Demonstr. Math. 2019; 52:20–28 DE GRUYTER

Research Article Open Access

Akram Teymouri, Abasalt Bodaghi*, and Davood Ebrahimi Bagha

Derivations into annihilators of the ideals of Banach algebras

https://doi.org/10.1515/dema-2019-0004
Received August 17, 2018; accepted November 26, 2018

Abstract: In this article, following Gorgi and Yazdanpanah, we define two new concepts of the ideal amenability for a Banach algebra $\mathcal{A}$. We compare these notions with $\mathcal{I}$-weak amenability and ideal amenability, where $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{A}$. We also study the hereditary properties of quotient ideal amenability for Banach algebras. Some examples show that the concepts of $\frac{\mathcal{A}}{\mathcal{I}}$-weak amenability and of $\mathcal{I}$-weak amenability do not coincide for Banach algebras in general.

Keywords: amenability, ideal amenability, weak amenability

MSC: Primary 46H25, 46H20; Secondary 46H35

1 Introduction

The notion of amenability for Banach algebras was first introduced by B.E. Johnson [1] in 1972. An amenable Banach algebra $\mathcal{A}$ has been known to satisfy $H^1(\mathcal{A}, X^\star) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X, where $X^\star$ is the dual module of X and $H^1(\mathcal{A}, X^\star)$ is the first cohomology group of $\mathcal{A}$ with coefficients in $X^\star$. Johnson showed that for a locally compact group G, the group algebra $L^1(G)$ is amenable if and only if G is amenable [1]. Since the notion of amenability was considered, several generalizations of this concept related to ideals such as weak amenability [2], ideal amenability [3], ideal Connes-amenability [4] etc. were introduced for Banach algebras.

The notion of weak amenability for Banach algebras was introduced by Bade, Curtis and Dales in [2]. A Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation from $\mathcal{A}$ into $\mathcal{A}^\star$ is inner, or equivalently if $H^1(\mathcal{A}, \mathcal{A}^\star) = \{0\}$. However, the class of weakly amenable Banach algebra is considerably larger than that of amenable Banach algebras. For example, every $C^\star$-algebra $\mathcal{A}$ is always weakly amenable [5] while $\mathcal{A}$ is amenable if and only if it is nuclear [6]. In [3], Gorgi and Yazdanpanah introduced and studied the concepts of $\mathcal{I}$-weak amenability and ideal amenability for a Banach algebra and showed that every $C^\star$-algebra is ideally amenable. In fact, a Banach algebra $\mathcal{A}$ is $\mathcal{I}$-weakly amenable if $H^1(\mathcal{A}, \mathcal{I}^\star) = \{0\}$ and is ideally amenable if it is $\mathcal{I}$-weakly amenable for every closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$. Obviously, an ideally amenable Banach algebra is weakly amenable and an amenable Banach algebra is ideally amenable. However, there are plenty of known examples of weakly amenable Banach algebras which are neither amenable nor ideally amenable.

In this paper, we consider derivations into annihilators of the closed ideals of Banach algebras. In other words, we introduce two notions of $\frac{\mathcal{A}}{\mathcal{I}}$-weak amenability and quotient ideal amenability for a Banach algebra $\mathcal{A}$, where $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{A}$. We relate these concepts to the weak amenability and the ideal

Akram Teymouri: Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran;
E-mail: teymuri.math@gmail.com
***Corresponding Author: Abasalt Bodaghi:** Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran; E-mail: abasalt.bodaghi@gmail.com
Davood Ebrahimi Bagha: Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran;
E-mail: e_bagha@yahoo.com

amenability of Banach algebras. We also study these new notions of amenability on the projective tensor product of Banach algebras.

2 Quotient ideal amenability

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. Then, $X^\star$ is a Banach $\mathcal{A}$-bimodule by the usual actions. If X is a Banach $\mathcal{A}$-bimodule, then a derivation from $\mathcal{A}$ into X is a linear operator D with

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

The set of all continuous derivations from $\mathcal{A}$ into X is denoted by $Z^1(\mathcal{A}, X)$ which is a linear subspace of $\mathcal{L}(\mathcal{A}, X)$; the space of all bounded linear operators from $\mathcal{A}$ into X. For each $x \in X$, we define a derivation δ_x via

$$\delta_x(a) = a \cdot x - x \cdot a, \qquad (a \in \mathcal{A}).$$

Such derivations are called inner, and the space of inner derivations is denoted by $N^1(\mathcal{A}, X)$. The quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$ is called *the first cohomology group* of $\mathcal{A}$ with coefficients in X.

We say that Y is a dual $\mathcal{A}$-bimodule when for some $\mathcal{A}$-bimodule X, Y is isometrically isomorphic to $X^\star$. It is known that for every closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$, the quotient space $\frac{\mathcal{A}}{\mathcal{I}}$ is a Banach algebra and Banach $\mathcal{A}$-bimodule such that $\left(\frac{\mathcal{A}}{\mathcal{I}}\right)^\star \cong \mathcal{I}^\perp$, where $\mathcal{I}^\perp$ is the annihilator of $\mathcal{I}$. Note that $\mathcal{I}^\perp$ is an $\mathcal{A}$-submodule of $\mathcal{A}^\star$. The last isomorphism leads us to the next definition.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. Then, $\mathcal{A}$ is called $\frac{\mathcal{A}}{\mathcal{I}}$-*weakly amenable* if $H^1(\mathcal{A}, \mathcal{I}^\perp) = \{0\}$. Also, $\mathcal{A}$ is said to be *quotient ideally amenable* if it is $\frac{\mathcal{A}}{\mathcal{I}}$-weakly amenable for every closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$.

Throughout this paper, the subsets $\{0\}$ and $\mathcal{A}$ are considered as the trivial ideals of a Banach algebra $\mathcal{A}$. We have the following trivial observation:

(i) an amenable Banach algebra is quotient ideally amenable;
(ii) a quotient ideally amenable Banach algebra is weakly amenable.

For a Banach algebra $\mathcal{A}$, we denote the space of characters on $\mathcal{A}$ by $\Phi_\mathcal{A}$. Let $\varphi \in \Phi_\mathcal{A} \cup \{0\}$ such that $\varphi = 0$ on $\mathcal{I}$. Then, the set of complex numbers $\mathbb{C}$ is symmetric $\mathcal{A}$-bimodule with the products $a \cdot z = z \cdot a = \varphi(a)z$ $(a \in \mathcal{A}, z \in \mathbb{C})$. We denote $\mathbb{C}$, equipped with this module structure by $\mathbb{C}_{(\varphi,\mathcal{I})}$. Similar to what has been defined in [2], we introduce a point derivation depended on $\mathcal{I}$ and φ as follows.

