Demonstr. Math. 2019; 52:10–19 **DE GRUYTER**

Research Article **Open Access**

Esma Yıldız Özkan*

Approximation properties of Kantorovich type q-Balázs-Szabados operators

https://doi.org/10.1515/dema-2019-0002
Received July 23, 2018; accepted November 10, 2018

Abstract: In this paper, we introduce a new kind of q-Balázs-Szabados-Kantorovich operators called q-BSK operators. We give a weighted statistical approximation theorem and the rate of convergence of the q-BSK operators. Also, we investigate the local approximation results. Further, we give some comparisons associated with the convergence of q-BSK operators.

Keywords: Balázs-Szabados operators, q-calculus, rate of convergence, Peetre's K-functional

MSC: 41A25, 41A35, 41A36

1 Introduction and auxiliary results

Approximation theory is an important area of research. Recently, several interesting studies have been conducted (see [1–7]). The statistical approximation properties of the some operators have also been recently investigated by several authors. For example, in [8] Meyer-König and Zeller operators based on q-integers; in [9] q-analogues of Bernstein-Kantrovich operators; in [10] q-Bleimann, Butzer and Hahn operators; in [11] q-Baskokov-Kantrovich operators; in [12] Kantrovich type q-Bernstein operators; in [13] q-Stancu-Beta operators; in [14] Kantorovich-type q-Bernstein-Stancu operators were defined and their statistical approximation properties were investigated.

Firstly, we recall some basic definitions used in q-calculus. Details can be found in [15–17]. For any non-negative integer r, the q-integer of the number r is defined by

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q} & if \;\; q \neq 1; \\ r & if \;\; q = 1, \end{cases}$$

where q is a fixed positive real number. The q-factorial is defined by

$$[r]_q! = \begin{cases} [1]_q \, [2]_q \dots [r]_q & if \;\; r = 1, 2, \dots; \\ 1 & if \;\; r = 0. \end{cases}$$

For integers n, r with $0 \leq r \leq n$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ r \end{array} \right]_q = \frac{[n]_q!}{[r]_q! \, [n-r]_q!}.$$

For fixed $0 < q < 1$, we denote the q-derivative of a function f with respect to x

$$D_q \left[f(x) \right] = \begin{cases} \frac{f(qx)-f(x)}{(q-1)z} & if \;\; x \neq 0; \\ f^{'}(0) & if \;\; x = 0, \end{cases}$$

***Corresponding Author: Esma Yıldız Özkan:** Gazi University, Faculty of Science, Department of Mathematics, 06500, Ankara/Turkey; E-mail: esmayildiz@gazi.edu.tr

and we get

$$\lim_{q\to 1} D_q[f(x)] = f'(x).$$

The definite q-integral is defined by

$$\int_0^b f(t)\,d_qt = (1-q)\,b\sum_{j=0}^{\infty} f\left(q^j b\right) q^j, \quad 0<q<1,\ b>0,$$

and

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt, \quad 0<a<b.$$

Bernstein-type rational functions are defined by Balázs [18]. Balázs and Szabados modified and studied the approximation properties of these operators [19]. The q-analogue of the Balázs-Szabados operators is defined by Doğru [20] as follows

$$R_n(f;q,x) = \frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n x)}\sum_{j=0}^{n} q^{j(j-1)/2} f\left(\frac{[j]_q}{b_n}\right)\begin{bmatrix} n \\ j\end{bmatrix}_q (a_n x)^j, \tag{1.1}$$

where $x\in[0,\infty)$, $a_n = [n]_q^{\beta-1}$ and $b_n = [n]_q^{\beta}$ for all $n\in\mathbb{N}$ and $0<\beta\le\frac{2}{3}$. Also, Doğru gave the following equalities

$$R_n(e_0;q,x) = 1, \tag{1.2}$$

$$R_n(e_1;q,x) = \frac{x}{1+a_n x}, \tag{1.3}$$

$$R_n(e_2;q,x) = \frac{[n-1]_q}{[n]_q}\frac{q^2x^2}{(1+a_nx)(1+a_nqx)} + \frac{x}{b_n(1+a_nx)}. \tag{1.4}$$

In (1.4), using the equality $[n]_q = q\,[n-1]_q + 1$, we get

$$R_n(e_2;q,x) = \frac{\left(1-\frac{a_n}{b_n}\right)qx^2}{(1+a_nx)(1+a_nqx)} + \frac{x}{b_n(1+a_nx)}. \tag{1.5}$$

We will use (1.5) instead of (1.4) throughout this paper.

In [21], a kind of real and complex q-Balázs-Szabados-Kantorovich operators were defined, and it was given an upper estimate on compact disks.

