Demonstr. Math. 2019; 52:56–60 DE GRUYTER

Research Article Open Access

Mamoru Nunokawa, Janusz Sokół*, and Huo Tang

Length problems for Bazilevič functions

https://doi.org/10.1515/dema-2019-0007
Received October 4, 2018; accepted December 30, 2018

Abstract: Let $C(r)$ denote the curve which is image of the circle $|z| = r < 1$ under the mapping f. Let $L(r)$ be the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. Furthermore $M(r) = \max_{|z|=r} |f(z)|$. We present some relations between these notions for Bazilevič functions.

Keywords: Bazilevič function, close-to-convex functions, convex functions, starlike function, convex function

MSC: 30C45, 30C80

1 Preliminaries

Let $\mathcal{H}$ denote the class of functions f which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of functions univalent (i.e. one-to-one) in $\mathbb{D}$. Denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ of starlike functions, i.e. the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike with respect to the origin. It is well-known, since the work of [1], that $f \in \mathcal{S}^*$ if, and only if, $f \in \mathcal{A}$ and

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ z \in \mathbb{D}.$$

Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to to the origin if, and only if, the linear segment joining 0 to every other point $w \in E$ lies entirely in E. By $\mathcal{P}$ we denote the class of Carathéodory functions p which are analytic in $\mathbb{D}$, satisfying the condition $\mathfrak{Re}\left\{p(z)\right\} > 0$ for $z \in \mathbb{D}$, with $p(0) = 1$.

Suppose now that $f \in \mathcal{A}$, then f is close-to-convex if, and only if, there exists $\alpha \in (-\pi/2, \pi/2)$, and a function $g \in \mathcal{S}^*$ such that

$$\mathfrak{Re}\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}\right\} > 0, \ z \in \mathbb{D}. \tag{1.1}$$

This class of close-to-convex functions was introduced in [2]. Functions defined by (1.1) with $\alpha = 0$ were considered earlier by Ozaki [3], see also Umezawa [4, 5]. Moreover, Lewandowski [6, 7] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{U})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski class is identical with the class of close-to-convex functions. Here, we denote this class by $\mathcal{K}$, and note that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. The class of close-to-convex functions forms an important subclass of $\mathcal{S}$. Length problems for close-to-convex functions were recently considered in [8]. A proper subset of $\mathcal{K}$ is the class of bounded boundary rotation of f such that $f'(z) \neq 0$ in the unit disc and

$$4\pi \le \lim_{r\to 1}\int_0^{2\pi}\left|\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| \mathrm{d}\theta, \quad z = re^{i\theta}.$$

Mamoru Nunokawa: University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan;
E-mail: mamoru_nuno@doctor.nifty.jp
***Corresponding Author: Janusz Sokół:** University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland; E-mail: jsokol@ur.edu.pl
Huo Tang: School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China;
E-mail: thth2009@163.com

Another even larger subset of $\mathcal{S}$ is formed by the Bazilevič functions. Bazilevič [9] introduced a class of functions $f \in \mathcal{A}$ which are defined by the following

$$f(z) = \left\{ \frac{\beta}{1+\alpha^2} \int_0^z (h(z) - i\alpha)\zeta^{(-\alpha\beta/(1+\alpha^2))-1} g^{\beta/(1+\alpha^2)}(\zeta) d\zeta \right\}^{(1+i\alpha)/\beta},$$

where $h \in \mathcal{P}$ and $g \in \mathcal{S}^\star$, α is any real number and $\beta > 0$. Bazilevič showed that all such functions are univalent in $\mathbb{D}$. Putting $\alpha = 0$ in (1) and differentiating it, we have

$$zf'(z) = (f(z))^{1-\beta}(g(z))^\beta h(z)$$

and

$$\mathfrak{Re}\{h(z)\} = \mathfrak{Re}\left\{ \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} \right\} > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

Thomas [10] called a function satisfying condition (1.2) a Bazilevič function of type β. For further works on Bazilevič functions we refer to [11]-[15]. It is easy to see that Bazilevič functions of type $\beta = 1$ are close-to-convex functions, univalent in $\mathbb{D}$. Furthermore, the set of starlike functions is contained in the set of Bazilevič functions of type β.