Definition 2.2. Let $\mathcal{A}$ be a non-zero algebra, and $\varphi \in \Phi_\mathcal{A} \cup \{0\}$ such that $\varphi = 0$ on an ideal $\mathcal{I}$ of $\mathcal{A}$. A linear functional $d : \mathcal{A} \longrightarrow \mathbb{C}_{(\varphi,\mathcal{I})}$ is called a $\mathcal{I}^\perp$-point derivation at φ if

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b) \;\; (a, b \in \mathcal{A}).$$

In the above definition, we note that $\mathcal{I} \subseteq \ker\varphi$. If $\mathcal{I} = \mathcal{A}$, then the only $\mathcal{I}^\perp$-point derivation is zero. Also, in the case that $\mathcal{I} = \ker\varphi = \{0\}$, the dimension of Banach algebra $\mathcal{A}$ should be one. This means that $\mathcal{A} \cong \mathbb{C}$.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. If $\mathcal{A}$ is $\frac{\mathcal{A}}{\mathcal{I}}$-weakly amenable and $\varphi \in \Phi_\mathcal{A}$ with $\varphi|_\mathcal{I} = 0$, then there is no non-zero $\mathcal{I}^\perp$-point derivation at φ.*

Proof. Let $d : \mathcal{A} \longrightarrow \mathbf{C}_{(\varphi,\mathcal{I})}$ be a non-zero $\mathcal{I}^\perp$-point derivation. Consider the map $D : \mathcal{A} \longrightarrow \mathcal{I}^\perp$ defined via $D(a) = d(a)\varphi$. It is easy to check that D is derivation. Since $\mathcal{A}$ is $\frac{\mathcal{A}}{\mathcal{I}}$-weakly amenable, there exists $\lambda \in \mathcal{I}^\perp$ with $D(a) = a \cdot \lambda - \lambda \cdot a \, (a \in \mathcal{A})$. Take $a_1 + \mathcal{I} \in \frac{\mathcal{A}}{\mathcal{I}}$ and $a_2 \in \ker\varphi$ with $\varphi(a_1) = 1$ and $d(a_2) = 1$. Note that $\ker\varphi \neq 0$ since $\varphi = 0$ on $\mathcal{I}$. Set $a_0 = a_1 + (1 - d(a_1))a_2$. Then, $\varphi(a_0) = d(a_0) = 1$ and so $1 = \langle Da_0, a_0 \rangle = \langle a_0 \cdot \lambda, a_0 \rangle - \langle \lambda \cdot a_0, a_0 \rangle = 0$, which is a contradiction. □

Here, we recall the result which is proved in [7, Proposition 2.1.3].

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity, and let X be a Banach $\mathcal{A}$-bimodule such that $\mathcal{A} \cdot X = \{0\}$. Then $H^1(\mathcal{A}, X^*) = \{0\}$.*

The next result is a direct consequence of Proposition 2.4.

Corollary 2.5. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. If $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{A}$, then $H^1(\mathcal{I}, \mathcal{I}^\perp) = \{0\}$.*

Proof. Putting $X = \frac{\mathcal{A}}{\mathcal{I}}$ in Proposition 2.4, we have $\mathcal{I} \cdot \frac{\mathcal{A}}{\mathcal{I}} = \{0\}$, and so one can obtain the desired result. □

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$ that has a bounded approximate identity. Then, $\mathcal{A}$ is $\frac{\mathcal{A}}{\mathcal{I}}$-weakly amenable if and only if $\frac{\mathcal{A}}{\mathcal{I}}$ is weakly amenable.*

Proof. Suppose that $\mathcal{A}$ is $\frac{\mathcal{A}}{\mathcal{I}}$-weakly amenable. Let $D : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \mathcal{I}^\perp$ be a derivation and $\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{I}}$ be the canonical quotient map. Then, $\frac{\mathcal{A}}{\mathcal{I}}$ becomes an $\mathcal{A}$-bimodule with the following actions

$$a \cdot x := \pi(a)x, \quad x \cdot a := x\pi(a) \qquad \left(a \in \mathcal{A}, x \in \frac{\mathcal{A}}{\mathcal{I}}\right).$$

Clearly, $D \circ \pi : \mathcal{A} \to \mathcal{I}^\perp$ is a bounded derivation. By assumption, $D \circ \pi$ is inner, and thus D is inner. Therefore, $\frac{\mathcal{A}}{\mathcal{I}}$ is weakly amenable.

Conversely, assume that $\frac{\mathcal{A}}{\mathcal{I}}$ is weakly amenable and $D : \mathcal{A} \longrightarrow \mathcal{I}^\perp$ is a derivation. Since $\mathcal{I}$ has approximate identity, by Corollary 2.5, $D|_\mathcal{I}$ is inner. Thus, there exists $\phi \in \mathcal{I}^\perp$ such that $D = \delta_\phi$ on $\mathcal{I}$. Obviously, $D - \delta_\phi : \mathcal{A} \to \mathcal{I}^\perp$ is a derivation. Also, $D - \delta_\phi = 0$ on $\mathcal{I}$. So, the mapping $\Delta : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \mathcal{I}^\perp$ defined via $\Delta(a + \mathcal{I}) = (D - \delta_\phi)(a)$ is well-defined and moreover it is derivation. By hypotheses, there exists $\varphi \in \mathcal{I}^\perp$ such that $\Delta = \delta_\varphi$. Hence, $D - \delta_\phi = \Delta \circ \pi = \delta_\varphi \circ \pi$. Therefore, $D = \delta_\psi$ where $\psi = \phi + \varphi$. □

Let $\mathcal{A}$ be a non-unital Banach algebra. Then, $\mathcal{A}^\# = \mathcal{A} \oplus \mathbb{C}$, the unitization of $\mathcal{A}$, is a unital Banach algebra which contains $\mathcal{A}$ as a closed ideal.

