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On the existence of complex Hadamard submatrices of the Fourier matrices

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Abstract: We use a theorem of Lam and Leung to prove that a submatrix of a Fourier matrix cannot be Hadamard for particular cases when the dimension of the submatrix does not divide the dimension of the Fourier matrix. We also make some observations on the trace-spectrum relationship of dephased Hadamard matrices of low dimension.

Keywords: Hadamard matrix, trace, spectrum, eigenvalue, Fourier matrix, discrete Fourier transform

MSC: 15B02, 42C05

1 Introduction

Definition 1. We say an $N \times N$ complex-valued matrix H is Hadamard when it is of the form

$$H = \begin{bmatrix} e^{2\pi i\lambda_{11}} & e^{2\pi i\lambda_{12}} & \cdots & e^{2\pi i\lambda_{1N}} \\ e^{2\pi i\lambda_{21}} & e^{2\pi i\lambda_{22}} & \cdots & e^{2\pi i\lambda_{2N}} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ e^{2\pi i\lambda_{N1}} & e^{2\pi i\lambda_{N2}} & \cdots & e^{2\pi i\lambda_{NN}} \end{bmatrix}$$

such that $\lambda_{jk} \in \mathbb{R}$ and

$$H^* H = HH^* = N \cdot I_N,$$

where I_N is the $N \times N$ identity matrix.

In other words, a matrix H is Hadamard if its entries are of unit modulus and $\frac{1}{\sqrt{N}} H$ is unitary.

Definition 2. A Hadamard matrix H is said to be *dephased* if it is of the form

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{2\pi i\lambda_{22}} & \cdots & e^{2\pi i\lambda_{2N}} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & e^{2\pi i\lambda_{N2}} & \cdots & e^{2\pi i\lambda_{NN}} \end{bmatrix}$$

Definition 3. Two Hadamard matrices H_1 and H_2 are said to be *equivalent* if

$$H_1 = P_1 D_1 H_2 D_2 P_2,$$

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where P_i and D_i are permutation and diagonal matrices, respectively.

It is easy to see that every Hadamard matrix is equivalent to a dephased Hadamard matrix. Hadamard matrices up to and including dimension $N = 5$ have been classified up to equivalence [1], but even for dimension $N = 6$, no complete classification exists. Dutkay, Haussermann, and Weber proved that through dimension $N = 5$, two dephased Hadamard matrices with the identical traces will be spectrally equivalent [2]. However, a counterexample was known for the 12×12 case. At the end of this paper we make a note that counterexamples in fact exist in the $N = 6$ case.

A subclass of the dephased $N \times N$ Hadamard matrices are the *Fourier matrices* $\mathcal{F}_N$, given by

$$\mathcal{F}_N := \left(e^{2\pi i(j-1)(k-1)/N}\right)_{jk}.$$

The Fourier matrices and their Hadamard submatrices are a subclass of the Butson-type Hadamard matrices. The Butson class $BH(n, q)$ consists of $n \times n$ Hadamard matrices whose entries are qth roots of unity. Thus, $\mathcal{F}_N \in BH(N, N)$. We refer the reader to [3] for further information on Butson-type Hadamard matrices.

Definition 4. A *Hadamard triple* is a triple (N, B, L) where N is a positive integer and B and L are finite sets of integers, such that

- $0 \in B, 0 \in L$;
- $|B| = |L|$;
- $H_{B,L} := \left(e^{2\pi i b\ell/N}\right)_{b\in B,\ell\in L}$ is Hadamard .

Note that in the trivial case $B = L = \{0, 1, 2, \ldots, N-1\}$, $H_{B,L} = \mathcal{F}_N$, and so (N, B, L) is a Hadamard triple.

