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The analytical solution of Van der Pol and Lienard differential equations within conformable fractional operator by retarded integral inequalities

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Abstract: In this study we introduced and tested retarded conformable fractional integral inequalities utilizing non-integer order derivatives and integrals. In line with this purpose, we used the Katugampola type conformable fractional calculus which has several practical properties. These inequalities generalize some famous integral inequalities which provide explicit bounds on unknown functions. The results provided here had been implemented to the global existence of solutions to the conformable fractional differential equations with time delay.

Keywords: global existence, retarded integral inequalities, conformable fractional Van der Pol differential equation, conformable fractional Liènard differential equation

MSC: 26A42, 26A51, 26D15, 26A33

1 Introduction

Being important tools in the analysis of differential equations, integral equations and integro-differential equations, a number of generalizations of Gronwall inequality and their utilizations have greatly attracted the interests of several mathematicians. In 1995, Pachpatte [1] provided a generalization of an interesting integral inequality thanks to Ou-Iang [2]. Later on Lipovan [3] proposed a retarded type of Pachpatte and Ou-Iang integral inequalities. Then Sun [4] made a generalization for results given by Lipovan.

A number of new definitions have been proposed in academia to provide a better method for fractional calculus such as Riemann- Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud and Riesz among others. [5, 6].

Now, all these definitions satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all the definitions. However, all definitions do not provide some properties such as Product Rule (Leibniz Rule), Quotient Rule, Chain Rule, Rolls Theorem and Mean Value Theorem. In addition most of the fractional derivatives (except Caputo-type derivatives), do not satisfy D^{α} (f) (1) = if α is not a natural number.

Recently a new local, limit-based definition of a conformable derivative has been introduced in [7]. Among the others, we refer the readers to [8]-[12] and references therein. This new idea was quickly generalized by Katugampola [13], whose definition forms the basis for this work and is referred to here as the Katugampola derivative (D^{α} will henceforth be referred to the Katugampola derivative). This definition has several practical properties which are summarized below

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In this study, we presented a retarded conformable fractional integral inequalities using the Katugampola conformable fractional calculus. The remainder of this study is organized as follows: In Section 2, the related definitions and theorems are reviewed. In Section 3, the general versions of retarded integral inequalities utilizing conformable fractional calculus are obtained while some conclusions and remarks are discussed in Section 4.

2 Fundamental facts

In this section, we summarize the Katugampola conformable derivatives for $\alpha \in (0,1]$ and $\eta \in [0,\infty)$ given by

$$D^{\alpha}(\mathrm{f})(\eta) = \lim_{\varepsilon\to 0}\frac{\mathrm{f}\left(te^{\varepsilon\eta^{-\alpha}}\right)-\mathrm{f}(\eta)}{\varepsilon},\ D^{\alpha}(\mathrm{f})(0) = \lim_{\eta\to 0} D^{\alpha}(\mathrm{f})(\eta), \tag{2.1}$$

provided the limits exist (for details see, [13]). If f is fully differentiable at η, then

$$D^{\alpha}(\mathrm{f})(\eta) = \eta^{1-\alpha}\frac{df}{dt}(\eta). \tag{2.2}$$

Theorem 1. *Let $\alpha \in (0,1]$ and f,g be α–differentiable at a point $\eta > 0$. Then*
i. $D^{\alpha}(a\mathrm{f}+b\mathrm{g}) = aD^{\alpha}(\mathrm{f}) + bD^{\alpha}(\mathrm{g})$, *for all* $a,b\in\mathbb{R}$,
ii. $D^{\alpha}(\lambda) = 0$, *for all constant functions* $\mathrm{f}(\eta)=\lambda$,
iii. $D^{\alpha}(\mathrm{fg}) = \mathrm{f}D^{\alpha}(\mathrm{g}) + \mathrm{g}D^{\alpha}(\mathrm{f})$,
iv. $D^{\alpha}\left(\frac{\mathrm{f}}{\mathrm{g}}\right) = \frac{\mathrm{g}D^{\alpha}(\mathrm{f})-\mathrm{f}D^{\alpha}(\mathrm{g})}{\mathrm{g}^2}$,
v. $D^{\alpha}\left(\eta^{n}\right) = nt^{n-\alpha}$ *for all* $n\in\mathbb{R}$, *vi.* $D^{\alpha}(\mathrm{f}\circ\mathrm{g})(\eta) = \mathrm{f}'(\mathrm{g}(\eta))D^{\alpha}(\mathrm{g})(\eta)$ *for* f *is differentiable at* $\mathrm{g}(\eta)$.

