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# Notes on multidimensional fixed-point theorems

**Abstract:** In this paper we prove the existence and uniqueness of coincident (fixed) points for nonlinear mappings of any number of arguments under a $(\psi, \theta, \varphi)$-weak contraction condition without $O$-compatibility. The obtained results extend, improve and generalize some well-known results in the literature to be discussed below. Moreover, we present an example to show the efficiency of our results.

**Keywords:** Fixed-point, contractive mapping, partially ordered set, mixed monotone property

**MSC:** 47H09, 47H10, 54H25, 55H25

## 1 Introduction

The Banach contraction principle [1] is one of the fundamental results in fixed point theory. Because of its application in many disciplines such as computer science, physics, engineering, and many branches of mathematics, a lot of authors have improved, generalized, and extended this classical result in nonlinear analysis (for instance see [2–4]). In 1987, Guo and Lakshmikantham [5] introduced the notion of a *coupled fixed point* with some applications. They also proved some related theorems for certain types of mappings. Recently, Berine and Borcut [6, 7] have introduced the concept of a *triple fixed point* and Karapinar [8] has extended this concept to a *quadruple fixed point*. The remarkable results on the advancement of fixed points have been contributed by Roldán et al. [9, 10] by introducing the notion of a multidimensional *Y-fixed point* which covers the concepts of coupled, tripled and quadruple fixed points up to *n*-tuple.

However, Samet et al. [11], Rad et al. [12] and Roldán et al. [13] have discovered that most of coupled, triple, quadruple and multidimensional *Y*-fixed point results in the context of (ordered) metric spaces are, in fact, immediate consequence of well-known fixed point results of the one dimensional case. In this paper we present some multidimensional *Y*- fixed point theorems under the $(\psi, \theta, \varphi)$- *weak contractive* condition of which the result cannot be obtained using immediate consequence.

Note that the methods which are used in the proofs of [11–13] do not work for our weak contractive condition. Generally speaking, the proofs of our main theorems do not follow immediately from some well-known fixed point results.

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## 2 Preliminaries

Let $(X, d, \preceq)$ be a *partially ordered metric space*, $k$ be a positive integer and $\{\mathcal{A}, \mathcal{B}\}$ be a partition of $\Lambda_k$; that is, $\mathcal{A}, \mathcal{B} \neq \emptyset$, $\mathcal{A} \cup \mathcal{B} = \Lambda_k$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. We define a $k$-dimensional partially ordered metric space $(X^k, \mathbf{d}_k, \preceq_k)$ as follows. Denote by $X^k = \underbrace{X \times X \times \cdots \times X}_{k}$ the Cartesian power of the set $X$. Define a partial order $\preceq_k$ over the set $X^k$ as follows: for any $\mathbf{x} = (x_1, x_2, ..., x_k), \mathbf{y} = (y_1, y_2, ..., y_k) \in X^k$, we say $\mathbf{x} \preceq_k \mathbf{y}$ if and only if $x_i \preceq_i y_i$ for all $i \in \Lambda_k$, where

$$x \preceq_i y \Leftrightarrow \begin{cases} x \preceq y, & \text{if} \quad i \in \mathcal{A}, \\ x \succeq y, & \text{if} \quad i \in \mathcal{B}. \end{cases}$$

The mapping $\mathbf{d}_k : X^k \times X^k \to [0, +\infty)$ given by

$$\mathbf{d}_k(\mathbf{x}, \mathbf{y}) = \max_{i \in \Lambda_k}\{d(x_i, y_i)\},$$

defines a metric on $X^k$, where $\mathbf{x} = (x_1, x_2, ..., x_k), \mathbf{y} = (y_1, y_2, ..., y_k)$. It is obvious that $(X^k, \mathbf{d}_k, \preceq_k)$ is a partially ordered metric space and $\mathbf{d}_k(\mathbf{x}^n, \mathbf{x}) \to 0$ as $n \to \infty$ if and only if $d(x_i^n, x_i) \to 0$ as $n \to \infty$ for all $i \in \Lambda_k$, where $\mathbf{x}^n = (x_1^n, x_2^n, ..., x_k^n)$, $\mathbf{x} = (x_1, x_2, ..., x_k) \in X^k$. Let $F : X^k \to X$ and $g : X \to X$ be two mappings.