Let G be a locally compact group. It is easily verified that the augmentation $L^1_0(G) = \{f \in L^1(G) : \int_G f(G) dm_G(g) = 0\}$ is an ideal of $L^1(G)$. Recall that the special linear group of degree n over a field $\mathbb{F}$ is the set of $n \times n$ matrices with determinant 1 which is denoted by $SL(n, \mathbb{F})$. It is proved in [3, Proposition 1.14] that a Banach algebra $\mathcal{A}$ is ideally amenable if and only if $\mathcal{A}^\#$ is ideally amenable. This result is also valid for the amenability of Banach algebras [1] but does not hold for weak amenability. For example, the augmentation ideal $\mathcal{I}$ of $L^1(SL(2, \mathbb{R}))$ is not weakly amenable while $\mathcal{I}^\#$ is weakly amenable [8]. We remember that similar to the proof of [3, Proposition 1.14], one can show that for a Banach algebra $\mathcal{A}$, if $\mathcal{A}^\#$ quotient ideally amenable, then $\mathcal{A}$ is quotient ideally amenable. However, the next corollary is a direct consequence of Theorem 2.6.

Corollary 2.7. *If $\mathcal{A}$ has a bounded approximate identity, then $\mathcal{A}^\#$ is $\frac{\mathcal{A}^\#}{\mathcal{A}}$- weakly amenable.*

Proof. By Theorem 2.6, we have $H^1(\mathcal{A}^\#, \mathcal{A}^\perp) = H^1\left(\frac{\mathcal{A}^\#}{\mathcal{A}}, \mathcal{A}^\perp\right) \cong H^1(\mathbb{C}, \mathbb{C}) = \{0\}$. □

It is proved in [7, Theorem 2.3.9] that the amenability of a locally compact group G is equivalent to that $L^1_0(G)$ has a bounded approximate identity. Let $G = SL(2, \mathbb{R})$. We know that G is not amenable and $\frac{L^1(G)}{L^1_0(G)} \cong \mathbb{C}$ (see [7, Exercise 1.2.6]). Therefore, $H^1(L^1(G), L^1_0(G)^\perp) = H^1(L^1(G), \mathbb{C}) \neq \{0\}$, and so $L^1(G)$ is not $\frac{L^1(G)}{L^1_0(G)}$-ideally amenable while $L^1(G)$ is weakly amenable.

Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule and Y be a closed $\mathcal{A}$-submodule of X. We say that the short exact sequence $\{0\} \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \longrightarrow \{0\}$ of $\mathcal{A}$-bimodules is *admissible* if π has a bounded right inverse and *splits* if such inverse is also $\mathcal{A}$-bimodule homomorphism. The following theorem is well-known [9, p. 56].

Theorem 2.8. *Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule and Y be a closed $\mathcal{A}$-submodule of X. Then, the following conditions are equivalent:*

(i) The short exact sequence $\{0\} \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \longrightarrow \{0\}$ splits.
(ii) i has a bounded left inverse which is also $\mathcal{A}$-bimodules homomorphism.
(iii) There exists a continuous projection of X onto Y which is also $\mathcal{A}$-bimodules homomorphism.

It is shown in [10, Theorem 2.5] that the concepts of weak amenability and ideal amenability for a Banach algebra $\mathcal{A}$ coincide when for each closed ideal $\mathcal{I}$ of $\mathcal{A}$, one of the following exact sequences of Banach $\mathcal{A}$-bimodules splits

$$\{0\} \to \mathcal{I}^{\perp} \xrightarrow{\pi^{\star}} \mathcal{A}^{\star} \xrightarrow{i^{\star}} \mathcal{I}^{\star} \longrightarrow \{0\}, \tag{2.1}$$

$$\{0\} \to \mathcal{I} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \{0\}. \tag{2.2}$$

So, in this case $\mathcal{A}^{\star} \cong \mathcal{I}^{\perp} \dot{\oplus} \mathcal{I}^{\star}$ where $\dot{\oplus}$ denotes the direct l^{∞}-sum as Banach spaces. We have

$$H^1(\mathcal{A}, \mathcal{A}^{\star}) \cong H^1\left(\mathcal{A}, \mathcal{I}^{\perp}\right) \oplus H^1(\mathcal{A}, \mathcal{I}^{\star}). \tag{2.3}$$

Therefore, for a weakly amenable Banach algebra $\mathcal{A}$, the notions of quotient ideally amenable and ideally amenable are the same when one of the exact sequences Eqs. (2.1) and (2.2) splits. However, for the commutative case, it is proved in [3] that every weakly amenable commutative Banach algebra is ideally amenable, and thus every weakly amenable commutative Banach algebra is quotient ideally amenable provided that one of the exact sequences of Banach $\mathcal{A}$-bimodules Eqs. (2.1) and (2.2) splits. Obviously, Eq. (2.2) is admissible if and only if $\mathcal{I}$ is complemented in $\mathcal{A}$. In other words, Eq. (2.2) splits whenever $\mathcal{I}$ or $\frac{\mathcal{A}}{\mathcal{I}}$ is finite-dimensional. In what follows, we present some examples of weakly amenable Banach algebras which are quotient ideally amenable.

Example 2.9. (i) Let $\mathcal{A}$ be a $C^{\star}$-algebra and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. It is well-known that every $C^{\star}$-algebra is weakly amenable [5]. Also, any ideal $\mathcal{I}$ in a $C^{\star}$-algebra is closed under the adjoint operation [11, Proposition 1.30] and so it is a $C^{\star}$-algebra. Once more, $\frac{\mathcal{A}}{\mathcal{I}}$ is a $C^{\star}$-algebra with the induced norm and involution [11, Proposition 1.31]. Therefore $\mathcal{I}$ has a bounded approximate identity [12]. Thus, $\frac{\mathcal{A}}{\mathcal{I}}$ is weakly amenable. Now, by Theorem 2.6, $\mathcal{A}$ is quotient ideally amenable.

(ii) Consider the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded linear operators on some infinite-dimensional separable Hilbert space $\mathcal{H}$. Then, $\mathcal{A}$ has exactly two non-zero closed ideals $\mathcal{I}_0 = \mathcal{K}(\mathcal{H})$, the compact operators on $\mathcal{H}$, and $\mathcal{I}_1 = \mathcal{B}(\mathcal{H})$. We know that $\mathcal{B}(\mathcal{H})$ is a $C^{\star}$-algebra [13]. Therefore, $H^1(\mathcal{A}, {\mathcal{I}_0}^{\perp}) = \{0\}$. Thus, $\mathcal{A}$ is quotient ideally amenable.