Now, we give the following new kinds of q-Balázs-Szabados-Kantorovich operators:

Definition 1. *A new kind of q-Balázs-Szabados-Kantorovich operators is defined as follows:*

$$\tilde{R}_n(f;q,x) = \frac{b_n}{\prod_{s=0}^{n-1}(1+q^s a_n x)}\sum_{j=0}^{n} q^{j(j-1)/2}\begin{bmatrix} n \\ j\end{bmatrix}_q (a_n x)^j \int_{\frac{q[j]_q}{b_n}}^{\frac{q[j+1]_q}{b_n}} f(t)\,d_qt, \tag{1.6}$$

where f is a nondecreasing and continuous function on $[0,\infty)$, $a_n = [n]_q^{\beta-1}$ *and* $b_n = [n]_q^{\beta}$ *for all* $n\in\mathbb{N}$, $q\in(0,1)$ *and* $0<\beta\le\frac{2}{3}$.

q-Balázs-Szabados-Kantorovich operators can be called as q-BSK operators for convenience. Since f is nondecreasing and from the definition of q-integral, q-BSK operator is a positive operator. And also, q-BSK operator is linear, so q-BSK operator is a linear and positive operator.

We have the following lemma for the q-BSK operators.

Lemma 1. *The following equalities hold for the q-BSK operators*

$$\tilde{R}_n(e_0;q,x)=1, \tag{1.7}$$

$$\tilde{R}_n(e_1;q,x)=\frac{2qx}{[2]_q(1+a_nx)}+\frac{1}{[2]_q b_n}, \tag{1.8}$$

$$\tilde{R}_n(e_2;q,x) = \frac{\left(1-\frac{a_n}{b_n}\right)3q^3x^2}{[3]_q(1+a_nx)(1+a_nqx)}+\frac{3q[2]_q x}{[3]_q b_n(1+a_nx)}+\frac{1}{[3]_q b_n^2}. \tag{1.9}$$

Proof. Using (1.2), (1.3), (1.5) and the equality $[n]_q=q[n-1]_q+1$, we have

$$\int_{\frac{q[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} d_qt=\frac{1}{b_n}, \tag{1.10}$$

$$\int_{\frac{q[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t d_qt=\frac{1}{[2]_q b_n^2}\left(2q[j]_q+1\right), \tag{1.11}$$

$$\int_{\frac{q[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^2 d_qt=\frac{1}{[3]_q b_n^3}\left(3q^2[j]_q^2+3q[j]_q+1\right). \tag{1.12}$$

Using (1.2) and (1.10), we get

$$\tilde{R}_n(e_0;q,x)=1.$$

Similarly, using (1.3) and (1.11), we obtain

$$\begin{aligned}\tilde{R}_n(e_1;q,x) &= \frac{1}{[2]_q b_n}R_n(e_1;q,x)+\frac{2q}{[2]_q}R_n(e_0;q,x)\\ &= \frac{2qx}{1+a_nx}+\frac{1}{[2]_q b_n}.\end{aligned}$$

Finally, from (1.2), (1.3), (1.5) and (1.12), we find that

$$\begin{aligned}\tilde{R}_n(e_2;q,x) &= \frac{3q^2}{[3]_q}R_n(e_2;q,x)+\frac{3q}{[3]_q b_n}R_n(e_1;q,x)+\frac{1}{[3]_q b_n^2}R_n(e_0;q,x)\\ &= \frac{\left(1-\frac{a_n}{b_n}\right)3q^3x^2}{[3]_q(1+a_nx)(1+a_nqx)}+\frac{3qx}{[3]_q b_n(1+a_nx)}+\frac{1}{[3]_q b_n^2}.\end{aligned}$$

□

Lemma 2. *The following equalities hold for the q-BSK operators*

$$\tilde{R}_n((e_1-x);q,x) = -\frac{a_nx^2}{1+a_nx}-\frac{(1-q)x}{[2]_q(1+a_nx)}+\frac{1}{[2]_q b_n}, \tag{1.13}$$

$$\tilde{R}_n\left(\left(e_2-x^2\right);q,x\right) = -\frac{a_n^2qx^4+a_n[2]_q x^3}{(1+a_nx)(1+a_nqx)}+\frac{\left[3q^3-[3]_q-3q^3\frac{a_n}{b_n}\right]x^2}{[3]_q(1+a_nx)(1+a_nqx)}+\frac{3q[2]_q x}{[3]_q b_n(1+a_nx)}+\frac{1}{[3]_q b_n^2}, \tag{1.14}$$

$$\tilde{R}_n\left((e_1-x)^2;q,x\right) = \frac{a_n^2 q x^4}{(1+a_n x)(1+a_n q x)} + \frac{a_n(1-q)\left(2q+[2]_q\right)x^3}{[2]_q(1+a_n x)(1+a_n q x)} - \frac{(1-q)3q^2x^2}{[3]_q(1+a_n x)(1+a_n q x)}$$
$$+\frac{\left[(1-q)^2-3q^3\frac{a_n}{b_n}\right]x^2}{[2]_q[3]_q(1+a_n x)(1+a_n q x)} + \frac{3q[2]_q x}{[3]_q b_n(1+a_n x)} - \frac{2x}{[2]_q b_n} + \frac{1}{[3]_q b_n^2}. \quad (1.15)$$