**Definition 2.1.** [9] We say that $F$ has the *mixed g-monotone* property with respect to (w.r.t.) the partition $\{\mathcal{A}, \mathcal{B}\}$ if $F$ is $g$-monotone non-decreasing in arguments of $\mathcal{A}$ and $g$-monotone non-increasing in arguments of $\mathcal{B}$; that is, for all $x_1, x_2 \ldots, x_k, y, z \in X$ and $i \in \Lambda_k$ we have,

$$g(y) \preceq g(z) \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k) \preceq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_k). \tag{2.1}$$

If $g$ is the identity mapping on $X$, then we say $F$ has the *mixed monotone* property w.r.t. the partition $\{\mathcal{A}, \mathcal{B}\}$.

Let us denote by $\Omega_{\mathcal{A}, \mathcal{B}}$ and $\Omega'_{\mathcal{A}, \mathcal{B}}$ the sets of mappings defined as

$$\Omega_{\mathcal{A}, \mathcal{B}} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(\mathcal{A}) \subseteq \mathcal{A}, \ \sigma(\mathcal{B}) \subseteq \mathcal{B}\},$$

$$\Omega'_{\mathcal{A}, \mathcal{B}} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(\mathcal{A}) \subseteq \mathcal{B}, \ \sigma(\mathcal{B}) \subseteq \mathcal{A}\}.$$

Henceforth, let $\tau$ be a mapping from $\Lambda_k$ into itself and $Y = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be a $k$ tuple such that $\sigma_i \in \Omega_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{B}$.

**Definition 2.2.** [9] A point $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$ is called a *Y-coincident point* of the mappings $F$ and $g$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(k)}) = g(x_i) \quad \text{for all} \quad i \in \Lambda_k. \tag{2.2}$$

If $g$ is the identity mapping on $X$, then $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$ is called a *Y-fixed point* of the mapping $F$.

## 3 Notes on Roldàn's theorems

In this section we formulate two $Y$-fixed point theorems which were obtained by Roldàn *et. al.* in [13]. These theorems play an important role in proving our main theorems.

**Theorem 3.1.** *[13] Let $(X, d, \preceq)$ be a complete partially ordered metric space and $Y = (\sigma_1, \sigma_2, ..., \sigma_k)$ be a $k$-tuple of mappings verifying $\sigma_i \in \Omega_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{B}$. Define $T_Y : X^k \to X^k$ as*

$$T_Y(x_1, x_2, \ldots, x_k) = \Big( F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(k)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(k)}),$$

$$\ldots, F(x_{\sigma_k(1)}, x_{\sigma_k(2)}, \ldots, x_{\sigma_k(k)}) \Big).$$

*Then the following properties hold true:*

- *if F has the mixed monotone property, then $T_Y$ is monotone non-decreasing (w.r.t. $\preceq_k$);*
- *if F is continuous (w.r.t. $\boldsymbol{d}_k$) then $T_Y$ is also continuous (w.r.t. $\boldsymbol{d}_k$);*
- *a point $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$ is a Y-fixed point of F if, and only if, $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is a fixed point of $T_Y$.*

Before formulating the second theorem let us recall two definitions.

**Definition 3.2.** The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an *altering distance function* if it is continuous, non-decreasing and $\psi^{-1}(\{0\}) = \{0\}$.

**Definition 3.3.** The metric space $(X, d, \preceq)$ is called *regular* if it verifies the following conditions:
- if $\{x_m\}$ is a non-decreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $x_m \preceq x$ for all $m \geq 1$;
- if $\{y_m\}$ is a non-increasing sequence and $\{y_m\} \xrightarrow{d} y$, then $y_m \succeq y$ for all $m \geq 1$.

