse functions, *Journal Appl. Math.*, 2011, Article ID 846736, <http://dx.doi.org/10.1155/2011/846736>
- [13] Fisher B., Özçağ E., Some results on the neutrix composition of the delta function, *Filomat*, 2012, 26(6), 1247–1256
- [14] Kraiweeradechachai T., Orankitjaroen S., Fisher B., Özçağ E., Further results on the neutrix composition of the delta function, *East-West J. Math.*, 2009, 11(2), 151–164