More generally, we may take $L, B \subseteq \mathbb{R}^\nu$, $\mathbf{0} \in L, B$, and then (N, B, L) is a Hadamard triple if $H_{B,L} := \left(e^{2\pi i\langle b,l\rangle/N}\right)_{b\in B,\ell\in L}$ is Hadamard. However, in this paper we will stick to the $\nu = 1$ case, as it affords the following equivalence between Hadamard triples and submatrices of Fourier matrices:

Proposition 1. *For given $N, n \in \mathbb{N}$, there exists a Hadamard triple (N, B, L) with $|B| = |L| = n$ if and only if there exists an $n \times n$ submatrix of $\mathcal{F}_N$ that is Hadamard.*

The following lemma is straightforward and will allow us to prove this proposition:

Lemma 1. *Suppose H is a submatrix of $\mathcal{F}_m$ that is Hadamard, and let $J \subseteq \{0, 1, \ldots, m-1\}$ index the set of rows and $K \subseteq \{0, 1, \ldots, m-1\}$ index the set of columns used to form it, so that $H = \left(e^{2\pi ijk/m}\right)_{j\in J,k\in K}$. Then for any integers a and b, the submatrix $H' = \left(e^{2\pi ijk/m}\right)_{j\in J',k\in K'}$ is also Hadamard, where $J' = \{j + a \mod m : j \in J\}$ and $K' = \{k + b \mod m : k \in K\}$.*

Proof. Let $k_1', k_2' \in K'$ be distinct columns of H'. Then there exist $k_1, k_2 \in K$ and integers ℓ_1, ℓ_2 such that $k_1' = k_1 + b + \ell_1 m$ and $k_2' = k_2 + b + \ell_2 m$. Thus $k_1' - k_2' = k_1 - k_2 + (\ell_1 - \ell_2)m$, and k_1 and k_2 are distinct since k_1' and k_2' are not congruent modulo m. Likewise, for every $j' \in J'$ there exists a unique $j \in J$ and integer ℓ_j such that $j' = j + a + \ell_j m$. Then by virtue of the fact that $H^\star H = nI_n$, where $n = |J| = |K|$, we have that:

$$\begin{aligned}\sum_{j'\in J'} e^{2\pi ij'k_1'/m}e^{-2\pi ij'k_2'/m} &= \sum_{j'\in J'} e^{2\pi ij'(k_1'-k_2')/m}\\ &= \sum_{j\in J} e^{2\pi i(j+a+\ell_j m)(k_1-k_2+(\ell_1-\ell_2)m)/m}\\ &= \sum_{j\in J} e^{2\pi i(j+a)(k_1-k_2)/m}e^{2\pi i(j+a)(\ell_1-\ell_2)m/m}e^{2\pi i\ell_j m(k_1-k_2+(\ell_1-\ell_2)m)/m}\\ &= \sum_{j\in J} e^{2\pi i(j+a)(k_1-k_2)/m}e^{2\pi i(j+a)(\ell_1-\ell_2)}e^{2\pi i\ell_j(k_1-k_2+(\ell_1-\ell_2)m)}\end{aligned}$$

$$= \sum_{j\in J} e^{2\pi i(j+a)(k_1-k_2)/m}$$
$$= e^{2\pi i a(k_1-k_2)/m} \sum_{j\in J} e^{2\pi i j(k_1-k_2)/m}$$
$$= 0.$$

This shows that $(H')^\star H' = nI_n$, and so H' is Hadamard. □

We now prove Proposition 1:

Proof. Suppose (N, B, L) is a Hadamard triple with $|B| = |L| = n$. If any two distinct members $b, b' \in B$ were congruent modulo N, then we would have

$$\sum_{\ell\in L} e^{2\pi i b\ell/N} e^{-2\pi i b'\ell/N} = \sum_{\ell\in L} e^{2\pi i(b-b')\ell/N} = n \neq 0,$$

contrary to the assumption that $H_{B,L}$ is Hadamard. Similar reasoning shows that L does not contain any members congruent modulo N. Let $b = k_1N + \tilde{b}$ and $\ell = k_2N + \tilde{\ell}$, where $0 \le \tilde{b} \le N-1$ and $0 \le \tilde{\ell} \le N-1$. Then

$$\begin{aligned} e^{2\pi i b\ell/N} &= e^{2\pi i(k_1N+\tilde{b})(k_2N+\tilde{\ell})/N} \\ &= e^{2\pi i(k_1k_2N^2+k_1N\tilde{\ell}+k_2N\tilde{b}+\tilde{b}\tilde{\ell})/N} \\ &= e^{2\pi i\tilde{b}\tilde{\ell}/N} \\ &= (\mathcal{F}_N)_{\tilde{b}+1,\tilde{\ell}+1}\,. \end{aligned}$$