**Theorem 3.4.** *[13] Let $(X, d, \preceq)$ be a complete partially ordered metric space and $Y = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be a k-tuple mapping verifying $\sigma_i \in \Omega_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{B}$. Assume that the mapping $F : X^k \to X$ satisfies the following conditions:*
*(i) there exist altering distance functions $\psi$, $\varphi$ verifying*

$$\psi(d(F(\mathbf{x}), F(\mathbf{y}))) \leq \psi(\boldsymbol{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\boldsymbol{d}_k(\mathbf{x}, \mathbf{y}))$$

*for all $\mathbf{x} = (x_1, x_2, \ldots, x_k), \mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ for which $\mathbf{x} \preceq_k \mathbf{y}$;*
*(ii) there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ verifying $x_i^0 \preceq_i F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ for all $i \in \Lambda_k$;*
*(iii) F has the mixed monotone property w.r.t. $\{\mathcal{A}, \mathcal{B}\}$;*
*(iv) for all $i \in \Lambda_k$, the mapping $\sigma_i$ is a permutation of $\Lambda_k$;*
*(v) (a) F is continuous or*
*(b) $(X, d, \preceq)$ is regular.*

*Then F has, at least, one Y-fixed point.*

# 4 Main results

In this section first we prove a $Y$-fixed point theorem for a mapping $F : X^k \to X$ satisfying a $(\psi, \theta, \varphi)$-weak contractive condition in the setup of partially ordered metric spaces. Then using this theorem we prove a $Y$-coincident point theorem for the mappings $F : X^k \to X$ and $g : X \to X$ satisfying the $(\psi, \theta, \varphi)$-weak contraction condition in the partially ordered metric spaces. Note that in the second main theorem we do not require the $O$-compatibility of the mappings $F$ and $g$.

## 4.1 Y-fixed point theorem

Before we formulate our results we would like to highlight the main contributions of the work. Note that, after Roldàn's theorems, many researchers have preferred that the multidimensional fixed point results are not used explicitly. This is because the multidimensional case can be reduced to the unidimensional case, by using Theorem 3.1.

Indeed, the reduction is possible. Nevertheless, it cannot ensure the existence of fixed points of $T_Y$ as well as $Y$-fixed points of $F$. Of course, one can prove the existence of fixed points of $T_Y$ while the contraction condition of $F : X^k \to X$ implies a well known contraction condition for the mapping $T_Y : Y \to Y$ in the unidimensional case. Such strategies have been used in [11–13]. In our main theorem we use a $(\psi, \theta, \varphi)$- contraction condition as in Theorem 3.4 although, our contraction condition will be given in a weak form. More

precisely, we assume that there exists an altering distance function $\psi$, an upper semi-continuous function $\theta$ and a lower semi-continuous function $\varphi$ verifying

$$\psi(d(F(\mathbf{x}), F(\mathbf{y}))) \le \theta(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\mathbf{d}_k(\mathbf{x}, \mathbf{y})). \tag{4.1}$$

In addition, in our theorem, we do not require the condition (*iv*) of Theorem 3.4. Therefore, generally speaking, relation (4.1) does not imply

$$\psi(d(T_Y(\mathbf{x}), T_Y(\mathbf{y}))) \le \theta(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) \tag{4.2}$$

which is an analogical contraction condition in the unidimensional case. Moreover, the methods which are used in [11–13] cannot be used in our case, since $\theta$ and $\varphi$ are weak functions and the mappings $\sigma_i$, $i \in \Lambda_k$ are not permutations of $\Lambda_k$. The following is our first main theorem.

**Theorem 4.1.** *Let $(X, d, \preceq)$ be a complete partially ordered metric space and $Y = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be a k-tuple mapping verifying $\sigma_i \in \Omega_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{B}$. Assume that the mapping $F : X^k \to X$ satisfies the following conditions:*

(i) *there exists an altering distance function $\psi$, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\theta(0) = \varphi(0) = 0$ and $\psi(x) - \theta(x) + \varphi(x) > 0$ for each $x > 0$, verifying*

$$\psi(d(F(\mathbf{x}), F(\mathbf{y}))) \le \theta(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\mathbf{d}_k(\mathbf{x}, \mathbf{y}))$$

*for all $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ for which $\mathbf{x} \preceq_k \mathbf{y}$;*
(ii) *there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that $x_i^0 \preceq_i F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ for all $i \in \Lambda_k$;*
(iii) *F has the mixed monotone property w.r.t. $\{\mathcal{A}, \mathcal{B}\}$;*
(iv) *(a) F is continuous or*
    *(b) $(X, d, \preceq)$ is regular.*

*Then F has a Y-fixed point. Moreover*
(v) *if for any $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ there exists $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$ such that $\mathbf{x} \preceq_k \mathbf{z}$ and $\mathbf{y} \preceq_k \mathbf{z}$, then F has a unique Y-fixed point.*

*Proof.* The proof will be divided into five steps where the existence of the fixed point is proven in Steps 1 through 4 while the uniqueness of the fixed point is proven in Step 5.