Proof. Using Lemma 2, after a simple calculation, the proof can be obtained easily, so we omit it. □

2 Weighted statistical approximation properties

The concept of the statistical convergence was introduced by H. Fast [22]. We recall some definitions about the statistical convergence. The density of a set $K \subset \mathbb{N}$ is defined by

$$\delta\{k \le n : k \in K\}.$$

The natural density, δ, of a set $K \subset \mathbb{N}$ is defined by

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k \le n : k \in K\}\right|,$$

provided the limits exist [23]. A sequence $x = (x_k)$ is called statistically convergent to a number L if

$$\delta\{k : |x_k - L| \ge \varepsilon\} = 0$$

for every $\varepsilon > 0$ and it is denoted as $st - \lim_k x_k = L$. Any convergent sequence is statistically convergent but not conversely (see [20]). A real function ρ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x|\to\infty}\rho(x) = \infty$, $\rho(x) \ge 1$ for all $x \in \mathbb{R}$.

Let denote by $B_{\rho_0}(\mathbb{R}_+)$ the weighted space of the real valued functions f defined on $\mathbb{R}_+$ satisfying $|f(x)| \le M_f\rho_0(x)$ for all $x \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$, $\rho_0(x) = 1 + x^2$ and M_f is a constant depending on the function f. We also denote with $C_{\rho_0}(\mathbb{R}_+) := \{f \in B_{\rho_0}(\mathbb{R}_+) : f \text{ continuous on } \mathbb{R}_+\}$ the weighted subspace of $B_{\rho_0}(\mathbb{R}_+)$. $B_{\rho_0}(\mathbb{R}_+)$ and $C_{\rho_0}(\mathbb{R}_+)$ are Banach spaces with $\|f\|_{\rho_0} = \sup_{x\in\mathbb{R}_+}\frac{|f(x)|}{\rho_0(x)}$.

Let $q = (q_n)$ a sequence satisfying

$$st - \lim_n q_n = 1 \quad \text{and} \quad st - \lim_n q_n^n = c, \quad (0 \le c < 1). \quad (2.1)$$

Now we are ready to prove the following convergence theorems for the q-BSK operators:

Theorem 1. *Let $q = (q_n)$ be a sequence satisfying the conditions given as in (2.1). If f is a nondecreasing function in $C_{\rho_0}(\mathbb{R}_+)$, then it holds for the q-BSK operators*

$$st - \lim_n \left\|\tilde{R}_n(f;q_n,.) - f\right\|_{\rho_0} = 0.$$

Proof. From (1.7) in Lemma 1, it is clear that

$$st - \lim_n \left\|\tilde{R}_n(e_0;q_n,.) - e_0\right\|_{\rho_0} = 0. \quad (2.2)$$

Using (1.13) in Lemma 2, we get

$$\left\|\tilde{R}_n(e_1;q_n,.) - e_1\right\|_{\rho_0} \le a_n\|e_2\|_{\rho_0} + \frac{1-q_n}{[2]_{q_n}}\|e_1\|_{\rho_0} + \frac{1}{[2]_{q_n}b_n}.$$

For a given $\varepsilon > 0$, let us define the following sets:

$$A := \left\{k : \left\|\tilde{R}_k(e_1;q_k,.) - e_1\right\|_{\rho_0} \ge \varepsilon\right\},$$
$$A_1 := \left\{k : \alpha_k \ge \frac{\varepsilon}{3}\right\},$$
$$A_2 := \left\{k : \zeta_k \ge \frac{\varepsilon}{3}\right\},$$
$$A_3 := \left\{k : \gamma_k \ge \frac{\varepsilon}{3}\right\}$$

with $\alpha_k = a_k \|e_2\|_{\rho_0}$, $\zeta_k = \frac{1-q_k}{[2]_{q_k}} \|e_1\|_{\rho_0}$ and $\gamma_k = \frac{1}{[2]_{q_k} b_k}$. Since $A \subseteq A_1 \cup A_2 \cup A_3$, we get

$$\delta\left\{k \le n : \left\|\tilde{R}_n(e_1; q_k, .) - e_1\right\|_{\rho_0} \ge \varepsilon\right\} \le \delta\left\{k \le n : \alpha_k \ge \frac{\varepsilon}{3}\right\} + \delta\left\{k \le n : \zeta_k \ge \frac{\varepsilon}{3}\right\} + \delta\left\{k \le n : \gamma_k \ge \frac{\varepsilon}{3}\right\}.$$

Under the conditions given in (2.1), it is clear that

$$st - \lim_n \alpha_n = st - \lim_n \zeta_n = st - \lim_n \gamma_n = 0,$$

which implies

$$st - \lim_n \left\|\tilde{R}_n(e_1; q_n, .) - e_1\right\|_{\rho_0} = 0 \tag{2.3}$$