It follows that (after suitable permutations of rows and columns), $H_{B,L}$ is an $n \times n$ submatrix of $\mathcal{F}_N$.

Conversely, if one has an $n \times n$ Hadamard submatrix H of $\mathcal{F}_N$, then there exist $J, K \subseteq \{0, 1, \ldots, n-1\}$, $|J| = |K| = n$, such that $H = \left(e^{2\pi ijk/m}\right)_{j\in J,k\in K}$. There exist a and b such that $0 \in J' = \{j + a \mod m : j \in J\}$ and $0 \in K' = \{k + b \mod m : k \in K\}$. Then by Lemma 1, $H_{J',K'}$ is Hadamard. Thus, (N, J', K') is a Hadamard triple. □

Hadamard triples were used by Jorgensen and Pedersen in [4] to demonstrate that, for a measure μ induced by an iterated function system (IFS) with parameters N and B, if an L can be found so that (N, B, L) is a Hadamard triple, then the exponential functions $\left\{e^{2\pi i\lambda x}\right\}_{\lambda\in\Lambda}$ are orthogonal in $L^2(\mu)$, where

$$\Lambda = \left\{\sum_{k=0}^{K} \ell_k N^k \mid K \in \mathbb{N}_0, \ell_k \in L\right\}.$$

For example, because the quaternary Cantor measure μ_4 is induced by an IFS with parameters $N = 4$, and $B = \{0, 2\}$, and $(4, \{0, 2\}, \{0, 1\})$ turns out to be a Hadamard triple, Jorgensen and Pedersen were able to show that $\left\{e^{2\pi i\lambda x}\right\}_{\lambda\in\Lambda}$, where $\Lambda = \{0, 1, 4, 5, 16, 17, 20, 21, \ldots\}$, are orthogonal in $L^2(\mu_4)$. They then used the Ruelle transfer operator to demonstrate that $\left\{e^{2\pi i\lambda x}\right\}_{\lambda\in\Lambda}$ is complete in $L^2(\mu_4)$, and therefore an orthonormal basis of $L^2(\mu_4)$. The existence of an orthogonal basis of complex exponential functions means that μ_4 is a spectral measure with spectrum Λ. On the other hand, the famous ternary Cantor measure μ_3 is induced by $N = 3$ and $B = \{0, 2\}$, but there is no L so that $(3, \{0, 2\}, L)$ is a Hadamard triple. The nonexistence of the Hadamard triple does not *a priori* imply that μ_3 is not spectral, but Jorgensen and Pedersen went on to show that it is, in fact, not spectral.

Moreover, in [5] Dutkay and Jorgensen showed that the Fuglede Conjecture, which is still unresolved in both directions in dimensions 1 and 2, is in dimension 1 in the spectral-tile direction equivalent to a Universal Tiling Conjecture (UTC). The UTC conjectures that if equally-sized sets of integers all share a common spectrum (with respect to the counting measure), then there exists a single translation set $\mathcal{T}$ by which each of them will tile $\mathbb{Z}$. For a finite set $A \subseteq \mathbb{Z}$, one can test whether a set Λ, $|\Lambda| = |A|$, is a spectrum for A by checking whether its associated matrix $H = \left(e^{2\pi i\lambda a}\right)_{\lambda\in\Lambda,a\in A}$ is Hadamard, completeness not being at issue in finite dimensions. If $\Lambda \subset \mathbb{Q}$, then H will be a submatrix of a Fourier matrix, and for an integer N such that $N\Lambda \subset \mathbb{Z}$, $(N, N\Lambda, A)$ will be (after suitable translations and modulations) a Hadamard triple.