**Step 1.** Let $\mathbf{x}^n := T_Y^n(\mathbf{x}^0)$ be the *n*-th Picard iteration of $\mathbf{x}^0$ under $T_Y$; that is, $\mathbf{x}^n = T_Y^n(\mathbf{x}^0) = (x_1^n, x_2^n, \ldots, x_k^n)$ where

$$
\begin{aligned}
x_1^n &= F(x_{\sigma_1(1)}^{n-1}, x_{\sigma_1(2)}^{n-1}, \ldots, x_{\sigma_1(k)}^{n-1}), \\
x_2^n &= F(x_{\sigma_2(1)}^{n-1}, x_{\sigma_2(2)}^{n-1}, \ldots, x_{\sigma_2(k)}^{n-1}), \\
&\vdots \\
x_k^n &= F(x_{\sigma_k(1)}^{n-1}, x_{\sigma_k(2)}^{n-1}, \ldots, x_{\sigma_k(k)}^{n-1}).
\end{aligned}
\tag{4.3}
$$

We claim that $\mathbf{x}^{n-1} \preceq_k \mathbf{x}^n$ for all $n \ge 1$. Indeed, by condition (*ii*) and the definition of $T_Y$, it follows that $\mathbf{x}^0 \preceq_k \mathbf{x}^1$. Since $F$ has the mixed monotone property we know that $T_Y$ is monotone non-decreasing. Therefore

$$\mathbf{x}^{n-1} \preceq_k \mathbf{x}^n \quad \text{for all} \quad n \ge 1. \tag{4.4}$$

**Step 2.** In this step we show that $\lim_{n \to \infty} \mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n) = 0$. Set

$$\mathcal{D}_i^n = d(x_i^{n-1}, x_i^n), \quad i \in \Lambda_k \quad \text{and} \quad \mathcal{D}^n = \max_{i \in \Lambda_k} \{\mathcal{D}_i^n\} \overset{\text{def}}{=} \mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n).$$

If $\mathcal{D}^n = 0$ for some $n \ge 1$ then we get $T_Y \mathbf{x}^{n-1} = \mathbf{x}^{n-1}$ where this means $F$ has a $Y$-fixed point which completes the proof of the existence of a $Y$-fixed point. Therefore we assume $\mathcal{D}^n > 0$ for all $n \ge 1$. From (4.4) and $\sigma_i(\Lambda_k) \subseteq \Lambda_k$ it follows that

$$(x_{\sigma_i(1)}^{n-1}, x_{\sigma_i(2)}^{n-1}, \ldots, x_{\sigma_i(k)}^{n-1}) \preceq_k (x_{\sigma_i(1)}^n, x_{\sigma_i(2)}^n, \ldots, x_{\sigma_i(k)}^n)$$

for any $i \in \Lambda_k$ and $n \geq 1$. Using condition $(i)$ we get

$$\psi(\mathcal{D}_i^n) = \psi(d(F(x_{\sigma_i(1)}^{n-2}, x_{\sigma_i(2)}^{n-2}, \ldots, x_{\sigma_i(k)}^{n-2}), F(x_{\sigma_i(1)}^{n-1}, x_{\sigma_i(2)}^{n-1}, \ldots, x_{\sigma_i(k)}^{n-1})))$$

$$\leq \theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{n-2}, x_{\sigma_i(j)}^{n-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{n-2}, x_{\sigma_i(j)}^{n-1})\}) \tag{4.5}$$

for any $i \in \Lambda_k$. Since $\Lambda_k$ is a finite set, there exists an index $i(n) \in \Lambda_k$ such that $\max_{i \in \Lambda_k}\{\mathcal{D}_i^n\} = \mathcal{D}_{i(n)}^n$. From (4.5) it follows that

$$\psi(\mathcal{D}^n) = \psi(\mathcal{D}_{i(n)}^n) = \psi(d(F(x_{\sigma_{i(n)}(1)}^{n-2}, x_{\sigma_{i(n)}(2)}^{n-2}, \ldots, x_{\sigma_{i(n)}(k)}^{n-2}), F(x_{\sigma_{i(n)}(1)}^{n-1}, x_{\sigma_{i(n)}(2)}^{n-1}, \ldots, x_{\sigma_{i(n)}(k)}^{n-1})))$$