Using (1.14) in Lemma 2, we can write

$$\begin{aligned}\left\|\tilde{R}_n(e_2; q_n, .) - e_2\right\|_{\rho_0} \le\ & a_n^2 q_n \|e_4\|_{\rho_0} + a_n [2]_{q_n} \|e_3\|_{\rho_0} + \frac{3q_n^3 - [3]_{q_n} - 3q_n^3 \frac{a_n}{b_n}}{[3]_{q_n}} \|e_2\|_{\rho_0} + \frac{3q_n}{[2]_{q_n}} \|e_1\|_{\rho_0} \\ & + \frac{1}{b_n^2 [3]_{q_n}} \|e_0\|_{\rho_0}.\end{aligned}$$

Again for a given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned} B &: = \left\{k : \left\|\tilde{R}_k(e_2; q_k, .) - e_2\right\|_{\rho_0} \ge \varepsilon\right\}, \\ B_1 &: = \left\{k : \nu_k \ge \frac{\varepsilon}{5}\right\},\ B_2 := \left\{k : \varphi_k \ge \frac{\varepsilon}{5}\right\}, \\ B_3 &: = \left\{k : \eta_k \ge \frac{\varepsilon}{5}\right\},\ B_4 := \left\{k : \mu_k \ge \frac{\varepsilon}{5}\right\}, \\ B_5 &: = \left\{k : \sigma_k \ge \frac{\varepsilon}{5}\right\}\end{aligned}$$

by choosing $\nu_k = a_k^2 q_k \|e_4\|_{\rho_0}$, $\varphi_k = a_k [2]_{q_k} \|e_3\|_{\rho_0}$, $\eta_k = \frac{3q_k^3 - [3]_{q_k} - 3q_k^3 \frac{a_k}{b_k}}{[3]_{q_k}} \|e_2\|_{\rho_0}$, $\mu_k = \frac{3q_k}{[2]_{q_k}} \|e_1\|_{\rho_0}$ and $\sigma_k = \frac{1}{b_k^2 [3]_{q_k}} \|e_0\|_{\rho_0}$ .Since $B \subseteq B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$, we have

$$\begin{aligned}\delta\left\{k \le n : \left\|\tilde{R}_k(e_2; q_k, .) - e_2\right\|_{\rho_0} \ge \varepsilon\right\} \le & \delta\left\{k \le n : \nu_k \ge \frac{\varepsilon}{5}\right\} + \delta\left\{k \le n : \varphi_k \ge \frac{\varepsilon}{5}\right\} \\ & + \delta\left\{k \le n : \eta_k \ge \frac{\varepsilon}{5}\right\} + \delta\left\{k \le n : \mu_k \ge \frac{\varepsilon}{5}\right\} \\ & + \delta\left\{k \le n : \sigma_k \ge \frac{\varepsilon}{5}\right\}.\end{aligned}$$

Under the conditions given in (2.1), we have

$$\begin{aligned} st - \lim_n \upsilon_n &= st - \lim_n \varphi_n = 0, \\ st - \lim_n \eta_n &= 0, \\ st - \lim_n \mu_n &= st - \lim_n \sigma_n = 0. \end{aligned} \tag{2.4}$$

From (2.4), we obtain

$$st - \lim_n \left\|\tilde{R}_n(e_2; q_n, .) - e_2\right\|_{\rho_0} = 0. \tag{2.5}$$

Since

$$\left\|\tilde{R}_n(f; q_n, .) - f\right\|_{\rho_0} \le \left\|\tilde{R}_n(e_0; q_n, .) - e_0\right\|_{\rho_0} + \left\|\tilde{R}_n(e_1; q_n, .) - e_1\right\|_{\rho_0} + \left\|\tilde{R}_n(e_2; q_n, .) - e_2\right\|_{\rho_0},$$

from (2.2), (2.3) and (2.5), the proof of theorem is completed. □

In this part, we give the rates of convergence of the q-BSK operators by means of the weighted modulus of smoothness

The weighted modulus of smoothness for the functions f in $B_{\rho_0}(\mathbb{R}_+)$ is defined as

$$\Omega_{\rho_0}(f;\delta)=\sup_{x\geq 0,0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2} \tag{2.6}$$

for $\delta>0$. It is clear that for each f in $B_{\rho_0}(\mathbb{R}_+)$, $\Omega_{\rho_0}(f;.)$ is well-defined and satisfies the following properties (see [24])

$$\begin{aligned}\Omega_{\rho_0}(f;\lambda\delta)&\leq(\lambda+1)\,\Omega_{\rho_0}(f;\delta)\,,\ \ \delta>0, \\ \Omega_{\rho_0}(f;n\delta)&\leq n\Omega_{\rho_0}(f;\delta)\,,\ \ \delta>0,\ n\in\mathbb{N}, \\ \Omega_{\rho_0}(f;\delta)&\leq 2\,\|f\|_{\rho_0}\,,\ \delta>0,\ f\in B_{\rho_0}(\mathbb{R}_+)\,, \\ \lim_{\delta\to 0^+}\Omega_{\rho_0}(f;\delta)&=0,\end{aligned} \tag{2.7}$$

We give the following rate of convergence for the q-BSK operators.