Let $C(r)$ denote the curve which is image of the circle $|z| = r < 1$ under the mapping f. Let $L(r)$ be the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. Furthermore $M(r) = \max_{|z|=r} |f(z)|$. In [16], Thomas has shown the following:

Theorem 1.1. *[16, Th.1] If $g \in \mathcal{S}^\star$, then*

$$L(r) \leq 2\sqrt{\pi A(r)} \left(1 + \log \frac{1+r}{1-r}\right) \quad \text{as} \quad r \to 1.$$

Note that in [17], Thomas considered $L(r)$ for the class of bounded close-to-convex functions and asked the following question.
Does there exist a starlike function for which

$$\liminf_{r\to 1} \frac{L(r)}{M(r) \log \frac{1}{1-r}} > 0$$

or

$$\liminf_{r\to 1} \frac{L(r)}{\sqrt{A(r)} \log \frac{1}{1-r}} > 0? \tag{1.3}$$

Applying the result of [18], we give a negative partial result of the above open problem (1.3). Some related problems were considered in [19, 20].

2 On Bazilevič functions of bounded rotation

The following lemma is due to Pommerenke [21].

Lemma 2.1. *[21] Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ be analytic and univalent in $\mathbb{D}$. Then we have*

$$M(r) = \frac{4}{\sqrt{\pi}} \left(A(r) \log \frac{3}{1-r}\right)^{1/2} \quad as \quad r \to 1.$$

Lemma 2.2. *Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ be analytic in $\mathbb{D}$. Then we have*

$$M(r) \leq \mathcal{O}\left(S(\sqrt{r}) \log \frac{1}{1-r}\right)^{1/2} \quad as \quad r \to 1, \tag{2.1}$$

where $\mathcal{O}$ means the Landau's symbol and

$$S(r)=\frac{1}{2\pi}\int_0^r\int_0^{2\pi}\rho|f'(\rho e^{i\theta})|^2\mathrm{d}\theta\mathrm{d}\rho.$$

Proof. Assume that $0<r_1<r$, $\zeta=\sqrt{\rho}e^{i\varphi}$, $0\le|t|\le r$, $0<\rho<r$ and throughout C will denote an absolute constant not necessarily the same each time. We have

$$|f(z)| = \left|\int_0^z f'(t)\mathrm{d}t\right|.$$

Now, by using the substitution

$$t=\rho e^{i\theta},\ \mathrm{d}t=e^{i\theta}\mathrm{d}\rho,\ \zeta=\sqrt{\rho}e^{i\varphi},$$

this becomes

$$\begin{aligned}
|f(z)| &= \left|\int_0^r f'(\rho e^{i\theta})e^{i\theta}\mathrm{d}\rho\right|\\
&\le \int_0^r |f'(\rho e^{i\theta})|\mathrm{d}\rho\\
&\le \frac{1}{2\pi}\int_0^r\int_{|\zeta|=\sqrt{\rho}}\frac{|f'(\zeta)|}{|\zeta-\rho e^{i\theta}|}|\mathrm{d}\zeta|\mathrm{d}\rho\\
&\le \frac{1}{2\pi}\int_0^{r_1}\int_0^{2\pi}\sqrt{\rho}\frac{|f'(\zeta)|}{|\zeta-\rho e^{i\theta}|}\mathrm{d}\varphi\mathrm{d}\rho+\frac{1}{2\pi}\int_{r_1}^r\int_0^{2\pi}\sqrt{\frac{\rho}{r_1}}\frac{|f'(\zeta)|}{|\zeta-\rho e^{i\theta}|}\mathrm{d}\varphi\mathrm{d}\rho\\
&\le C+\frac{1}{2\pi\sqrt{r_1}}\int_{r_1}^r\int_0^{2\pi}\sqrt{\rho}\frac{|f'(\zeta)|}{|\zeta-\rho e^{i\theta}|}\mathrm{d}\varphi\mathrm{d}\rho.
\end{aligned}$$

Further, because

$$\left(\iint_D|f(x,y)g(x,y)|\mathrm{d}x\mathrm{d}y\right)^2\le\left(\iint_D|f(x,y)|^2\mathrm{d}x\mathrm{d}y\right)\left(\iint_D|g(x,y)|^2\mathrm{d}x\mathrm{d}y\right),$$