$$\leq \theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\}). \tag{4.6}$$

On the other hand, it can be easily shown that

$$0 < \max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\} \leq \mathcal{D}^{n-1}$$

for all $n \geq 1$. Therefore, from the inequality $\psi(x) > \theta(x) - \varphi(x) \; x > 0$, it implies that

$$\theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\}) \tag{4.7}$$

$$< \psi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}}^{n-1}(j))\}) \leq \psi(\max_{j \in \Lambda_k}\{d(x_j^{n-2}, x_j^{n-1})\}) = \psi(\mathcal{D}^{n-1}).$$

Combining (4.6) with (4.7) we can conclude that

$$\psi(\mathcal{D}^n) < \psi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\}) \leq \psi(\mathcal{D}^{n-1}) \tag{4.8}$$

for all $n \geq 1$. Since $\psi$ is an altering distance function, the inequality (4.8) yields the following inequalities

$$\mathcal{D}^n < \mathcal{D}^{n-1}, \qquad \max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n+1)}(j)}^{n-1}, x_{\sigma_{i(n+1)}(j)}^{n})\} < \max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\} \tag{4.9}$$

and

$$\mathcal{D}^n < \max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\} \leq \mathcal{D}^{n-1}.$$

Hence the sequences $\{\mathcal{D}^n\}$ and $\{\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\}\}$ are monotone decreasing and bounded below. Therefore, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} \mathcal{D}^n = \lim_{n \to \infty} \max_{j \in \Lambda_k}\{d(x_{\sigma_{i(n)}(j)}^{n-2}, x_{\sigma_{i(n)}(j)}^{n-1})\} = r.$$

We shall prove $r = 0$. Suppose $r > 0$. Then by letting $n \to \infty$ in (4.6) and using the properties of $\psi(r)$, $\theta(r)$ and $\varphi(r)$ we get $\psi(r) - \theta(r) + \varphi(r) \leq 0$ which is a contradiction. Hence

$$\lim_{n \to \infty} \mathcal{D}^n = \lim_{n \to \infty} \mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n) = 0. \tag{4.10}$$

**Step 3.** In this step we show that the sequence $\{\mathbf{x}^n = (x_1^n, x_2^n, \ldots, x_k^n)\} \in X^k$ is a Cauchy sequence in $(X^k, \mathbf{d}_k)$. Conversely, suppose that $\{\mathbf{x}^n\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{\mathbf{x}^{n_s}\}$, $\{\mathbf{x}^{m_s}\}$ of $\{\mathbf{x}^n\}$ with $n_s > m_s > s$ such that

$$\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s}) \geq \varepsilon. \tag{4.11}$$

Let $n_s$ be the smallest integer satisfying (4.11) and $n_s > m_s > s$. Then

$$\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}) < \varepsilon. \tag{4.12}$$

By using the triangle inequality, we get

$$\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s}) \le \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}) + \mathbf{d}_k(\mathbf{x}^{n_s-1}, \mathbf{x}^{n_s}). \tag{4.13}$$

Relations (4.10)-(4.13) imply

$$\varepsilon \le \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s}) \le \varepsilon + \mathbf{d}_k(\mathbf{x}^{n_s-1}, \mathbf{x}^{n_s})$$

and

$$\lim_{s \to \infty} \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s}) = \varepsilon.$$

Next, we show $\lim_{s \to \infty} \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}) = \varepsilon$. The triangle inequality implies

$$\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}) \le \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s}) + \mathbf{d}_k(\mathbf{x}^{n_s}, \mathbf{x}^{n_s-1}).$$

Taking $s \to \infty$ in (4.13) and the above inequality we get

$$\varepsilon \le \lim_{s \to \infty} \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}) \le \varepsilon.$$

Thus

$$\lim_{s \to \infty} \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}) = \varepsilon. \tag{4.14}$$

Similarly it can be shown that

$$\lim_{s \to \infty} \mathbf{d}_k(\mathbf{x}^{m_s+1}, \mathbf{x}^{n_s}) = \varepsilon. \tag{4.15}$$

In the sequel we use the relations in (4.14) and (4.15). From (4.4) it follows that

$$\mathbf{x}^{m_s} \preceq_k \mathbf{x}^{m_s+1} \preceq_k \cdots \preceq_k \mathbf{x}^{n_s-1}.$$