Hadamard triples are therefore instrumental in the harmonic analysis of measures, especially those induced by iterated function systems and those that are fractal, and we would like to understand better when they can and cannot be found. In this paper we will ask the following question: For a given N and n, does there exist a Hadamard triple (N, B, L) such that $|B| = |L| = n$? In light of Proposition 1, and because of the potentially broader interest, we choose to reframe the question in terms of Hadamard submatrices of Fourier matrices as follows: For a given N and n, does there exist an $n \times n$ Hadamard submatrix of $\mathcal{F}_N$?

2 The existence of Hadamard submatrices of the Fourier matrix

The following observation is immediate:

Proposition 2. *If $n \mid m$, then there exists an $n \times n$ Hadamard submatrix of $\mathcal{F}_m$.*

Proof. Let $J = K = \{0, 1, \dots, n-1\}$. If $n \mid m$, then $\mathcal{F}_n = \left(e^{2\pi ijk/n}\right)_{j\in J,k\in K} = \left(e^{2\pi ijk(m/n)/m}\right)_{j\in J,k\in K}$ is a submatrix of $\mathcal{F}_m$, and Fourier matrices are Hadamard. □

So, Hadamard submatrices exist whenever the submatrix dimension divides the dimension of the Fourier matrix. At this point in time, we are unaware of any examples of an $n \times n$ Hadamard submatrix of $\mathcal{F}_m$ where $n \nmid m$. This raises the question as to whether the condition $n \mid m$ is not just sufficient, but also necessary. The theorem of Lam and Leung [6] dealing with zero-sums of roots of unity, which we will utilize below, seems to suggest that counterexamples could possibly be found when m is sufficiently composite. On the other hand, Hadamard matrices have additional orthogonality structure that may preclude this possibility. The remainder of this section eliminates a wide range of subcases of the $n \nmid m$ case, but by no means all of them.

We first check off our list the following basic impossibility:

Proposition 3. *Let m be a positive integer. If n is an integer such that $m/2 < n < m$, then there does not exist an $n \times n$ submatrix of the Fourier matrix $\mathcal{F}_m$ that is Hadamard.*

Proof. Assume there were such a submatrix, call it H. Let J be the set of rows of $\mathcal{F}_m$ selected to make H, and likewise let K be the set of columns selected. Let r_1 and r_2 be distinct members of J. Then because $\mathcal{F}_m$ and H consist of orthogonal rows, we have

$$\begin{aligned}0 &= \sum_{k=0}^{m-1} e^{2\pi ir_1k/m}e^{-2\pi ir_2k/m}\\ &= \sum_{k\in K} e^{2\pi ir_1k/m}e^{-2\pi ir_2k/m} + \sum_{k\in K^{\mathrm{c}}} e^{2\pi ir_1k/m}e^{-2\pi ir_2k/m}\\ &= \sum_{k\in K^{\mathrm{c}}} e^{2\pi ir_2k/m}e^{-2\pi ir_2k/m}.\end{aligned}$$

This shows that the $n \times (m-n)$ matrix $\left(e^{2\pi ijk/m}\right)_{j\in J,k\in K^{\mathrm{c}}}$ has orthogonal rows. However, this is impossible, since there cannot be n nonzero vectors of dimension less than n that are mutually orthogonal. □

Our main results are based on the following seminal theorem, proven by Lam and Leung [6]:

Theorem ([6]). *Let m be a positive integer, and let $p_1, \dots, p_s$ be the distinct prime divisors of m. Then there exist mth roots of unity $x_1, x_2, \dots, x_n$ such that $x_1+\cdots+x_n = 0$ if and only if n is of the form $n = k_1p_1+\cdots+k_sp_s$, where each k_j is a nonnegative integer.*

Our main theorem, which eliminates many instances of the $n \nmid m$ case, is as follows:

Theorem 1. *Let m be a positive integer, and let n be an integer such that $1 \le n \le m$. Let ℓ be a divisor of m such that $1 \le \ell < n$, and let $p_1, \dots, p_s$ be the distinct prime divisors of m/ℓ. If there do not exist nonnegative integers $k_1, k_2, \dots, k_s$ such that $n = k_1p_1 + k_2p_2 + \cdots + k_sp_s$, then there does not exist an $n \times n$ submatrix of the Fourier matrix $\mathcal{F}_m$ that is Hadamard.*

Proof. Assume, to the contrary, that such a submatrix does exist, say $H = \left(e^{2\pi ijk/m}\right)_{j\in J,k\in K}$ where $J, K \subseteq \{0, \dots, m-1\}$, $|J| = |K| = n$. Now, each of the n elements of J is a representative of only one congruence class of integers modulo ℓ, but there are ℓ such congruence classes, and since $\ell < n$, there are more elements of J than there are congruence classes modulo ℓ. It follows by the Pigeonhole Principle that there must exist two members of J, say r_1 and r_2, that are congruent modulo ℓ. Without loss of generality, we may suppose $r_1 < r_2$, with $r_2 = r_1 + b\ell$, where b is a positive integer. By the Lam-Leung Theorem, there do not exist $\frac{m}{\ell}$th roots of unity $x_1, \dots, x_n$ such that $x_1 + \cdots + x_n = 0$. However, because the rows r_1 and r_2 of H must be orthogonal, we have

$$\begin{aligned} 0 &= \sum_{k\in K} e^{2\pi ir_1k/m}e^{-2\pi ir_2k/m} \\ &= \sum_{k\in K} e^{2\pi ir_1k/m}e^{-2\pi i(r_1+b\ell)k/m} \\ &= \sum_{k\in K} e^{-2\pi ib\ell k/m} \\ &= \sum_{k\in K} e^{-2\pi ibk/(m/\ell)}. \end{aligned}$$

This is a contradiction. Hence, no such H exists. □

Corollary 1. *If the prime divisors of m are all larger than n and $n > 1$, then $\mathcal{F}_m$ does not have a Hadamard submatrix of size $n \times n$.*

Proof. Take $\ell = 1$ in Theorem 1. Since the prime divisors $p_1, \dots, p_s$ are all larger than n, certainly n cannot be of the form $n = k_1p_1 + \cdots + k_sp_2$ for nonnegative integers k_j. □

Corollary 2. *If $n \notin \mathbb{N}_0p_1 + \mathbb{N}_0p_2 + \cdots + \mathbb{N}_0p_s$, where $p_1, \dots, p_s$ are the prime divisors of m, then $\mathcal{F}_m$ does not have a Hadamard submatrix of size n.*

Proof. Take $\ell = 1$ in Theorem 1. □

Example 1. *Consider the Fourier matrix $\mathcal{F}_{1000}$. So $m = 1000 = 2^35^3$. Let $n = 12$. Let $\ell = 8$. Then $m/\ell = 5^3$. Since 12 cannot be written as a sum of 5's, Theorem 1 shows that $\mathcal{F}_{1000}$ does not have a Hadamard submatrix of dimension 12×12, even though the Lam-Leung Theorem shows that there are twelve 1000th roots of unity that sum to 0.*

While Theorem 1 applies to a wide variety of cases, it does not apply to every case where $n \nmid m$. For example, consider the case $n = 6$, $m = 27 = 3^3$. The only choices for ℓ are $\ell = 1$ and $\ell = 3$, but since 3 is a factor of m/ℓ either way and $6 = 2 \cdot 3$, Theorem 1 cannot eliminate this case. A corollary of the Lam-Leung Theorem will allow us to achieve another result that will eliminate this case, too.

Corollary (Lam & Leung, 2000). *Let $m = p^aq^b$, where p, q are primes. Then, up to a rotation, the only minimal vanishing sums of mth roots of unity are $1 + \zeta_p + \cdots + \zeta_p^{p-1} = 0$ and $1 + \zeta_q + \cdots + \zeta_q^{q-1} = 0$.*

Here $\zeta_p = e^{2\pi i/p}$. By a "minimal vanishing sum," it is meant that no proper subsum of the terms also sums to zero.