Definition 1. *Let $\alpha \in (0,1]$ and $0 \le a < b$. A map $\mathrm{f}:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b \mathrm{f}(x)\,d_{\alpha}x := \int_a^b \mathrm{f}(x)\,x^{\alpha-1}dx,$$

exists and is finite. Whole α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$.

We will also take advantage of the following significant consequences, which can be derived from the results above.

Lemma 1. *Let the conformable differential operator D^{α} be given as in (2.1), where $\alpha\in(0,1]$ and $\eta\ge 0$, and assume the maps* f *and* g *are α-differentiable as needed. Then*
i. $D^{\alpha}(\ln\eta) = \eta^{-\alpha}$ *for* $\eta > 0$,
ii. $D^{\alpha}\left[\int_a^{\eta}\mathrm{f}(\eta,\mathrm{s})\,d_{\alpha}\mathrm{s}\right] = \mathrm{f}(\eta,\eta) + \int_a^{\eta}D^{\alpha}[\mathrm{f}(\eta,\mathrm{s})]\,d_{\alpha}\mathrm{s}$,
iii. $\int_a^b \mathrm{f}(x)D^{\alpha}(\mathrm{g})(x)\,d_{\alpha}x = fg|_a^b - \int_a^b \mathrm{g}(x)D^{\alpha}(\mathrm{f})(x)\,d_{\alpha}x$.

In this manuscript, by using the Katugampola type conformable fractional calculus, we introduced retarded conformable fractional integrals inequalities. The results provided here can be implemented to the global existence of solutions to the conformable fractional differential equations with time delay.

3 Main findings and cumulative results

Theorem 2. *Let* $v, \mathrm{f}, k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, *assume that* φ *and* k *are non-decreasing with* $\varphi(\eta) \leq \eta$ *for* $\eta \geq 0$ *and* $k(v) > 0$. *If* $v \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies*

$$v(\eta) \leq \mathrm{m} + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s}, \quad \eta \geq 0, \tag{3.1}$$

where m *is a non-negative constant, then*

$$v(\eta) \leq \mathcal{G}^{-1}\left(\mathcal{G}(\mathrm{m}) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s}\right), \tag{3.2}$$

where $\mathcal{G}^{-1}$ *is inverse of* $\mathcal{G}$ *such that*

$$\mathcal{G}(\xi) =: \int_1^{\xi} \frac{1}{k(\mathrm{s})} d_\alpha \mathrm{s}, \quad \xi \geq 0,$$

and

$$\mathcal{G}(\mathrm{m}) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s} \in Dom(\mathcal{G}^{-1}), \quad \forall \eta \geq 0.$$

Proof. Let us first assume that m > 0. Define the non-decreasing positive function $\mathrm{z}(\eta)$ by the right-hand side of (3.1). Then $v(\eta) \leq \mathrm{z}(\eta)$ and $\mathrm{z}(0) = \mathrm{m}$, and

$$D^\alpha \mathrm{z}(\eta) = \mathrm{f}(\varphi(\eta))k(v(\varphi(\eta)))D^\alpha \varphi(\eta) \leq \mathrm{f}(\varphi(\eta))k(\mathrm{z}(\varphi(\eta)))D^\alpha \varphi(\eta) \leq \mathrm{f}(\varphi(\eta))k(\mathrm{z}(\eta))D^\alpha \varphi(\eta)$$

as $\varphi(\eta) \leq \eta$. Then from the definition of $\mathcal{G}$ we have

$$\mathcal{G}(\mathrm{z}(\eta)) = \int_1^{\mathrm{z}(\eta)} \frac{1}{k(\mathrm{s})} d_\alpha \mathrm{s}.$$

Then by taking the α–th order of conformable derivative of $\mathcal{G}(\mathrm{z}(\eta))$, we have

$$D^\alpha \mathcal{G}(\mathrm{z}(\eta)) = \frac{1}{k(\mathrm{z}(\eta))} D^\alpha \mathrm{z}(\eta) \leq \mathrm{f}(\varphi(\eta))D^\alpha \varphi(\eta).$$