(iii) Let $\mathbb{N}$ be the commutative semigroup of positive integers. Consider $(\mathbb{N}, \vee)$ with maximum operation $m \vee n = max\{m, n\}$, then each element of $\mathbb{N}$ is an idempotent. All ideals of $(\mathbb{N}, \vee)$ are exactly the sets $\mathcal{I}_n = \{m \in \mathbb{N} : m \geq n\}$, and so $l^1(\mathcal{I}_n)$ are ideals of $l^1(\mathbb{N})$. In fact, for any element $f = \sum_{r \in \mathcal{I}_n} \alpha_r \delta_r$ and $g = \sum_{s \in \mathbb{N}} \beta_s \delta_s$, we have

$$f \star g = \left(\sum_{r \in \mathcal{I}_n} \alpha_r \delta_r\right) \left(\sum_{s \in \mathbb{N}} \beta_s \delta_s\right) = \sum_{r \vee s = t \in \mathcal{I}_n} (\alpha_r \beta_s) \delta_t \in l^1(\mathcal{I}_n).$$

Similarly, $g \star f \in l^1(\mathcal{I}_n)$. Since $E(\mathbb{N}) = \mathbb{N}$ and $\mathbb{N}$ is a commutative semigroup with maximum operation, by [14, Proposition 10.5], $l^1(\mathbb{N})$ is weakly amenable, and thus $l^1(\mathbb{N})$ is $l^1(\mathcal{I}_n)$-weakly amenable by [2, Theorem 1.5]. Going back to the case where $\mathcal{A} = l^1(\mathbb{N})$ and $\mathcal{I} = l^1(\mathcal{I}_n)$. Since $\frac{\mathcal{A}}{\mathcal{I}}$ is finite-dimensional, the exact sequences Eq. (2.2) splits and hence the relation (2.3) implies that $l^1(\mathbb{N})$ is quotient ideally amenable.

Let $\mathcal{A}$ be a Banach algebra. It is well-known that $\mathcal{A}$ is weak*-dense in $\mathcal{A}^{\star\star}$, the second dual of $\mathcal{A}$. Suppose that $(a_\alpha), (b_\beta) \subseteq \mathcal{A}$ such that $F = w^{\star} - \lim_\alpha a_\alpha$ and $G = w^{\star} - \lim_\beta b_\beta$. We denote by $\square$ and $\lozenge$ the first and second Arens products on $\mathcal{A}^{\star\star}$, respectively which are defined as follows:

$$F\Box G = w^* - \lim_\alpha w^* - \lim_\beta a_\alpha b_\beta,$$

$$F\Diamond G = w^* - \lim_\beta w^* - \lim_\alpha a_\alpha b_\beta.$$

A Banach algebra $\mathcal{A}$ is said to be *Arens regular* if both Arens products coincide on $\mathcal{A}^{**}$. For the general theory of Arens regularity, see [9].

For a non-weakly amenable Banach algebra $\mathcal{A}$, the concepts of $\frac{\mathcal{A}}{\mathcal{I}}$-weak amenability and $\mathcal{I}$-weak amenability (specially, ideal and quotient ideal amenability) are different. We show this by the examples.

Example 2.10. Let G be a non discrete commutative group. By [15, Corollary 2.2], $(L^1(G)^{**}, \Box)$ is not weakly amenable. Put $\mathcal{J} = \{F \in L^1(G)^{**}; L^1(G)^{**}F = 0\}$ which is a closed two-sided ideal of $L^1(G)^{**}$ having zero product. It is proved in [3] that $H^1(L^1(G)^{**}, \mathcal{J}^*) = \{0\}$. Now, Corollary 1.8 of [3] implies that $H^1(L^1(G)^{**}, \mathcal{J}^\perp) \neq \{0\}$.

Example 2.11. Let $A(G)$ be the Fourier algebra of a locally compact group G. Consider the closed ideal $\mathcal{I}(E) = \{u \in A(G) : u(x) = 0, \ \forall x \in E\}$ of $A(G)$, where E is a (closed) coset ring of G of the form

$$E = \bigcup_{i=1}^{n} \left(a_i H_i \backslash \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j} \right),$$

where $a_i, b_{i,j} \in G, H_i$ is a closed subgroup of G and $K_{i,j}$ is an open subgroup of H_i $(n, m_i \in \mathbb{N}_0, 1 \le i \le n, 1 \le j \le m_i)$. We also suppose that $\mathbb{C}$ be equipped with corresponding $A(G)$-bimodule structure obtained by identification with $\frac{A(G)}{\mathcal{I}(E)}$. It is proved in [16, Theorem 2.3] that if G is amenable, then $\mathcal{I}(E)$ has a bounded approximate identity. By Theorem 2.6, we have

$$H^1(A(G), \mathcal{I}(E)^\perp) = H^1\left(\frac{A(G)}{\mathcal{I}(E)}, \mathcal{I}(E)^\perp\right) = H^1(\mathbb{C}, \mathbb{C}) = \{0\}.$$

Hence, $A(G)$ is $\frac{A(G)}{\mathcal{I}(E)}$-weakly amenable. On the other hand, Johnson [17] showed that $A(G)$ is weakly amenable whenever G is abelian. However, for the non-commutative case, if G is either $SO(3)$, the group of all rotations about the origin of three-dimensional Euclidean space $\mathbb{R}^3$ or $SU(2)$, the Lie group of 2×2 unitary matrices with determinant 1, then $A(G)$ is not weakly amenable. Note that $A(G)$ is not $\mathcal{I}(E)$-weakly amenable by the relation (2.3).

In two upcoming results, we study the hereditary properties of quotient ideal amenability.

Theorem 2.12. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$ with a bounded approximate identity. If $\mathcal{A}$ is quotient ideally amenable, then so is $\mathcal{I}$.*

Proof. The same proof of [3, Theorem 1.9] can be repeated. □

Theorem 2.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $\mathcal{I}, \mathcal{J}$ be closed ideals in $\mathcal{A}, \mathcal{B}$, respectively. If $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous surjective homomorphism such that $\phi(\mathcal{I}) \subseteq \mathcal{J}$ and $H^1(\mathcal{A}, \mathcal{I}^\perp) = \{0\}$, then $H^1(\mathcal{B}, \mathcal{J}^\perp) = \{0\}$.*