Since $\mathbf{x}^{m_s} \preceq_k \mathbf{x}^{n_s-1}$ and $\sigma_i(\Lambda_k) \subseteq \Lambda_k$, it implies

$$(x_{\sigma_i(1)}^{m_s}, x_{\sigma_i(2)}^{m_s}, \ldots, x_{\sigma_i(k)}^{m_s}) \preceq_k (x_{\sigma_i(1)}^{n_s-1}, x_{\sigma_i(2)}^{n_s-1}, \ldots, x_{\sigma_i(k)}^{n_s-1})$$

for any $i \in \Lambda_k$ and $s \ge 1$. By condition (*i*) we get

$$\psi(d(x_i^{m_s+1}, x_i^{n_s})) = \psi(d(F(x_{\sigma_i(1)}^{m_s}, x_{\sigma_i(2)}^{m_s}, \ldots, x_{\sigma_i(k)}^{m_s}), F(x_{\sigma_i(1)}^{n_s-1}, x_{\sigma_i(2)}^{n_s-1}, \ldots, x_{\sigma_i(k)}^{n_s-1})))$$

$$\le \theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{m_s}, x_{\sigma_i(j)}^{n_s-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{m_s}, x_{\sigma_i(j)}^{n_s-1})\}) \tag{4.16}$$

for any $i \in \Lambda_k$. Since $\Lambda_k$ is a finite set, there exists an index $i(s) \in \Lambda_k$ such that

$$\max_{i \in \Lambda_k}\{d(x_i^{m_s+1}, x_i^{n_s})\} = d(x_{i(s)}^{m_s+1}, x_{i(s)}^{n_s})$$

and

$$\psi(\mathbf{d}_k(\mathbf{x}^{m_s+1}, \mathbf{x}^{n_s})) = \psi(d(F(x_{\sigma_{i(s)}(1)}^{m_s}, x_{\sigma_{i(s)}(2)}^{m_s}, \ldots, x_{\sigma_{i(s)}(k)}^{m_s}), F(x_{\sigma_{i(s)}(1)}^{n_s-1}, x_{\sigma_{i(s)}(2)}^{n_s-1}, \ldots, x_{\sigma_{i(s)}(k)}^{n_s-1})))$$

$$\le \theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\}). \tag{4.17}$$

Relation (4.15) implies that there exists a sufficiently large $s_0$ such that $\mathbf{d}_k(\mathbf{x}^{m_s+1}, \mathbf{x}^{n_s}) > 0$ for all $s \ge s_0$. This implies $\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\} > 0$ for all $s \ge s_0$. On the other hand, $\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\} \le \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1})$. Therefore, combining this together with the inequality $\psi(x) > \theta(x) - \varphi(x)$, $x > 0$ we obtain

$$\theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\}) \tag{4.18}$$

$$< \psi(\max_{j \in \Lambda_k}\{d(x_{\sigma_{i(s)}(j)}^{m_s}, x_{\sigma_{i(s)}(j)}^{n_s-1})\}) \le \psi(\max_{j \in \Lambda_k}\{d(x_j^{m_s}, x_j^{n_s-1})\}) = \psi(\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}))$$

for $s \geq s_0$. The inequalities (4.17) and (4.18) imply

$$\psi(\mathbf{d}_k(\mathbf{x}^{m_s+1}, \mathbf{x}^{n_s})) < \psi(\max_{j \in \Lambda_k}\{d(x^{m_s}_{\sigma_{i(s)}(j)}, x^{n_s-1}_{\sigma_{i(s)}(j)})\}) \leq \psi(\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1}))$$

for $s \geq s_0$. Since $\psi$ is an altering distance function we get

$$\mathbf{d}_k(\mathbf{x}^{m_s+1}, \mathbf{x}^{n_s}) < \max_{j \in \Lambda_k}\{d(x^{m_s}_{\sigma_{i(s)}(j)}, x^{n_s-1}_{\sigma_{i(s)}(j)})\} \leq \mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s-1})$$

for $s \geq s_0$. Therefore relations (4.14) and (4.15) imply

$$\lim_{s \to \infty} \max_{j \in \Lambda_k}\{d(x^{m_s}_{\sigma_{i(s)}(j)}, x^{n_s-1}_{\sigma_{i(s)}(j)})\} = \varepsilon.$$

Taking $s \to \infty$ in (4.17) we get

$$\psi(\varepsilon) \leq \theta(\varepsilon) - \varphi(\varepsilon)$$

which is a contradiction. Hence the sequence $\{\mathbf{x}^n\}$ is a Cauchy sequence in $(X^k, \mathbf{d}_k)$.