we have

$$\begin{aligned}
&C+\frac{1}{2\pi\sqrt{r_1}}\int_{r_1}^r\int_0^{2\pi}\sqrt{\rho}\frac{|f'(\zeta)|}{|\zeta-\rho e^{i\theta}|}\mathrm{d}\varphi\mathrm{d}\rho\\
\le\ & C+\frac{1}{\sqrt{r_1}}\left(\frac{1}{2\pi}\int_{r_1}^r\int_0^{2\pi}\sqrt{\rho}|f'(\sqrt{\rho}e^{i\varphi})|^2\mathrm{d}\varphi\mathrm{d}\rho\right)^{1/2}\left(\frac{1}{2\pi}\int_{r_1}^r\int_0^{2\pi}\frac{\sqrt{\rho}}{|\sqrt{\rho}e^{i\varphi}-\rho e^{i\theta}|^2}\mathrm{d}\varphi\mathrm{d}\rho\right)^{1/2}\\
=\ & C+\frac{1}{\sqrt{r_1}}\left(\frac{1}{2\pi}\int_0^r\int_0^{2\pi}\sqrt{\rho}|f'(\sqrt{\rho}e^{i\varphi})|^2\mathrm{d}\varphi\mathrm{d}\rho\right)^{1/2}\left(\frac{1}{2\pi}\int_{r_1}^r\frac{\sqrt{\rho}}{\rho-\rho^2}\mathrm{d}\rho\right)^{1/2}\\
\le\ & C+\frac{1}{\sqrt{r_1}}\left(\frac{1}{2\pi}\int_0^{\sqrt{r}}\int_0^{2\pi}\sqrt{\rho}|f'(\sqrt{\rho}e^{i\varphi})|^2\mathrm{d}\varphi\mathrm{d}\rho\right)^{1/2}\left(\frac{1}{2\pi\sqrt{r_1}}\int_{r_1}^r\frac{1}{1-\rho}\mathrm{d}\rho\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C + \frac{1}{\sqrt{r_1}} \sqrt{S(\sqrt{r})} \left(\frac{1}{2\pi\sqrt{r_1}} \int_0^r \frac{1}{1-\rho} \mathrm{d}\rho \right)^{1/2} \\
&= \mathcal{O}\left\{ \sqrt{S(\sqrt{r})} \sqrt{\log \frac{1}{1-r}} \right\} \quad \text{as } r \to 1,
\end{aligned}
$$

where $0 < r_1 < r < 1$. Because $M(r) = \max_{|z|=r} |f(z)|$, we finally obtain (2.1). □

Remark 1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and univalent in $\mathbb{D}$, then it is trivial that

$$S(r) = A(r) \text{ for } 0 < r < 1$$

so in this case (2.1) becomes

$$M(r) \leq \mathcal{O}\left(A(\sqrt{r}) \log \frac{1}{1-r} \right)^{1/2} \quad \textit{as } r \to 1.$$

Theorem 2.3. *Let f be a Bazilevič function of type β and let f be a function of bounded rotation on $0 < |z| = r < 1$, and suppose that*

$$M(r) = \mathcal{O}\left\{ (1-r)^{-\alpha} p(r) \right\} \quad \text{as } r \to 1 \tag{2.2}$$

for all α, where $0 < \alpha \leq 2$, while $p(r)$ is monotone increasing function of r in a wider sense, and $\mathcal{O}$ in (2.2) cannot be replaced by o. We then have

$$L(r) = \mathcal{O}\left(A(r) \log \frac{1}{1-r} \right)^{1/2} \quad \text{as } r \to 1.$$

Proof. From (2.2) and applying the same method as in the proof of Theorem 1 in [22], we have

$$
\begin{aligned}
L(r) &= \int_0^{2\pi} |zf'(z)| \mathrm{d}\theta \\
&= \int_0^{2\pi} |(f(z))^{(1-\beta)} (g(z))^{\beta} h(z)| \mathrm{d}\theta \\
&\leq \int_0^r \int_0^{2\pi} |(1-\beta) f'(z) f^{-\beta} (g(z))^{\beta} h(z)| \mathrm{d}\theta \mathrm{d}\rho \\
&\quad + \int_0^r \int_0^{2\pi} |f^{1-\beta} \beta (g'(z))^{\beta-1} h(z)| \mathrm{d}\theta \mathrm{d}\rho + \int_0^r \int_0^{2\pi} |f^{1-\beta} (g(z))^{\beta} h'(z)| \mathrm{d}\theta \mathrm{d}\rho \\
&= I_1 + I_2 + I_3, \quad \text{say.}
\end{aligned}
$$

Then, from [18, p.338], and from Lemma 2.1, we have the following

$$I_1 \leq 2\pi |1-\beta| M(r) = \mathcal{O}\left(A(r) \log \frac{1}{1-r} \right)^{1/2} \quad \text{as } r \to 1.$$