Proof. Suppose that $D : \mathcal{B} \longrightarrow (\frac{\mathcal{B}}{\mathcal{J}})^*$ is a derivation. Consider the mapping $\tilde{D} := \tilde{\phi} \circ D \circ \phi : \mathcal{A} \longrightarrow (\frac{\mathcal{A}}{\mathcal{I}})^*$, where $\tilde{\phi} : (\frac{\mathcal{B}}{\mathcal{J}})^* \longrightarrow (\frac{\mathcal{A}}{\mathcal{I}})^*$ is defined by $\langle \tilde{\phi}(f), a + \mathcal{I} \rangle := \langle f, \phi(a) + \mathcal{J} \rangle$, for all $a \in \mathcal{A}$ and $f \in (\frac{\mathcal{B}}{\mathcal{J}})^*$. For each $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}\langle \tilde{\phi} \circ D \circ \phi(ab), c + \mathcal{I} \rangle &= \langle \tilde{\phi}(D(\phi(a)\phi(b)), c + \mathcal{I} \rangle \\ &= \langle \tilde{\phi}(\phi(a)D(\phi(b)) + D(\phi(a))\phi(b)), c + \mathcal{I} \rangle \\ &= \langle \tilde{\phi}(a \cdot D(\phi(b)) + D(\phi(a)) \cdot b), c + \mathcal{I} \rangle \\ &= \langle \tilde{\phi}(a \cdot D(\phi(b))), c + \mathcal{I} \rangle + \langle \tilde{\phi}(D(\phi(a)) \cdot b), c + \mathcal{I} \rangle \\ &= \langle a \cdot D(\phi(b)), \phi(c) + \mathcal{J} \rangle + \langle D(\phi(a)) \cdot b, \phi(c) + \mathcal{J} \rangle \\ &= \langle D(\phi(b)), (\phi(c) + \mathcal{J}) \cdot a \rangle + \langle D(\phi(a)), b \cdot (\phi(c) + \mathcal{J}) \rangle \\ &= \langle D(\phi(b)), (\phi(c)\phi(a) + \mathcal{J}) \rangle + \langle D(\phi(a)), (\phi(b)\phi(c) + \mathcal{J}) \rangle \\ &= \langle D(\phi(b)), (\phi(ca) + \mathcal{J}) \rangle + \langle D(\phi(a)), (\phi(bc) + \mathcal{J}) \rangle \\ &= \langle \tilde{\phi}(D(\phi(b))), ca + \mathcal{I} \rangle + \langle \tilde{\phi}(D(\phi(a))), bc + \mathcal{I} \rangle \\ &= \langle a \cdot \tilde{\phi}(D(\phi(b))), c + \mathcal{I} \rangle + \langle \tilde{\phi}(D(\phi(a))) \cdot b, c + \mathcal{I} \rangle.\end{aligned}$$

Hence, $\tilde{D}$ is a derivation. By assumption, there exists $f \in (\frac{\mathcal{A}}{\mathcal{J}})^{\star}$ such that $\tilde{D}(a) = a \cdot f - f \cdot a$ for all $a \in \mathcal{A}$. Now, it is not hard to show that $\tilde{D}(b) = b \cdot \tilde{f} - \tilde{f} \cdot b$ for all $b \in \mathcal{B}$, where $\tilde{f} = \tilde{\phi}^{-1}(f)$. □

Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Phi_{\mathcal{A}}$. We consider a new product $\bullet_{\varphi}$ on $\mathcal{A}$ as follows:

$$a \bullet_{\varphi} b = \varphi(a)b \qquad (a, b \in \mathcal{A}).$$

It is clear that $\mathcal{A}$ equipped with this product is a Banach algebra and we denote this Banach algebra by $(\mathcal{A}, \bullet_{\varphi})$ or briefly $(\mathcal{A}, \bullet)$ if there is no ambiguity. In the upcoming result, we show that $(\mathcal{A}, \bullet)$ can be $\frac{\mathcal{A}}{\mathcal{J}}$-weakly amenable, where $\mathcal{I}$ is a closed two-sided ideal of $\mathcal{A}$.

Theorem 2.14. *Let $\mathcal{I}$ be a closed two-sided ideal of a Banach algebra $\mathcal{A}$ and φ belongs to $\Phi_{\mathcal{A}}$ such that it vanishes on $\mathcal{I}$. Then, $(\mathcal{A}, \bullet)$ is $\frac{\mathcal{A}}{\mathcal{J}}$-weakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{I}^{\perp}$ be a continuous derivation. For each $a, b, c, \in \mathcal{A}$, we have

$$\begin{aligned}\varphi(a)\langle D(b), c + \mathcal{I} \rangle = \langle D(ab), c + \mathcal{I} \rangle &= \langle a \cdot D(b) + D(a) \cdot b, c + \mathcal{I} \rangle \\ &= \langle D(b), ca + \mathcal{I} \rangle +, \langle D(a), bc + \mathcal{I} \rangle \\ &= \varphi(c)\langle D(b), a + \mathcal{I} \rangle + \varphi(b)\langle D(a), c + \mathcal{I} \rangle.\end{aligned} \tag{2.4}$$

Let $\lambda \in \mathcal{I}^{\perp}$, and $\delta_{\lambda} : \mathcal{A} \longrightarrow \mathcal{I}^{\perp}$ be the inner derivation specified by λ. Then

$$\begin{aligned}\langle \delta_{\lambda}(a), b + \mathcal{I} \rangle = \langle a \cdot \lambda - \lambda \cdot a, b + \mathcal{I} \rangle &= \langle \lambda, ba + \mathcal{I} \rangle - \langle \lambda, ab + \mathcal{I} \rangle \\ &= \varphi(b)\langle \lambda, a + \mathcal{I} \rangle - \varphi(a)\langle \lambda, b + \mathcal{I} \rangle.\end{aligned} \tag{2.5}$$

Choose $a_0 \in \mathcal{A} \backslash \mathcal{I}$ with $\varphi(a_0) = 1$ and set $\lambda(a + \mathcal{I}) = \langle D(a), a_0 + \mathcal{I} \rangle$ for each $a \in \mathcal{A}$. Obviously, λ is a linear functional. Using Eqs. (2.4) and (2.5), we obtain

$$\begin{aligned}\langle \delta_{\lambda}(a), b + \mathcal{I} \rangle &= \varphi(b)\langle \lambda, a + \mathcal{I} \rangle - \varphi(a)\langle \lambda, b + \mathcal{I} \rangle \\ &= \varphi(b)\langle D(a), a_0 + \mathcal{I} \rangle - \varphi(a)\langle D(b), a_0 + \mathcal{I} \rangle \\ &= \varphi(a_0)\langle D(a), b + \mathcal{I} \rangle = \langle D(a), b + \mathcal{I} \rangle.\end{aligned}$$

Therefore, $D = \delta_{\lambda}$, and thus $\mathcal{A}$ is $\frac{\mathcal{A}}{\mathcal{J}}$-weakly amenable. □

Definition 2.15. [18] Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. We say $\mathcal{I}$ has the trace extension property, if every $m \in \mathcal{I}^{\star}$ such that $am = ma$ for each $a \in \mathcal{A}$, can be extended to an $a^{\star} \in \mathcal{A}^{\star}$ such that $aa^{\star} = a^{\star}a$ for every $a \in \mathcal{A}$.