Theorem 2. *Let $q=(q_n)$ be a sequence satisfying the conditions given as in (2.1). For all nondecreasing functions f in $B_{\rho_0}(\mathbb{R}_+)$, we have*

$$\left|\tilde{R}_n(f;q_n,x)-f\right|\leq 2\sqrt{\tilde{R}_n\left(\kappa_x^2(t);q_n,x\right)}\,\Omega_{\rho_0}(f;\mu_n(x))\,,$$

$x\geq 0$, $\delta>0$, $n\in\mathbb{N}$, *where* $\kappa_x(t):=1+(x+|t-x|)^2$ *for* $t\geq 0$ *and* $\mu_n(x)=\left(\tilde{R}_n\left((e_1-x)^2;q_n,x\right)\right)^{1/2}$.

Proof. Let $n\in\mathbb{N}$ and $f\in B_{\rho_0}(\mathbb{R}_+)$. From (2.6) and (2.7), we can write

$$\begin{aligned}|f(t)-f(x)| &\leq \left(1+(x+|t-x|)^2\right)\left(1+\frac{1}{\delta}|t-x|\right)\Omega_{\rho_0}(f;\delta) \\ &= \kappa_x(t)\left(1+\frac{1}{\delta}|t-x|\right)\Omega_{\rho_0}(f;\delta).\end{aligned} \tag{2.8}$$

From the linearity and positivity of the q-BSK operators and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\tilde{R}_n(f;q_n,.)-f\right| &\leq \tilde{R}_n(|f(t)-f(x)|;q_n,x) \\ &\leq \left\{\tilde{R}_n(\kappa_x(t);q_n,x)+\frac{1}{\delta}\tilde{R}_n(|t-x|;q_n,x)\right\}\Omega_{\rho_0}(f;\delta) \\ &\leq \sqrt{\tilde{R}_n\left(\kappa_x^2(t);q_n,x\right)}\left(1+\frac{1}{\delta}\sqrt{\tilde{R}_n\left((t-x)^2;q_n,x\right)}\right)\Omega_{\rho_0}(f;\delta)\end{aligned}$$

Finally, choosing $\delta=\mu_n(x)$, the proof is completed. □

3 Local approximation

Let $C_B(\mathbb{R}_+)$ be the space of all real valued continuous bounded functions defined on $\mathbb{R}_+$. The norm on the space $C_B(\mathbb{R}_+)$ is the supremum norm $\|f\|=\sup\{|f(x)|:x\in\mathbb{R}_+\}$. Also, Peetre's K-functional is defined $K_2(f,\delta)=\inf_{g\in W^2}\left\{\|f-g\|+\delta\left\|g''\right\|\right\}$, where $W^2=\left\{g\in C_B(\mathbb{R}_+):g',g''\in C_B(\mathbb{R}_+)\right\}$.

By [25] (p.117), there exists a positive constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2\left(f,\delta^{1/2}\right),\delta>0,$$

where

$$\omega_2\left(f,\delta^{1/2}\right)=\sup\left\{|f(x+2h)-2f(x+h)+f(x)|:x\in\mathbb{R}_+,0<|h|<\delta^{1/2}\right\}$$

is the second order modulus of continuity of functions f in $C_B[0,\infty)$. Further, the usual modulus of continuity is defined by

$$\omega\left(f,\delta^{1/2}\right)=\sup\left\{|f(x+h)-f(x)|:x\in\mathbb{R}_+,\ \ 0<|h|<\delta^{1/2}\right\}.$$

Now, we give local results for the q-BSK operators.

Theorem 3. *Let $q=(q_n)$ be a sequence satisfying the conditions given as in (2.1) and $f\in C_B(\mathbb{R}_+)$. Then for all $n\in\mathbb{N}$, there exists a positive constant $C>0$ such that*

$$\left|\tilde{R}_n(f;q_n,x)-f(x)\right|\leq C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega(f,\alpha_n(x)),$$

where $\delta_n(x)=\tilde{R}_n\left((t-x)^2;q_n,x\right)+\left(\tilde{R}_n((t-x);q_n,x)\right)^2$, $\alpha_n(x)=\left|\tilde{R}_n((t-x);q_n,x)\right|$, where $\tilde{R}_n((t-x);q_n,x)$ and $\tilde{R}_n\left((t-x)^2;q_n,x\right)$ are given as in and (1.13) and (1.15).