Then by taking integration from 0 to $\varphi(\eta)$, we get

$$\mathcal{G}(\mathrm{z}(\eta)) \leq \mathcal{G}(\mathrm{m}) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s}.$$

Then we obtain

$$\mathrm{z}(\eta) \leq \mathcal{G}^{-1}\left(\mathcal{G}(\mathrm{m}) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s}\right).$$

Since $v(\eta) \leq \mathrm{z}(\eta)$ and $\mathcal{G}^{-1}$ is increasing on $\mathrm{Dom}(\mathcal{G}^{-1})$, we get the desired inequality, that is

$$v(\eta) \leq \mathcal{G}^{-1}\left(\mathcal{G}(\mathrm{m}) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s}\right).$$

□

Remark 1. *If we take $\varphi(\eta) = \eta$, then the inequality given by Theorem 2 reduces to Bihari-LaSalle type inequality for conformable integrals.*

Remark 2. *If we take $\varphi(\eta) = \eta$ and $k(v) = v$ in Theorem 2, then the inequality given by Theorem 2 reduces to Gronwall's inequality for conformable integrals in [7].*

Corollary 1. *Let $v, \mathrm{f}, \mathrm{g}, k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, assume that φ and k are non-decreasing with $\varphi(\eta) \leq \eta$ for $\eta \geq 0$ and $k(v) > 0$. If $v \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$v(\eta) \leq \mathrm{m} + \int_0^{\eta} \mathrm{f}(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + \int_0^{\varphi(\eta)} \mathrm{g}(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s}, \quad \eta \geq 0, \tag{3.3}$$

where m is a non-negative constant. Then

$$v(\eta) \leq \mathcal{G}^{-1}\left(\mathcal{G}(\mathrm{m}) + \int_0^{\eta} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s} + \int_0^{\varphi(\eta)} \mathrm{g}(\mathrm{s})d_\alpha \mathrm{s}\right), \tag{3.4}$$

with $\mathcal{G}$ as in Theorem 2.

Theorem 3. *Let $v, \mathrm{f}, \mathrm{g}, k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$. Assume that φ and k are non-decreasing with $\varphi(\eta) \leq \eta$ for $\eta \geq 0$ and $k(v) > 0$. If $v \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$v^2(\eta) \leq \mathrm{m}^2 + 2\int_0^{\varphi(\eta)} [\mathrm{f}(\mathrm{s})v(\mathrm{s})k(v(\mathrm{s})) + \mathrm{g}(\mathrm{s})v(\mathrm{s})]d_\alpha \mathrm{s}, \quad \eta \geq 0, \tag{3.5}$$

where m is a non-negative constant. Then

$$v(\eta) \leq \mathcal{G}^{-1}\left[\mathcal{G}\left(\mathrm{m} + \int_0^{\varphi(\eta)} \mathrm{g}(\mathrm{s})d_\alpha \mathrm{s}\right) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s}\right],$$

where $\mathcal{G}^{-1}$ is inverse of $\mathcal{G}$ such that

$$\mathcal{G}(\xi) =: \int_1^{\xi} \frac{1}{k(\mathrm{s})} d_\alpha \mathrm{s}, \quad \xi \geq 0,$$

and

$$\mathcal{G}\left(\mathrm{m} + \int_0^{\varphi(\eta)} \mathrm{g}(\mathrm{s})d_\alpha \mathrm{s}\right) + \int_0^{\varphi(\eta)} \mathrm{f}(\mathrm{s})d_\alpha \mathrm{s} \in Dom(\mathcal{G}^{-1}), \quad \forall \eta \geq 0.$$

Proof. Similarly let us assume that $\mathrm{m} > 0$. Then define the non-decreasing positive function $\mathrm{z}(\eta)$ by the right-hand side of (3.7). Thus $v^2(\eta) \leq \mathrm{z}(\eta)$ and $\mathrm{z}(0) = \mathrm{m}^2$, and