Recall, that in a Banach algebra $\mathcal{A}$, a net $(e_{\alpha})_{\alpha}$ is quasi-central in $\mathcal{A}$ if for each element $a \in \mathcal{A}$; $\lim_{\alpha}(ae_{\alpha} - e_{\alpha}a) = 0$. It is clear that each approximate identity in $\mathcal{A}$ is a quasi-central in $\mathcal{A}$.

Proposition 2.16. *Let $\mathcal{A}$ be a weakly amenable Banach algebra and let $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$ with a bounded approximate identity $(e_\alpha)_\alpha$. Under one the following conditions, $H^1(\mathcal{A}, \mathcal{I}^\perp) = \{0\}$,*

(i) *$\mathcal{I}$ has the trace extension property;*
(ii) *$(e_\alpha)_\alpha$ is a quasi-central in $\mathcal{A}$;*
(iii) *$\mathcal{I}$ or $\mathcal{A}$ is Arens regular.*

Proof. (i) Since $\mathcal{I}$ has the trace extension property and $\mathcal{A}$ is weakly amenable, $\frac{\mathcal{A}}{\mathcal{I}}$ is weakly amenable [19]. Now, the result follows from Theorem 2.6.

(ii) Let J be an ultrafilter on the index set of $(e_\alpha)_\alpha$ such that dominates the order filter. By the proof of [10, Theorem 2.7], one can show that

$$P : \mathcal{A}^* \longrightarrow \mathcal{A}^*, \ \phi \longrightarrow w^* - lim_J(\phi - e_\alpha \cdot \phi)$$

is a projection of $\mathcal{A}^*$ onto $\mathcal{I}^\perp$ which is also a left $\mathcal{A}$-module homomorphism. Consequently, the exact sequence $\{0\} \longrightarrow \mathcal{I}^\perp \xrightarrow{\pi^*} \mathcal{A}^* \xrightarrow{i^*} \mathcal{I}^* \longrightarrow \{0\}$ splits and thus $H^1(\mathcal{A}, \mathcal{I}^\perp) = \{0\}$ by Theorem 2.3.

(iii) By the proof of Theorem 2.8 from [10], $\mathcal{I}$ is quasi-central in $\mathcal{A}$. Now, the part (ii) implies $H^1(\mathcal{A}, \mathcal{I}^\perp) = \{0\}$. □

3 Results on projective tensor products

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. The projective tensor product of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $(\mathcal{A}\hat{\otimes}\mathcal{B}, \|.\|_\pi)$. Each $z \in \mathcal{A}\hat{\otimes}\mathcal{B}$ has a representation $z = \sum_{j=1}^{\infty} a_j \otimes b_j$, where $a_j \in \mathcal{A}$ and $b_j \in \mathcal{B}$ for each $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty$; further, $\|z\|_\pi$ is equal to the infimum of $\sum_{j=1}^{\infty} \|a_j\|\|b_j\|$ over all such representations. $(\mathcal{A}\hat{\otimes}\mathcal{B}, \|\cdot\|_\pi)$ is a Banach algebra and a Banach $\mathcal{A}$-bimodule. For more details, see [9]. In this section, we prove the conditions under which $\mathcal{A}\hat{\otimes}\mathcal{B}$ is quotient ideally amenable.

Let X and Y be Banach spaces. We recall that the space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. Also, we say $X\hat{\otimes}$" *respects subspace isomorphically* if for every subspace G of Y, then $X\hat{\otimes}G$ is subspace of $X\hat{\otimes}Y$.

Definition 3.1. A Banach space Y is called *injective* if for every Banach space X, every subspace $G \subset X$ and every $T \in \mathcal{L}(G, Y)$ there is an extension $\widehat{T} \in \mathcal{L}(X, Y)$ of T.

The next proposition has been presented in [20, pp. 36-37].

Proposition 3.2. *Let X and Y be Banach spaces.*

(i) $X\hat{\otimes}$" respects subspace isomorphically if and only if X^* is an injective Banach space;
(ii) If G is complemented subspace of X, then $G\hat{\otimes}Y$ is a subspace of $X\hat{\otimes}Y$.

Here, we highlight the following lemma which is the key tool to achieve our aim in this section.

Lemma 3.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be closed two-sided ideals in Banach algebras $\mathcal{A}$ and $\mathcal{B}$, respectively.*

(i) If $\mathcal{A}^*$ is injective, then $\mathcal{A}\hat{\otimes}\mathcal{J}$ is a closed two-sided ideal in $\mathcal{A}\hat{\otimes}\mathcal{B}$ and is an $\mathcal{A}\hat{\otimes}\mathcal{B}$-bimodule;
(ii) If $\mathcal{I}$ is complemented in $\mathcal{A}$, then $\mathcal{I}\hat{\otimes}\mathcal{B}$ is a closed two-sided ideal in $\mathcal{A}\hat{\otimes}\mathcal{B}$ and is an $\mathcal{A}\hat{\otimes}\mathcal{B}$-bimodule.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras. Assume that $\varphi \in \Phi_{\mathcal{A}}$, and $\psi \in \Phi_{\mathcal{B}}$. From now on, we consider two Banach algebras $(\mathcal{A}, \bullet_\varphi)$ and $(\mathcal{B}, \bullet_\psi)$.

It is shown in [21, Theorem 3.1] that if $\mathcal{A}$ and $\mathcal{B}$ are ideally amenable Banach algebras with bounded approximate identity, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is ideally amenable. After that, Jabbari in [22, Theorem 2.2] obtained the same result for $(\mathcal{A}, \bullet_\varphi)\hat{\otimes}(\mathcal{B}, \bullet_\psi)$. In fact, there is a gap in the proof of Theorem 2.2 from [22]. In that proof, the author claims in the relation (2.5) that there exists $\lambda \in \mathcal{K}^*$ such that

$$\langle a \otimes b, \lambda \rangle = \langle a \otimes b_0, D(e_{\mathfrak{A}} \otimes b) \rangle \tag{3.1}$$

for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ in which $\psi(b_0) = 1$ (ψ is a character on $\mathfrak{B}$) and $\mathcal{K}$ is an arbitrary closed two-sided ideal of $(\mathfrak{A}, \bullet_\varphi)\hat{\otimes}(\mathfrak{B}, \bullet_\psi)$. We note that λ is defined on $\mathcal{K}$ and so the left side of Eq. (3.1) is meaningless. On the other hand, since $D(e_{\mathfrak{A}} \otimes e_{\mathfrak{B}}) = 0$, it follows from Eq. (3.1) that $\langle a \otimes e_{\mathfrak{B}}, \lambda \rangle = 0$ and so $\langle a \otimes b, \lambda \rangle = 0$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. This contradicts that $\lambda \neq 0$. In view of the proof of [14, Proposition 2.14], we present the quotient version of [22, Theorem 2.2] for $(\mathcal{A}, \bullet_\varphi)\hat{\otimes}(\mathcal{B}, \bullet_\psi)$ as follows. Indeed, the idea of the proof is taken from the proof of Proposition 2.14 from [14]. We include the proof for the sake of completeness.