**Step 4.** In this step we prove the existence of a $Y$-fixed point. By the assumption of the theorem the space $(X, d)$ is a complete metric space. It follows that $(X^k, \mathbf{d}_k)$ is complete. Therefore, there exists $\mathbf{x}^\star \in X^k$ such that $\lim_{n \to \infty} \mathbf{x}^n = \mathbf{x}^\star$; that is,

$$\lim_{n \to \infty} x^n_1 = \lim_{n \to \infty} F(x^{n-1}_{\sigma_1(1)}, x^{n-1}_{\sigma_1(2)}, \ldots, x^{n-1}_{\sigma_1(k)}) = x^\star_1,$$

$$\lim_{n \to \infty} x^n_2 = \lim_{n \to \infty} F(x^{n-1}_{\sigma_2(1)}, x^{n-1}_{\sigma_2(2)}, \ldots, x^{n-1}_{\sigma_2(k)}) = x^\star_2,$$

$$\vdots \tag{4.19}$$

$$\lim_{n \to \infty} x^n_k = \lim_{n \to \infty} F(x^{n-1}_{\sigma_k(1)}, x^{n-1}_{\sigma_k(2)}, \ldots, x^{n-1}_{\sigma_k(k)}) = x^\star_k.$$

Next we show that the point $\mathbf{x}^\star = (x^\star_1, x^\star_2, \ldots, x^\star_k) \in X^k$ is a $Y$-fixed point of $F$ if condition $(iv)$ holds. Suppose $F$ is continuous. Then we have

$$\lim_{n \to \infty} F(x^{n-1}_{\sigma_1(1)}, x^{n-1}_{\sigma_1(2)}, \ldots, x^{n-1}_{\sigma_1(k)}) = F(x^\star_{\sigma_1(1)}, x^\star_{\sigma_1(2)}, \ldots, x^\star_{\sigma_1(k)}),$$

$$\lim_{n \to \infty} F(x^{n-1}_{\sigma_2(1)}, x^{n-1}_{\sigma_2(2)}, \ldots, x^{n-1}_{\sigma_2(k)}) = F(x^\star_{\sigma_2(1)}, x^\star_{\sigma_2(2)}, \ldots, x^\star_{\sigma_2(k)}),$$

$$\vdots \tag{4.20}$$

$$\lim_{n \to \infty} F(x^{n-1}_{\sigma_k(1)}, x^{n-1}_{\sigma_k(2)}, \ldots, x^{n-1}_{\sigma_k(k)}) = F(x^\star_{\sigma_k(1)}, x^\star_{\sigma_k(2)}, \ldots, x^\star_{\sigma_k(k)}).$$

Relations (4.19) and (4.20) imply

$$F(x^\star_{\sigma_i(1)}, x^\star_{\sigma_i(2)}, \ldots, x^\star_{\sigma_i(k)}) = x^\star_i, \quad i \in \Lambda_k \tag{4.21}$$

which means the point $\mathbf{x}^\star = (x^\star_1, x^\star_2, \ldots, x^\star_k)$ is a $Y$-fixed point of $F$. Next suppose $(X, d, \preceq)$ is regular. Then relation (4.19) implies $\mathbf{x}^n = (x^n_1, x^n_2, \ldots, x^n_k) \preceq_k \mathbf{x}^\star = (x^\star_1, x^\star_2, \ldots, x^\star_k)$. On the other hand we have $(x^n_{\sigma_i(1)}, x^n_{\sigma_i(2)}, \ldots, x^n_{\sigma_i(k)}) \preceq_k (x^\star_{\sigma_i(1)}, x^\star_{\sigma_i(2)}, \ldots, x^\star_{\sigma_i(k)})$ since $\sigma_i(\Lambda_k) \subseteq \Lambda_k$ for any $i \in \Lambda_k$. By using $(i)$ we obtain

$$\psi(d(F(x^\star_{\sigma_i(1)}, x^\star_{\sigma_i(2)}, \ldots, x^\star_{\sigma_i(k)}), F(x^n_{\sigma_i(1)}, x^n_{\sigma_i(2)}, \ldots, x^n_{\sigma_i(k)})))$$
$$\leq \theta(\max_{j \in \Lambda_k}\{d(x^\star_{\sigma_i(j)}, x^n_{\sigma_i(j)})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x^\star_{\sigma_i(j)}, x^n_{\sigma_i(j)})\}).$$