$$\begin{aligned} D^\alpha \mathrm{z}(\eta) &= 2[\mathrm{f}(\varphi(\eta))v(\varphi(\eta))k(v(\varphi(\eta))) + \mathrm{g}(\varphi(\eta))v(\varphi(\eta))]D^\alpha \varphi(\eta) \\ &\leq 2[\mathrm{f}(\varphi(\eta))\sqrt{\mathrm{z}(\varphi(\eta))}k(\sqrt{\mathrm{z}(\varphi(\eta))}) + \mathrm{g}(\varphi(\eta))\sqrt{\mathrm{z}(\varphi(\eta))}]D^\alpha \varphi(\eta) \\ &\leq 2[\mathrm{f}(\varphi(\eta))\sqrt{\mathrm{z}(\eta)}k(\sqrt{\mathrm{z}(\varphi(\eta))}) + \mathrm{g}(\varphi(\eta))\sqrt{\mathrm{z}(\eta)}]D^\alpha \varphi(\eta) \end{aligned}$$

as $\varphi(\eta) \leq \eta$. The last relation given above gives us

$$\frac{D^\alpha \mathrm{z}(\eta)}{2\sqrt{\mathrm{z}(\eta)}} \leq [\mathrm{f}(\varphi(\eta))k(\sqrt{\mathrm{z}(\varphi(\eta))}) + \mathrm{g}(\varphi(\eta))]D^\alpha \varphi(\eta).$$

By setting $\varphi(\eta) = \mathrm{s}$ and integrating from 0 to $\varphi(\eta)$ with respect to s, we get

$$\sqrt{\mathrm{z}(\eta)} \leq \int_0^{\varphi(\eta)} [\mathrm{m} + \mathrm{f(s)}k(\sqrt{\mathrm{z(s)}}) + \mathrm{g(s)}] d_\alpha \mathrm{s}.$$

If we define an arbitrary number T such that $0 \leq T \leq \eta$, then we have

$$\sqrt{\mathrm{z}(\eta)} \leq \mathrm{m} + \int_0^{T} \mathrm{g(s)} d_\alpha \mathrm{s} + \int_0^{\varphi(\eta)} \mathrm{f(s)}k(\sqrt{\mathrm{z(s)}}) d_\alpha \mathrm{s}.$$

Now an application of Theorem 2 given above, we get

$$\sqrt{\mathrm{z}(\eta)} \leq \mathcal{G}^{-1}\left[\mathcal{G}\left(\mathrm{m} + \int_0^{T} \mathrm{g(s)} d_\alpha \mathrm{s}\right) + \int_0^{\varphi(\eta)} \mathrm{f(s)} d_\alpha \mathrm{s}\right], \quad 0 \leq T \leq \eta.$$

By using the fact that $v(\eta) \leq \sqrt{\mathrm{z}(\eta)}$ and taking $\eta = T$ in the above inequality, we obtain

$$v(T) \leq \mathcal{G}^{-1}\left[\mathcal{G}\left(\mathrm{m} + \int_0^{T} \mathrm{g(s)} d_\alpha \mathrm{s}\right) + \int_0^{\varphi(T)} \mathrm{f(s)} d_\alpha \mathrm{s}\right], \quad 0 \leq T \leq \eta.$$

Thus we get the desired result. □

Remark 3. *If we take $\varphi(\eta) = \eta$, then the inequality given by Theorem 3 reduces to Pachpatte's generalization of Ou-Iang type inequality for conformable integrals.*

Corollary 2. *Let $v, \mathrm{g} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$. Assume that φ is non-decreasing with $\varphi(\eta) \leq \eta$ for $\eta \geq 0$. If $v \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$v^2(\eta) \leq \mathrm{m}^2 + 2\int_0^{\varphi(\eta)} \mathrm{g(s)}v(\mathrm{s}) d_\alpha \mathrm{s}, \quad \eta \geq 0, \tag{3.6}$$

where m *is a non-negative constant. Then*

$$v(\eta) \leq \int_0^{\varphi(\eta)} \mathrm{g(s)} d_\alpha \mathrm{s}.$$

Remark 4. *If we take $\varphi(\eta) = \eta$, then the Corollary 2 reduces to Ou-Iang type inequality for conformable integrals.*

Corollary 3. *Let $v, \mathrm{f}, \mathrm{g} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$. Assume that φ is non-decreasing with $\varphi(\eta) \leq \eta$ for $\eta \geq 0$. If we take $k(v) = v$ in Theorem 3, that is*

$$v^2(\eta) \leq \mathrm{m}^2 + 2\int_0^{\varphi(\eta)} [\mathrm{f(s)}v^2(\mathrm{s}) + \mathrm{g(s)}v(\mathrm{s})] d_\alpha \mathrm{s}, \quad \eta \geq 0, \tag{3.7}$$

where m *is a non-negative constant. Then*

$$v(\eta) \leq \left[\mathrm{m} + \int_0^{\varphi(\eta)} \mathrm{g(s)} d_\alpha \mathrm{s}\right] e^{\int_0^{\varphi(\eta)} \mathrm{f(s)} d_\alpha \mathrm{s}}, \quad \eta \geq 0.$$