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras such that $\mathcal{A}^*$ is an injective Banach space, and $\mathcal{J}$ be a closed two-sided ideal of $\mathcal{B}$. If $\psi|_{\mathcal{J}} = 0$, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $\frac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{A}\hat{\otimes}\mathcal{J}}$-weakly amenable.*

Proof. Let $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ be the unit elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. Clearly, by Lemma 3.3, $\mathcal{K} = \mathcal{A}\hat{\otimes}\mathcal{J}$ is closed two-sided ideal of $\mathcal{A}\hat{\otimes}\mathcal{B}$. Assume that $D : \mathcal{A}\hat{\otimes}\mathcal{B} \longrightarrow \mathcal{K}^\perp$ be a continuous derivation. For each $a, c, e \in \mathcal{A}$ and $b, d, f \in \mathcal{B}$, we have

$$\begin{aligned}
\langle D(ae \otimes bf), c \otimes d + \mathcal{K} \rangle &= \langle D((a \otimes b)(e \otimes f)), c \otimes d + \mathcal{K} \rangle \\
&= \langle (a \otimes b) \cdot D(e \otimes f), c \otimes d + \mathcal{K} \rangle + \langle D(a \otimes b) \cdot (e \otimes f), c \otimes d + \mathcal{K} \rangle \\
&= \langle D(e \otimes f), ca \otimes db + \mathcal{K} \rangle + \langle D(a \otimes b), ec \otimes fd + \mathcal{K} \rangle \\
&= \varphi(c)\psi(d)\langle D(e \otimes f), a \otimes b + \mathcal{K} \rangle + \varphi(e)\psi(f)\langle D(a \otimes b), c \otimes d + \mathcal{K} \rangle \\
&= \varphi(a)\psi(b)\langle D(e \otimes f), c \otimes d + \mathcal{K} \rangle.
\end{aligned} \tag{3.2}$$

Fix $b_0 \in \mathcal{B}\backslash\mathcal{J}$ with $\psi(b_0) = 1$. By Eq. (3.2), we obtain

$$\begin{aligned}
\langle D(a \otimes b), e_{\mathcal{A}} \otimes e_{\mathcal{B}} + \mathcal{K} \rangle &= \langle D((a \otimes b_0)(e_{\mathcal{A}} \otimes b)), e_{\mathcal{A}} \otimes e_{\mathcal{B}} + \mathcal{K} \rangle \\
&= \varphi(a)\psi(b_0)\langle D(e_{\mathcal{A}} \otimes b), e_{\mathcal{A}} \otimes e_{\mathcal{B}} + \mathcal{K} \rangle \\
&= \langle D(e_{\mathcal{A}} \otimes b), a \otimes b_0 + \mathcal{K} \rangle
\end{aligned}$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence, there exists $0 \neq \lambda \in \mathcal{K}^\perp$ such that

$$\langle \lambda, a \otimes b + \mathcal{K} \rangle = \langle D(e_{\mathcal{A}} \otimes b), a \otimes b_0 + \mathcal{K} \rangle \tag{3.3}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}\backslash\{e_{\mathcal{B}}\}$. Let $\delta_\lambda : \mathcal{A}\hat{\otimes}\mathcal{B} \longrightarrow \mathcal{K}^\perp$, be an inner derivation specified by λ. Take $a \in \mathcal{A}$ and $b, c \in \mathcal{B}\backslash\{e_{\mathcal{B}}\}$. Then

$$\begin{aligned}
\langle \delta_\lambda(e_{\mathcal{A}} \otimes b), a \otimes c + \mathcal{K} \rangle &= \langle (e_{\mathcal{A}} \otimes b) \cdot \lambda - \lambda \cdot (e_{\mathcal{A}} \otimes b), a \otimes c + \mathcal{K} \rangle \\
&= \langle (e_{\mathcal{A}} \otimes b) \cdot \lambda, a \otimes c + \mathcal{K} \rangle - \langle \lambda \cdot (e_{\mathcal{A}} \otimes b), a \otimes c + \mathcal{K} \rangle \\
&= \langle \lambda, a \otimes cb + \mathcal{K} \rangle - \langle \lambda, a \otimes bc + \mathcal{K} \rangle \\
&= \langle \psi(c)D(e_{\mathcal{A}} \otimes b), a \otimes b_0 + \mathcal{K} \rangle - \langle \psi(b)D(e_{\mathcal{A}} \otimes c), a \otimes b_0 + \mathcal{K} \rangle \\
&= \langle \psi(c)D(e_{\mathcal{A}} \otimes b) - \psi(b)D(e_{\mathcal{A}} \otimes c), a \otimes b_0 + \mathcal{K} \rangle \\
&= \langle (e_{\mathcal{A}} \otimes c) \cdot D(e_{\mathcal{A}} \otimes b), a \otimes b_0 + \mathcal{K} \rangle \\
&= \langle D(e_{\mathcal{A}} \otimes b), a \otimes c + \mathcal{K} \rangle.
\end{aligned}$$

Thus, for every $b \in \mathcal{B}$, we have $D(e_{\mathcal{A}} \otimes b) = \delta_\lambda(e_{\mathcal{A}} \otimes b)$. Similarly, one can obtain for every $a \in \mathcal{A}$ that $D(a \otimes e_{\mathcal{B}}) = \delta_\lambda(a \otimes e_{\mathcal{B}})$. Hence, for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we find

$$\begin{aligned}
D(a \otimes b) &= D((a \otimes e_{\mathcal{B}})(e_{\mathcal{A}} \otimes b)) \\
&= (a \otimes e_{\mathcal{B}}) \cdot D(e_{\mathcal{A}} \otimes b) + D(a \otimes e_{\mathcal{B}}) \cdot (e_{\mathcal{A}} \otimes b) \\
&= (a \otimes b).\lambda - \lambda \cdot (a \otimes b) = \delta_\lambda(a \otimes b).
\end{aligned}$$