Next, we have (2.3) below from [22, p.277] and Lemma 2.1:

$$
\begin{aligned}
I_2 &= C \int_\delta^r \frac{M(\rho)}{1-\rho} \mathrm{d}\rho + C \qquad (2.3) \\
&\leq C \int_\delta^r \frac{p(\rho)}{(1-\rho)^{1+\alpha}} \mathrm{d}\rho + C \\
&\leq \frac{Cp(r)}{\alpha} (1-r)^{-\alpha} + C
\end{aligned}
$$

$$= \mathcal{O}(M(r)) = \mathcal{O}\left(A(r)\log\frac{1}{1-r}\right)^{1/2} \quad \text{as } r \to 1,$$

where δ is fixed $0 < \delta < \rho \le r < 1$. Applying the result of [22, p.277] and the same method as in the calculation (2.3), we have

$$\begin{aligned} I_3 &= 2\pi\{|1-\beta|C+|\beta|\}\int_0^r \frac{M(\rho)}{1-\rho}\mathrm{d}\rho \\ &= \mathcal{O}\left(A(r)\log\frac{1}{1-r}\right)^{1/2} \quad \text{as } r \to 1. \end{aligned}$$

This completes the proof of Theorem 2.3. □

From Theorem 2.3, we easily have the following corollary.

Corollary 2.4. *Let f be a Bazilevič function of type β and let f be a function of bounded rotation on* $0 < |z| = r < 1$ *and suppose that*

$$M(r) = \mathcal{O}\left\{(1-r)^{-\alpha}\left(\log\frac{1}{1-r}\right)^{1/2}\right\} \quad \text{as } r \to 1.$$

Then there is no Bazilevič function of type β satisfying the condition (1.3).

References

[1] Study E., Konforme Abbildung Einfachzusammenhangender Bereiche, B. C. Teubner, Leipzig und Berlin, 1913
[2] Kaplan W., Close to convex schlicht functions, Michigan Math. J., 1952, 1, 169–185
[3] Ozaki S., On the theory of multivalent functions, Sci. Rep. Tokyo Bunrika Daig. A, 1935, 2, 167–188
[4] Umezawa T., On the theory of univalent functions, Tohoku Math. J., 1955, 7, 212–228
[5] Umezawa T., Multivalently close-to-convex functions, Proc. Amer. Math. Soc., 1957, 8, 869–874
[6] Lewandowski Z., Sur l'identité de certaines classes de fonctions univalentes, I, Ann. Univ. Mariae Curie-Skłodowska Sect. A, 1958, 12, 131–146
[7] Lewandowski Z., Sur l'identité de certaines classes de fonctions univalentes, II, Ann. Univ. Mariae Curie–Skłodowska Sect. A, 1960, 14, 19–46
[8] Nunokawa M., Sokół J., On some lenght problems for close-to-convex functions, Studia Scient. Math. Hungarica, 2018, 55(3), 293–304
[9] Bazilevič I. E., On the case of integrability in quadratures of Loewner-Kufariev equation, Mat. Sb., 1955, 37, 47–476 (in Russian)
[10] Thomas D. K., On Bazilevič functions, Trans. Amer. Math. Soc., 1968, 132, 353–361
[11] Arif M., Raza M., Noor K. I., Malik S. N., On strongly Bazilevič functions associated with generalized Robertson functions, J. Math. Comput. Model., 2011, 54(5-6), 1608–1612
[12] Haq W. U., Arif M., Khan A., Arc length inequality for a certain class of analytic functions related to conic regions, J. Complex Analysis, 2013 (2013), Article ID 407596
[13] Noor K. I., Al-Bany S. A., On Bazilevič functions, Inter. J. Math. Math. Sci., 1987, 10(1), 79–88
[14] Noor K. I., On generalized Bazilevic functions related with conic regions, Abstract Appl. Analysis, 2012(2012), Article ID 345261
[15] Noor K. I., Yousaf K., On uniformly Bazilevič and related functions, J. Appl. Math., 2012(2012), Article ID 982321
[16] Thomas D. K., A note on starlike functions, J. London Math. Soc., 1968, 43, 703–706
[17] Thomas D. K., On starlike and close-to-convex univalent functions, J. London Math. Soc., 1967, 42, 427–435
[18] Nunokawa M., On Bazilevič and convex functions, Trans. Amer. Math. Soc., 1969, 143, 337–341
[19] Nunokawa M., Sokół J., On some lenght problems for analytic functions, Osaka J. Math., 2014, 51, 695–707
[20] Nunokawa M., Sokół J., On some lenght problems for univalent functions, Math. Meth. Appl. Sci., 2016, 39(7), 1662–1666
[21] Pommerenke Ch., Über nahezu konvexe analitische Functionen, Arch. Math., 1965, 16, 344–347
[22] Nunokawa M., On Bazilevič functions of bounded boundary rotation, J. Math. Soc. Japan, 1972, 24(2), 275–278