Proof. For $x\in\mathbb{R}_+$, we introduce the auxiliary operator as follows:

$$\check{R}_n(f;q_n,x)=\tilde{R}_n(f;q_n,x)+f(x)-f(\xi_n(x)), \tag{3.1}$$

where $\xi_n(x)=x-\frac{a_nx^2}{1+a_nx}+\frac{(1-q_n)x}{[2]_{q_n}(1+a_nx)}+\frac{1}{[2]_{q_n}b_n}$. Using (3.1), we have

$$\check{R}_n((t-x);q_n,x)=\tilde{R}_n((t-x);q_n,x)+f(x)-f(\xi_n(x))=0.$$

Let $x\in\mathbb{R}_+$ and $g\in W^2$. Using the Taylor formula

$$g(t)=g(x)+g^{'}(x)(t-x)+\int_x^t(t-u)g^{''}(u)\,du.$$

Applying $\check{R}_n$ to both sides of the above equation, we obtain

$$\begin{aligned}\check{R}_n(g(t);q_n,x)-g(x) &= \check{R}_n\left((t-x)g^{'}(x);q_n,x\right)+\check{R}_n\left(\int_x^t(t-u)g^{''}(u)\,du;q_n,x\right)\\ &= g^{'}(x)\check{R}_n((t-x);q_n,x)\\ &\quad+\tilde{R}_n\left(\int_x^t(t-u)g^{''}(u)\,du;q_n,x\right)-\int_x^{\xi_n(x)}(\xi_n(x)-u)g^{''}(u)\,du\\ &= \tilde{R}_n\left(\int_x^t(t-u)g^{''}(u)\,du;q_n,x\right)-\int_x^{\xi_n(x)}(\xi_n(x)-u)g^{''}(u)\,du.\end{aligned}$$

On the other hand,

$$\left|\int_x^t(t-u)g^{''}(u)\,du\right|\leq\int_x^t|t-u|\left|g^{''}(u)\right|du\leq\left\|g^{''}\right\|\int_x^t|t-u|\,du\leq\left\|g^{''}\right\|(t-u)^2$$

and

$$\left|\int_x^{\xi_n(x)}(\xi_n(x)-u)g^{''}(u)\,du\right|\leq\left\|g^{''}\right\|(\xi_n(x)-x)^2=\left\|g^{''}\right\|\left(\tilde{R}_n((t-x);q_n,x)\right)^2,$$

which implies

$$\begin{aligned}\left|\check{R}_n(g;q_n,x)-g(x)\right| &\leq \left|\tilde{R}_n\left(\int_x^t(t-u)g^{''}(u)\,du;q_n,x\right)\right|+\left|\int_x^{\xi_n(x)}(\xi_n(x)-u)g^{''}(u)\,du\right|\\ &\leq \left\|g^{''}\right\|\left\{\tilde{R}_n\left((t-x)^2;q_n,x\right)+\left(\tilde{R}_n((t-x);q_n,x)\right)^2\right\}\\ &\leq \left\|g^{''}\right\|\delta_n(x).\end{aligned}$$

Considering $\tilde{R}_n(1; q_n, x) = 1$, we have also

$$\left|\breve{R}_n(f; q_n, x)\right| \leq \left|\tilde{R}_n(f; q_n, x)\right| + |f(x)| + |f(\xi_n(x))| \leq \tilde{R}_n(|f|; q_n, x) + 2\|f\| \leq 3\|f\|.$$

Therefore

$$\begin{aligned}\left|\tilde{R}_n(f; q_n, x) - f(x)\right| &\leq \left|\breve{R}_n((f-g); q_n, x) - (f-g)(x)\right| + |f(\xi_n(x)) - f(x)| + \left|\breve{R}_n(g; q_n, x) - g(x)\right| \\ &\leq 4\|f-g\| + \omega(f; \alpha_n(x)) + \left\|g''\right\| \delta_n(x).\end{aligned}$$

Taking infimum on the right hand side over all $g \in W^2$, we obtain

$$\left|\tilde{R}_n(f; q_n, x) - f(x)\right| \leq 4K_2(f, \delta_n(x)) + \omega(f; \alpha_n(x)).$$

By the inequality $K_2(f, \delta) \leq C\omega_2\left(f, \delta^{1/2}\right)$ for $\delta > 0$, we obtain the desired result. □

Let E be any subset of $\mathbb{R}_+$ and $\alpha \in (0, 1]$. Then $Lip_{M_f}(E, \alpha)$ denotes the space of functions f in $C_B(\mathbb{R}_+)$ satisfying the condition

$$|f(t) - f(x)| \leq M_f |t-x|^\alpha, \quad t \in \overline{E} \text{ and } x \in [0, \infty),$$

where M_f is a constant depending on f and $\overline{E}$ denotes the closure of E in $\mathbb{R}_+$.

Theorem 4. *Let $q = (q_n)$ be a sequence satisfying the conditions given as in (2.1) and the function f in $C_B[0, \infty) \cap Lip_{M_f}(E, \alpha)$, $\alpha \in (0, 1]$ and E be any bounded subset of $\mathbb{R}_+$. Then for each $x \in \mathbb{R}_+$, we have*

$$\left|\tilde{R}_n(f; q_n, x) - f(x)\right| \leq M_f\left\{(\mu_n(x))^\alpha + 2(d(x, E))^\alpha\right\},$$

where $\mu_n(x) = \left(\tilde{R}_n\left((t-x)^2; q_n, x\right)\right)^{1/2}$ and M_f is a constant depending on f, and $d(x, E)$ is a distance between points x and E, that is $d(x, E) = \inf\{|t-x| : t \in E\}$.