Remark 5. *Corollary 3 is called a retarded version of a conformable fractional integral inequality whose classical version given in [1]. Similarly if we take* $g(\eta) = 0$*, Corollary 3 becomes a retarded Gronwall type of conformable fractional integral inequality whose classical version given in [3].*

Theorem 4. *Let* $v, f, g, k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$*. Assume that* φ *and* k *are non-decreasing with* $\varphi(\eta) \le \eta$ *for* $\eta \ge 0$ *and* $k(v) > 0$*. If* $v \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies*

$$v^2(\eta) \le m^2 + 2\int_0^{\varphi(\eta)} f(s)v(s)k(v(s)) + 2\int_0^{\varphi(\eta)} g(s)v(s)k(v(s))d_\alpha s, \quad \eta \ge 0, \tag{3.8}$$

where m *is a non-negative constant. Then*

$$v(T) \le \mathcal{G}^{-1}\left[\mathcal{G}(m) + \int_0^{\varphi(\eta)} f(s)d_\alpha s + \int_0^{\varphi(\eta)} g(s)d_\alpha s\right], \quad 0 \le \eta,$$

where $\mathcal{G}^{-1}$ *the is inverse of* $\mathcal{G}$ *such that*

$$\mathcal{G}(\xi) =: \int_1^{\xi} \frac{1}{k(s)} d_\alpha s, \quad \xi \ge 0.$$

Proof. By following the similar steps of the proof of Theorem 3, we obtain

$$\frac{D^\alpha z(\eta)}{2\sqrt{z(\eta)}} \le [f(\varphi(\eta)) + g(\varphi(\eta))]k(\sqrt{z(\varphi(\eta))})D^\alpha\varphi(\eta).$$

By setting $\varphi(\eta) = s$ and integrating from 0 to $\varphi(\eta)$ with respect to s, we get

$$\sqrt{z(\eta)} \le \int_0^{\varphi(\eta)} [m + f(s)k(\sqrt{z(s)}) + g(s)k(\sqrt{z(s)})]d_\alpha s.$$

By using the fact that $v(\eta) \le \sqrt{z(\eta)}$ in the above inequality, we obtain

$$v(T) \le \mathcal{G}^{-1}\left[\mathcal{G}(m) + \int_0^{\varphi(\eta)} f(s)d_\alpha s + \int_0^{\varphi(\eta)} g(s)d_\alpha s\right], \quad 0 \le \eta.$$

Thus we get the desired result. □

4 Applications

In this part, we give some applications of our results to obtain the solution of several specific non-linear conformable fractional differential equations.

Problem 1. ***Conformable fractional Van der Pol differential equation***
The fractional order of Van der Pol equation can be expressed as follows:

$$D_2^\alpha x + \kappa[(D^\alpha x)^2 - 1]D^\alpha x + x = 0, \quad 0.5 < \alpha < 1, \tag{4.1}$$

where $\kappa > 0$*. In other words, one can consider the conformable fractional Van der Pol equation with time delay*

$$\begin{aligned} D^\alpha x &= y, \\ D^\alpha y &= -\mathcal{F}(y) - \mathcal{H}(x(\varphi(\eta))), \end{aligned} \tag{4.2}$$

where $\mathcal{F}, \mathcal{H} \in C(\mathbb{R}, \mathbb{R})$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\varphi(\eta) \leq \eta$ for $\eta \geq 0$. If φ is increasing diffeormorphism of $\mathbb{R}^+$, and

$$-x^{2-\alpha}\mathcal{F}(\xi) \leq |x|\phi(|\xi|),$$
$$\mathcal{H}^{2/\alpha}(\xi) \leq |x|\phi(|\xi|),$$