We are prepared to prove the following result:

Theorem 2. *Suppose $m = p^a$ where p is an odd prime, and $a \in \mathbb{N}$. Then $\mathcal{F}_m$ does not have a Hadamard submatrix of size $2p$.*

Proof. The $a = 1$ case is trivial, so let $a \geq 2$. Assume, for the sake of contradiction, that $\mathcal{F}_m$ does have a Hadamard matrix of size $2p$. Then by Lemma 1, it has a $2p \times 2p$ submatrix H that uses the first row and first column of $\mathcal{F}_m$, so that the first row and first column of H consist entirely of 1's. Let us say $H_{r,s} = e^{2\pi i j_r k_s/m}$, where $J = \{j_1, j_2, \dots, j_{2p}\} \subset \mathbb{Z}_m$, $K = \{k_1, k_2, \dots, k_{2p}\} \subset \mathbb{Z}_m$, and $j_1 = k_1 = 0$. In light of the orthogonality of H, this means that the entries of any row besides the first row must sum to zero, and likewise the entries of any column besides the first column must sum to zero. Since such sums are zero sums of p^ath roots of unity, by the Lam-Leung Corollary, each of them must allow a partition into minimal subsums, each being a sum of a rotation of the pth roots of unity. In fact, since the number 1 is in each row and column, every row and column of H (except the first row and column) has at least one set of the unrotated pth roots of unity.

Assume that there exist two columns of H besides the first column, say $s, s' \in \{2, 3, \dots, 2p\}$, that both consist only of the unrotated pth roots of unity, so that each partitions into two sets of the pth roots. So there is some $r \neq 1$ such that $H_{r,s} = 1$. For a positive integer x, let $\alpha(x)$ denote the number of times p divides into x. It follows that $\alpha(j_r) + \alpha(k_s) \geq a$, because $1 = H_{r,s} = e^{2\pi i j_r k_s/p^a}$. Now, the rth row of H cannot contain any more 1's, so for a different row $r' \notin \{1, r\}$, we have $H_{r',s'} = 1$. It follows that $\alpha(j_{r'}) + \alpha(k_{s'}) \geq a$. Then $\alpha(j_r) + \alpha(j_{r'}) + \alpha(j_s) + \alpha(j_{s'}) \geq 2a$. It follows that either $\alpha(j_r) + \alpha(k_{s'}) \geq a$ or $\alpha(j_{r'}) + \alpha(k_s) \geq a$. Therefore, either $m \mid j_{r'} k_s$ or $m \mid j_r k_{s'}$. So either $H_{r',s} = 1$ or $H_{r,s'} = 1$, and either way, this is a contradiction to the fact that H cannot contain more than two 1's in any column besides the first column. Therefore, H cannot have two columns in addition to the first column that consist entirely of unrotated pth roots of unity. A similar argument holds for the rows of H.

Therefore, there must exist some row of H, say R, that contains a copy of the pth roots rotated by (an mth root) ω, where $\omega^p \neq 1$. Let $C \subset \{2, 3, \dots, 2p\}$ be the set of columns of H for which $c \in C$ means $H_{R,c}$ is a pth root rotated by ω. We claim that every entry in columns $C^{\mathbb{C}} := \{1, 2, \dots, 2p\} \setminus C$ is an unrotated pth root of unity. This is obviously true for the first row. Let $r \in \{2, 3, \dots, 2p\}$ be any other row, and let $s \in C^{\mathbb{C}}$. If $H_{r,c}$ is a rotated pth root for all $c \in C$, then the fact that row r must contain the unrotated pth roots leaves no choice but for $H_{r,s}$ to be an unrotated pth root. Otherwise, there must exist some column $c \in C$ such that $H_{r,c}$ is an unrotated pth root. It follows that $\alpha(j_r) + \alpha(k_c) \geq a - 1$. In addition, since $H_{R,s}$ is an unrotated pth root, we have $\alpha(j_R) + \alpha(k_s) \geq a - 1$. Thus $\alpha(j_r) + \alpha_{k_c} + \alpha(j_R) + \alpha(k_s) \geq 2(a-1)$, and it follows that either $\alpha(j_r) + \alpha(k_s) \geq a - 1$ or $\alpha(j_R) + \alpha(k_c) \geq a - 1$. The latter cannot be the case, or else $H_{R,c}$ would be a pth root. Therefore, the former is the case, which implies $H_{r,s}$ is a pth root.