So, $D = \delta_\lambda$ on $\mathcal{A}\hat{\otimes}\mathcal{B}$. Therefore, $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $\frac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{K}}$-weakly amenable. □

Remark 3.5. We recall that the relation (3.3) can not hold for $b = e_{\mathcal{B}}$. Suppose contrary to our claim, that (3.3) is true for $b = e_{\mathcal{B}}$. We have

$$\langle \lambda, a \otimes e_{\mathcal{B}} + \mathcal{K} \rangle = \langle D(e_{\mathcal{A}} \otimes e_{\mathcal{B}}), a \otimes b_0 + \mathcal{K} \rangle = 0. \tag{3.4}$$

The relation (3.4) implies that

$$\langle \lambda, a \otimes b + \mathcal{K} \rangle = \langle \lambda, a \otimes \psi(b) e_{\mathcal{B}} + \mathcal{K} \rangle = \psi(b) \langle \lambda, a \otimes e_{\mathcal{B}} + \mathcal{K} \rangle = 0,$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Therefore, $\lambda = 0$ which is a contradiction.

In analogy with Theorem 3.4, we have the following result which shows that $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $\frac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{I}\hat{\otimes}\mathcal{B}}$-weakly amenable, where $\mathcal{I}$ is a closed two-sided ideal of Banach algebra $\mathcal{A}$. Since the proof is similar, it is therefore omitted.

Theorem 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{I}$ be a closed two-sided ideal of $\mathcal{A}$ such that $\mathcal{I}$ is complemented in $\mathcal{A}$. If $\varphi|_{\mathcal{I}} = 0$, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $\frac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{I}\hat{\otimes}\mathcal{B}}$-weakly amenable.*

For the case that $\mathcal{A}$ and $\mathcal{B}$ are non-unital Banach algebras, suppose $\mathcal{A}$ and $\mathcal{B}$ have bounded approximate identity and are both quotient ideally amenable. Then, $\mathcal{A}\hat{\otimes}\mathcal{B}$ has a bounded approximate identity and it is a closed two-sided ideal of $\mathcal{A}^{\#}\hat{\otimes}\mathcal{B}^{\#}$ and by Theorem 2.12, $\mathcal{A}\hat{\otimes}\mathcal{B}$ is $\frac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{A}\hat{\otimes}\mathcal{J}}$-weakly amenable ($\frac{\mathcal{A}\hat{\otimes}\mathcal{B}}{\mathcal{I}\hat{\otimes}\mathcal{B}}$-weakly amenable) if $\mathcal{A}^{\#}\hat{\otimes}\mathcal{B}^{\#}$ is $\frac{\mathcal{A}^{\#}\hat{\otimes}\mathcal{B}^{\#}}{\mathcal{A}^{\#}\hat{\otimes}\mathcal{J}^{\#}}$-weakly amenable ($\frac{\mathcal{A}^{\#}\hat{\otimes}\mathcal{B}^{\#}}{\mathcal{I}^{\#}\hat{\otimes}\mathcal{B}^{\#}}$-weakly amenable).

Acknowledgement: The authors sincerely thank the anonymous reviewers for their careful reading, constructive comments and fruitful suggestions to improve the quality of the first draft.

References

[1] Johnson B. E., Cohomology in Banach algebras, Mem. Amer. Math. Soc., 1972, 127
[2] Bade W. G., Curtis P. G., Dales H. G., Amenability and weak amenability for Beurling and Lipschitz algebras, Proc. London Math. Soc., 1987, 55(3), 359–377
[3] Gorgi M. E., Yazdanpanah T., Derivations into duals of ideals of Banach algebras, Proc. Indian Acad. Sci. (Math. Sci.), 2004, 114(4), 399–408
[4] Minapoor A., Bodaghi A., Ebrahimi Bagha D., Ideal Connes-amenability of dual Banach algebras, Mediterr. J. Math., 2017, 14:174. https://doi.org/10.1007/s00009-017-0970-2
[5] Haagerup U., Laustsen N. J., Weak amenability of C^*-algebras and a theorem of Goldstein, In: Albrecht E., Mathieu M. (Eds.), Banach algebras '97, Walter de Gruyter, Berlin, 1998, 223-244
[6] Haagerup U., All nuclear C^*-algebras are amenable, Invent. Math., 1983, 74, 30–319
[7] Runde V., Lectures on Amenability, Lecture Notes in Math., vol. 1774, Springer-Verlag Berlin Heidelberg, 2002
[8] Johnson B. E., White M. C., A non-weakly amenable augmentation ideal, preprint
[9] Dales H. G., Banach algebra and automatic continuity, Clarendon press, Oxford, 2000
[10] Gordji M. E., Hayati B., Hosseinioun S. A. R., Ideal amenability of Banach algebras and some hereditary properties, J. Sci., Islamic Rep. Iran., 2010, 21(4), 333–341
[11] Roe J., Notes on operator algebras, Fall 2000, http://www.personal.psu.edu/jxr57/520/notes.pdf
[12] Connes A., On the cohomology of operator algebras, J. Funct. Anal., 1978, 28, 243–253
[13] Wassermann S., On tensor products of certain group C^*-algebras, J. Funct. Anal., 1976, 23, 28–36
[14] Dales H. G., Lau A. T.-M., Strauss D., Banach algebras on semigroups and on their compactifications, Mem. Amer. Math. Soc., 2010, 205(966)
[15] Ghahramani F., Loy R. J., Willis G. A., Amenability and weak amenability of second conjugate Banach algebras, Proc. Amer. Math. Soc., 1996, 124, 1489–1497
[16] Forrest B., Kaniuth E., Lau A. T.-M., Spronk N., Ideals with bounded approximate identities in Fourier algebras, J. Funct. Math. Anal., 2003, 203, 286–304
[17] Johnson B. E., Non-amenability of the Fourier algebra of a compact group, J. London Math. Soc., 1994, 50(2), 361–374
[18] Grønbæk N., Weak and cyclic amenable for non-commutative Banach algebra, Proc. Edinburg Math. Soc., 1992, 35, 315–328
[19] Grønbæk N., Weak amenability of group algebras, Bull. London Math. Soc., 1991, 23, 231–284
[20] Defant A., Floret K., Tensor norms and operator ideals, vol. 176, North-Holland Mathematics studies, 1992
[21] Mewomo O. T., On ideal amenability in Banach algebras, Ann. Alex. Ioan Cuza Uni.-Mathematics, 2010, Tomul LVI, 273–278
[22] Jabbari A., On ideal ameability of Banach algebras, J. Math. Phy. Anal. Geom., 2012, 8(2), 135–143