where $\xi \in \mathbb{R}$ for some non-decreasing function $\xi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the properties $\xi(v) > 0$ for $v > 0$ and $\int_1^\infty (1/\phi(\mathrm{s}))d_\alpha \mathrm{s} = \infty$, then all solutions to conformable fractional Van der Pol equation given above are global. Then, if $(x(\eta), y(\eta))$ is a solution to (4.2) defined on the maximal existence interval $[0, T)$, let $v^2(\eta) = x^2(\eta)+y^2(\eta)$ for $\eta \in [0, T)$ and $y^{2/\alpha}(\eta) \leq v^2(\eta)$ then we have $|x(\eta)| \leq v(\eta)$, $|y(\eta)| \leq v(\eta)$. Then

$$\begin{aligned} D^\alpha v^2(\eta) &= 2x^{2-\alpha}D^\alpha x + 2y^{2-\alpha}D^\alpha y \\ &= 2x^{2-\alpha}y - 2y^{2-\alpha}\mathcal{F}(y) - 2y^{2-\alpha}\mathcal{H}(x \circ \varphi) \\ &\leq x^2 + y^{2/\alpha} - 2y^{2-\alpha}\mathcal{F}(y) + y^2 + \mathcal{H}^{2/\alpha}(x \circ \varphi) \\ &\leq 2u^2 + 2u\phi(v) + |(x \circ \varphi)|\phi(|x \circ \varphi|). \end{aligned}$$

By setting $k(v) := v+\phi(v)$ and integrating the above inequality from 0 to η with the help of conformable fractional calculus, we get

$$\begin{aligned} v^2(\eta) &\leq \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^\eta |x(\varphi(\mathrm{s}))|\phi(|x(\varphi(\mathrm{s}))|)d_\alpha \mathrm{s} \\ &\leq \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^\eta |x(\varphi(\mathrm{s}))|k(|x(\varphi(\mathrm{s}))|)d_\alpha \mathrm{s} \\ &\leq \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^{\varphi(\eta)} \frac{1}{D^\alpha\varphi(\varphi^{-1}(\mathrm{z}))}|x(\mathrm{z})|k(|x(\mathrm{z})|)d_\alpha \mathrm{z} \\ &\leq \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^{\varphi(\eta)} \frac{1}{D^\alpha\varphi(\varphi^{-1}(\mathrm{z}))}v(\mathrm{z})k(v(\mathrm{z}))d_\alpha \mathrm{z}, \end{aligned}$$

where $\mathrm{z} = \varphi(\mathrm{s})$. Then if

$$\mathcal{G}(\xi) =: \int_1^\xi \frac{1}{k(\mathrm{s})}d_\alpha \mathrm{s}, \quad \xi \geq 0,$$

the Theorem 4 concludes that

$$v(\eta) \leq \mathcal{G}^{-1}\left[\mathcal{G}(\mathrm{m}) + \int_0^\eta d_\alpha \mathrm{s} + \int_0^{\varphi(\eta)} \frac{1}{D^\alpha\varphi(\varphi^{-1}(\mathrm{z}))}d_\alpha \mathrm{s}\right] \leq \mathcal{G}^{-1}\left[\mathcal{G}(\mathrm{m}) + 2t\right].$$

This result proves that $v(\eta)$ does not blow up in finite time. In other words, all solutions of (4.2) have global existence.

Remark 6. *If we replaced the coefficient of $(D^\alpha x)^2$ in equation (4.2) by $1/3$, it gives the conformable fractional Rayleigh equation with time delay. Following the same steps above we get the similar results for that equation.*

Problem 2. ***Conformable fractional Liènard differential equation***
Consider the conformable fractional Liènard equation with time delay

$$\begin{aligned} D^\alpha x &= y - \mathcal{F}(x), \\ D^\alpha y &= -\mathcal{H}(x(\eta - \varphi(\eta))), \end{aligned} \tag{4.3}$$

where $\mathcal{F} \in C^1(\mathbb{R},\mathbb{R})$, $\mathcal{H} \in C(\mathbb{R}^+ \times \mathbb{R},\mathbb{R})$ $\varphi \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ and $\varphi(\eta) \le \varphi$ on $\mathbb{R}^+$. If $\varphi(\eta) \equiv \eta - \varphi(\eta)$ is increasing diffeormorphism of $\mathbb{R}^+$, and

$$-x^{2-\alpha}\mathcal{F}(\xi) \le |x|\phi(|\xi|),$$
$$\mathcal{H}^{2/\alpha}(\eta,\xi) \le \psi(\eta)|x|\phi(|\xi|),$$