Thus every entry in the columns $C^{\mathbb{C}}$ consists only of pth roots. However, this is a contradiction, since there cannot be two columns besides the first with all entries pth roots. Therefore, there is no $2p \times 2p$ Hadamard submatrix of $\mathcal{F}_m$. □

Example 2. *Let $m = 27 = 3^3$ and let $n = 6 = 2 \cdot 3$. Theorem 2 shows directly that there is no 6×6 Hadamard submatrix of $\mathcal{F}_{27}$.*

There are still many cases unhandled by either Theorem 1 or Theorem 2. For example, let $m = 45 = 5 \cdot 3^2$ and $n = 6$. Neither theorem applies to this case, but $n \nmid m$.

3 Trace and spectra of the 6 × 6 Fourier matrix and cyclic 6-roots matrix

While [2] showed that trace equivalence implies spectral equivalence (and, of course, vice versa) for dephased Hadamard matrices up to size 5×5, a counterexample was evidently known in dimension 12×12. We observe in this section that 6×6 is the first size for which trace-spectral equivalence for dephased Hadamard matrices does not hold. Consider the following dephased Hadamard matrices:

First, consider the 6×6 Fourier matrix, which may be written as:

$$\mathcal{F}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 & \gamma^3 & \gamma^4 & \gamma^5 \\ 1 & \gamma^2 & \gamma^4 & 1 & \gamma^2 & \gamma^4 \\ 1 & \gamma^3 & 1 & \gamma^3 & 1 & \gamma^3 \\ 1 & \gamma^4 & \gamma^2 & 1 & \gamma^4 & \gamma^2 \\ 1 & \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma \end{bmatrix}$$

where $\gamma = e^{\frac{2\pi i}{6}}$.

The so-called Cyclic 6-Roots Matrix is defined in [1] as:

$$C_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -d & -d^2 & d^2 & d \\ 1 & -d^{-1} & 1 & d^2 & -d^3 & d^2 \\ 1 & -d^{-2} & d^{-2} & -1 & d^2 & -d^2 \\ 1 & d^{-2} & -d^{-3} & d^{-2} & 1 & -d \\ 1 & d^{-1} & d^{-2} & -d^{-2} & -d^{-1} & -1 \end{bmatrix}$$

where $d = \frac{1-\sqrt{3}}{2} + i\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}}$.

Also, if we take the 2×2 Fourier matrix

$$\mathcal{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the 3×3 Fourier matrix

$$\mathcal{F}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$, we may form the following two Kronecker products of $\mathcal{F}_2$ and $\mathcal{F}_3$ to obtain two more 6×6 dephased Hadamard matrices:

$$K_{2,3} := F_2 \otimes F_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^{-1} & 1 & \omega & \omega^{-1} \\ 1 & \omega^{-1} & \omega & 1 & \omega^{-1} & \omega \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \omega & \omega^{-1} & -1 & -\omega & -\omega^{-1} \\ 1 & \omega^{-1} & \omega & -1 & -\omega^{-1} & -\omega \end{bmatrix}$$

and

$$K_{3,2} := F_3 \otimes F_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & \omega & \omega & \omega^{-1} & \omega^{-1} \\ 1 & -1 & \omega & -\omega & \omega^{-1} & -\omega^{-1} \\ 1 & 1 & \omega^{-1} & \omega^{-1} & \omega & \omega \\ 1 & -1 & \omega^{-1} & -\omega^{-1} & \omega & -\omega \end{bmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$.