Proof. Let E be the closure of the set E. Then there exists an $x_0 \in E$ such that $|x - x_0| = d(x, E)$, where $x \in \mathbb{R}_+$. Thus we can write

$$\begin{aligned}\left|\tilde{R}_n(f; q_n, x) - f(x)\right| &\leq \tilde{R}_n(|f(t) - f(x_0)|; q_n, x) + \tilde{R}_n(|f(x_0) - f(x)|; q_n, x) \\ &\leq M_f\left(\tilde{R}_n\left(|t-x_0|^\alpha; q_n, x\right) + |x-x_0|^\alpha\right) \\ &\leq M_f\left(\tilde{R}_n\left(|t-x|^\alpha; q_n, x\right) + 2|x-x_0|^\alpha\right).\end{aligned}$$

Using the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we obtain

$$\begin{aligned}\left|\tilde{R}_n(f; q_n, x) - f(x)\right| &\leq M_f\left\{\tilde{R}_n\left(|t-x|^{\alpha p}; q_n, x\right)\right\}^{1/p}\left[\left\{\tilde{R}_n(1; q_n, x)\right\}^{1/q} + 2(d(x, E))^\alpha\right] \\ &\leq M_f\left[\left(\mu_n(x)^\alpha + 2(d(x, E))^\alpha\right)\right],\end{aligned}$$

which completes the proof. □

Remark 1. *Let $q = (q_n)$ be a sequence satisfying the conditions given as in (2.1), then*

$$st - \lim_n a_n = st - \lim_n \frac{1}{b_n} = 0,$$

so the results in weighted space give us the weighted approximation degree of the q-BSK operators to f, and also the results of local approximation give us the approximation degree of the q-BSK operators to f.

We give an illustrative example which shows the rate of convergence of the operators q-BSK to certain functions in the following example:

Example 1. *In case of $\beta = 0.5$, the convergence of the q-BSK operators to $f(x) = x + \sin(3x)$ is illustrated in Figure 1 and Figure 2 according to increasing values of n and q, respectively.*

The figures clearly show that, for increasing values of n and q, the degree of approximation becomes better.

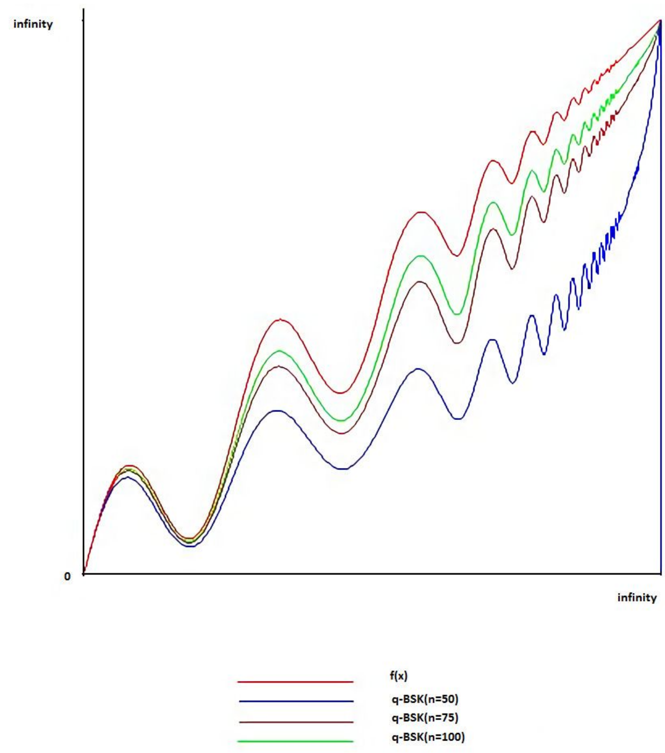

Figure 1: The convergence of the operators $\tilde{R}_n(f; 0.95, x)$ to $f(x)$ for increasing values of n.