where $\xi \in \mathbb{R}$ and $(\eta,\xi) \in \mathbb{R}^+ \times \mathbb{R}$ for some non-decreasing function $\xi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with the properties $\xi(v) > 0$ for $v > 0$ and $\int_1^\infty (1/\phi(\mathrm{s}))d_\alpha \mathrm{s} = \infty$, then all solutions to conformable fractional Liènard equation given above are global. Then, if $(x(\eta),y(\eta))$ is a solution to (4.3) defined on the maximal existence interval $[0,T)$, let $v^2(\eta) = x^2(\eta)+y^2(\eta)$ for $\eta \in [0,T)$ and $y^{2/\alpha}(\eta) \le v^2(\eta)$ then we have $|x(\eta)| \le v(\eta)$, $|y(\eta)| \le v(\eta)$. Then

$$\begin{aligned} D^\alpha v^2(\eta) &= 2x^{2-\alpha}D^\alpha x + 2y^{2-\alpha}D^\alpha y \\ &= 2x^{2-\alpha}y - 2x^{2-\alpha}\mathcal{F}(y) - 2y^{2-\alpha}\mathcal{H}(\eta, x\circ\varphi) \\ &\le x^2 + y^{2/\alpha} - 2x^{2-\alpha}\mathcal{F}(y) + y^2 + \mathcal{H}^{2/\alpha}(\eta, x\circ\varphi) \\ &\le 2u^2 + 2u\phi(v) + \psi(\eta)(|x\circ\varphi|)\phi(|x\circ\varphi|). \end{aligned}$$

By setting $k(v) := v+\phi(v)$ and integrating the above inequality from 0 to η with the help of conformable fractional calculus, we get

$$\begin{aligned} v^2(\eta) &\le \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^\eta \psi(\mathrm{s})|x(\varphi(\mathrm{s}))|\phi(|x(\varphi(\mathrm{s}))|)d_\alpha \mathrm{s} \\ &\le \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^\eta \psi(\mathrm{s})|x(\varphi(\mathrm{s}))|k(|x(\varphi(\mathrm{s}))|)d_\alpha \mathrm{s} \\ &\le \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^{\varphi(\eta)} \frac{\psi(\varphi^{-1}(\mathrm{z}))}{D^\alpha\varphi(\varphi^{-1}(\mathrm{z}))}|x(\mathrm{z})|k(|x(\mathrm{z})|)d_\alpha \mathrm{z} \\ &\le \mathrm{m}^2 + 2\int_0^\eta v(\mathrm{s})k(v(\mathrm{s}))d_\alpha \mathrm{s} + 2\int_0^{\varphi(\eta)} \frac{\psi(\varphi^{-1}(\mathrm{z}))}{D^\alpha\varphi(\varphi^{-1}(\mathrm{z}))}v(\mathrm{z})k(v(\mathrm{z}))d_\alpha \mathrm{z}, \end{aligned}$$

where $\mathrm{z} = \varphi(\mathrm{s})$. Then if

$$\mathcal{G}(\xi) =: \int_1^\xi \frac{1}{k(\mathrm{s})}d_\alpha \mathrm{s}, \quad \xi \ge 0,$$

Theorem 4 concludes that

$$\begin{aligned} v(\eta) &\le \mathcal{G}^{-1}\left[\mathcal{G}(\mathrm{m}) + \int_0^\eta d_\alpha \mathrm{s} + \int_0^{\varphi(\eta)} \frac{\psi(\varphi^{-1}(\mathrm{z}))}{D^\alpha\varphi(\varphi^{-1}(\mathrm{z}))}d_\alpha \mathrm{s}\right] \\ &\le \mathcal{G}^{-1}\left[\mathcal{G}(\mathrm{m}) + \eta + \int_0^\eta \psi(\mathrm{s})d_\alpha \mathrm{s}\right]. \end{aligned}$$

This result proves that $v(\eta)$ does not blow up in finite time. In other words, all solutions of (4.3) have global existence.

5 Concluding remark

In this paper, retarded conformable fractional integral inequalities is proposed and tested with the help of Katugampola type conformable fractional calculus. To verify the results given here, we applied them to the global existence of solutions to conformable fractional differential equations with time delay.

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