All four of the above 6×6 dephased Hadamard matrices have the same trace, namely

$$tr(C_6) = tr(\mathcal{F}_6) = tr(K_{2,3}) = tr(K_{3,2}) = 0.$$

However, the eigenvalues of C_6 are:

$$\sqrt{6} \text{ (algebraic multiplicity 3)},$$

$$-\sqrt{6} \text{ (algebraic multiply 3)}.$$

By contrast, the eigenvalues of $\mathcal{F}_6$, $K_{2,3}$, and $K_{3,2}$ are:

$$\sqrt{6} \text{ (algebraic multiplicity 2)},$$

$$-\sqrt{6} \text{ (algebraic multiplicity 2)},$$

$$i\sqrt{6} \text{ (algebraic multiplicity 1)},$$

$$-i\sqrt{6} \text{ (algebraic multiplicity 1)}.$$

Recall that Specht's Theorem states that two matrices A and B are unitarily equivalent if and only if $\mathrm{tr}(W(A, A^\star)) = \mathrm{tr}(W(B, B^\star))$ for any word W. It is known, however, that it is sufficient for this trace equality to hold only for words up to a certain finite length d that depends on the size n of the matrices. A theorem of Pappacena [7] shows that it is sufficient to check words of length no greater than $n\sqrt{\frac{2n^2}{n-1} + \frac{1}{4}} + \frac{n}{2} - 2$. It is possible that this bound may be reduced further.

Proposition 4. *Two $n \times n$ Hadamard matrices H_1 and H_2 are spectrally equivalent if and only if $\mathrm{tr}(H_1^d) = \mathrm{tr}(H_2^d)$ for all positive integers $d < n\sqrt{\frac{2n^2}{n-1} + \frac{1}{4}} + \frac{n}{2} - 2$.*

Proof. First, we claim that two Hadamard matrices are spectrally equivalent if and only if they are unitarily equivalent. The reverse implication is obvious. Suppose H_1 and H_2 are $n \times n$ Hadamard matrices with the same eigenvalues. Since Hadamard matrices are normal, they are unitarily diagonalizable. Hence, there exists a diagonal matrix Λ containing the eigenvalues of H_1 and H_2 and unitary matrices U and V such that $H_1 = U^\star \Lambda U$ and $H_2 = V^\star \Lambda V$. Then $H_1 = U^\star V H_2 V^\star U$.

Therefore, by Specht's Theorem, combined with the upper bound of Pappacena, H_1 and H_2 have the same spectra if and only if $\mathrm{tr}(W(H_1, H_1^\star)) = \mathrm{tr}(W(H_2, H_2^\star))$ for all words W of length at most $n\sqrt{\frac{2n^2}{n-1} + \frac{1}{4}} + \frac{n}{2} - 2$. However, since $H_1 H_1^\star = H_1^\star H_1 = H_2 H_2^\star = H_2^\star H_2 = nI_n$ and $\mathrm{tr}(H^\star) = \overline{\mathrm{tr}(H)}$, equality of trace for all such words is implied by equality of trace for words that are merely powers of the matrices. □

Using a nearly identical argument, the proposition can be relaxed to require only that there exist a nonzero constant α such that $H_1 H_1^\star = H_1^\star H_1 = H_2 H_2^\star = H_2^\star H_2 = \alpha I_n$, without H_1 and H_2 having to be Hadamard.

Of course, the upper bound on d in Proposition 4 is rather crude, for it is the same bound as that which works for matrices in general when invoking Specht's Theorem. As noted before, in the case of dephased Hadamard matrices of sizes $n \leq 5$, [2] showed that it is sufficient to check only $d = 1$. Our counterexample for $n = 6$ shows that checking higher values of d is necessary. An open question is to what the bound can be reduced in the special case of dephased Hadamard matrices. We note that $\mathrm{tr}(C_6^2) = 36 \neq 12 = \mathrm{tr}(\mathcal{F}_6^2)$, but we do not know whether checking $d \leq 2$ is sufficient for all 6×6 dephased Hadamard matrices.

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