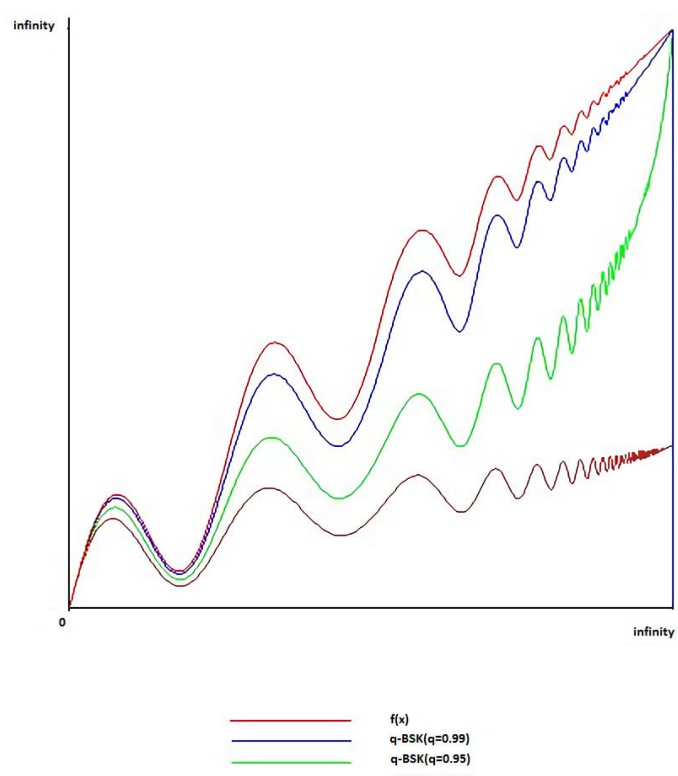

Figure 2: The convergence of the operators $\tilde{R}_{50}(f; q, x)$ to $f(x)$ for increasing values of q.

References

[1] Acar T., Quantitative q-Voronovskaya, q-Grüss-Voronovskaya-type results for q-Szasz operators, G. Math. J., 2016, 23(4), 459-468

[2] Acar T., (p,q)-Generalization of Szasz-Mirakyan operators, Math. Methods in the Appl. Sciences, Math. Methods in the Appl. Sciences, 2016, 39(10), 2685-2695

[3] Acar T., Aral A., Mohiuddine S. A., On Kantorovich modification of (p,q)-Baskakov operators, J. Inequal. Appl., 2016, 98

[4] Acar T., Mohiuddine S. A., Mursaleen M., Approximation by (p,q)-Baskakov-Durrmeyer-Stancu operators, Complex Anal. Oper. Theory, 2018, 12(6), 1453-1468

[5] Mohiuddine S. A., Acar T., Alotaibi A., Construction of a new family of Bernstein-Kantorovich operators, Math. Meth. Appl. Sci., 2017, 40(18), 7749-7759

[6] Mohiuddine S. A., Acar T., Alghamdi A., Genuine modified Bernstein-Durrmeyer operators, J. Inequal. Appl., 2018, 2018:104

[7] Bodur M., Yılmaz Ö. G., Aral A., Approximation by Baskakov-Szász-Stancu Operators Preserving Exponential Functions, Constructive Mathematical Analysis, 2018, 1(1), 1-8

[8] Doğru O., Duman O., Statistical approximation of Meyer-König and Zeller operators based on q-integers, Publ. Math. Debrecen, 2006, 68(1-2), 199-214

[9] Radu C., Statistical approximation properties of Kantorovich operators based on q-integers, Creative Math. Inf., 2008, 17(2), 75-84

[10] Ersan S., Doğru O., Statistical approximation properties of q-Bleimann, Butzer and Hahn operators, Math. Comput. Modelling, 2009, 49(7-8), 1595-1606

[11] Gupta V., Radu C., Statistical approximation properties of q-Baskokov-Kantrovich operators, Cent. Eur. J. Math., 2009, 7(4), 809-818

[12] Dalmanoğlu Ö., Doğru O., On statistical approximation properties of Kantrovich type q-Bernstein operators, Math. Comput. Modelling, 2010, 52(5-6), 760-771

[13] Mursaleen M., Khan A., Statistical approximation properties of modified q-Stancu-Beta operators, Bull. Malays. Math. Sci. Soc. (2), 2013, 36(3), 683-690

[14] Ren M.-Y., Zeng X.-M., Some statistical approximation properties of Kantorovich-type q-Bernstein-Stancu operators, J. Inequal. Appl., 2014, 2014:10

[15] Andrews G. E., Askey R., Roy R., Special functions, Cambridge University Press,1999

[16] Kac V., Cheung P., Quantum Calculus, Springer-Verlag, Newyork, 2002

[17] Aral A., Gupta V., Agarwal R. P., Applications of q-Calculus in Operator Theory, Springer, 2013

[18] Balázs K., Approximation by Bernstein type rational function, Acta Math. Acad. Sci. Hungar, 1975, 26(1-2), 123-134

[19] Balázs K., Szabados J., Approximation by Bernstein type rational function II, Acta Math. Acad. Sci. Hungar, 1982, 40(3-4), 331-337

[20] Doğru O., On statistical approximation properties of Stancu type bivariate generalization of q-Balázs-Szabados operators, In: Proceedings of the International Conference on Numerical Analysis and Approximation Theory, Cluj-Napoca, Romania, 2006, 179-194

[21] Ozkan E. Y., An upper estimate of q-Balázs-Szabados-Kantorovich operators on compact disks, G.U. J. Sci., 2016, 29(2), 479-486

[22] Fast H., Sur la convergence statistique, Collog. Math., 1951, 2(3-4), 241-244

[23] Niven I., Zuckerman H. S., Montgomery H., An Introduction to the Theory of Numbers, 5th edition, Wiley, New York, 1991

[24] Lopez-Moreno A-J, Weighted simultaneous approximation with Baskakov type operators, Acta Math. Hungar., 2004, 104(1-2), 143-151

[25] De Vore R. A., Lorentz G. G., Constructive approximation, Springer, Berlin, 1993