Taking into account (4.19) and letting $n \to \infty$ in the last inequality we obtain

$$\psi(d(F(x^\star_{\sigma_i(1)}, x^\star_{\sigma_i(2)}, \ldots, x^\star_{\sigma_i(k)}), x^\star_i)) = 0.$$

This implies

$$F(x^\star_{\sigma_i(1)}, x^\star_{\sigma_i(2)}, \ldots, x^\star_{\sigma_i(k)}) = x^\star_i, \quad i \in \Lambda_k. \tag{4.22}$$

This completes the proof of the existence of a $Y$- fixed point.

**Step 5.** In this step we prove the uniqueness of a $Y$-fixed point. Suppose $\mathbf{y}^{\star} = (y_1^{\star}, y_2^{\star}, \ldots, y_k^{\star}) \in X^k$ is another $Y$-fixed point of $F$. By condition ($v$) there exists $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$ such that $\mathbf{x}^{\star} \preceq_k \mathbf{z}$ and $\mathbf{y}^{\star} \preceq_k \mathbf{z}$. Putting $\mathbf{z}^0 := \mathbf{z}$ we define the $n$-th $Y$-iteration of $\mathbf{z}^0$ under $F$ as follows

$$
\begin{aligned}
z_1^n &= F(z_{\sigma_1(1)}^{n-1}, z_{\sigma_1(2)}^{n-1}, \ldots, z_{\sigma_1(k)}^{n-1}), \\
z_2^n &= F(z_{\sigma_2(1)}^{n-1}, z_{\sigma_2(2)}^{n-1}, \ldots, z_{\sigma_2(k)}^{n-1}), \\
&\vdots \\
z_k^n &= F(z_{\sigma_k(1)}^{n-1}, z_{\sigma_k(2)}^{n-1}, \ldots, z_{\sigma_k(k)}^{n-1}).
\end{aligned}
\tag{4.23}
$$

By the induction method we shall prove that

$$
\mathbf{x}^{\star} \preceq_k \mathbf{z}^n \quad \text{and} \quad \mathbf{y}^{\star} \preceq_k \mathbf{z}^n
\tag{4.24}
$$

for all $n \geq 0$. By condition ($v$) we have $\mathbf{x}^{\star} \preceq_k \mathbf{z}^0$. Assume (4.24) holds for $n - 1$. Utilizing this and the manner of proof of Step 1, it can be shown that

$$
x_i^{\star} = F(x_{\sigma_i(1)}^{\star}, x_{\sigma_i(2)}^{\star}, \ldots, x_{\sigma_i(k)}^{\star}) \preceq_i F(z_{\sigma_i(1)}^{n-1}, z_{\sigma_i(2)}^{n-1}, \ldots, z_{\sigma_i(k)}^{n-1}) = z_i^n,
\tag{4.25}
$$

for all $i \in \Lambda_k$; that is, $\mathbf{x}^{\star} \preceq_k \mathbf{z}^n$. The proof of the second inequality is similar. Further we prove

$$
\lim_{n \to \infty} \mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) = 0.
\tag{4.26}
$$

For this we first show that if $\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n_0}) = 0$ for some $n_0$ then $\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) = 0$ for all $n \geq n_0$. Indeed, from (4.24) it follows that

$$
(x_{\sigma_i(1)}^{\star}, x_{\sigma_i(2)}^{\star}, \ldots, x_{\sigma_i(k)}^{\star}) \preceq_k (z_{\sigma_i(1)}^n, z_{\sigma_i(2)}^n, \ldots, z_{\sigma_i(k)}^n),
$$

for all $i \in \Lambda_k$ and $n \geq 1$. Therefore condition ($i$) implies

$$
\begin{aligned}
\psi(d(x_i^{\star}, z_i^n)) &= \psi(d(F(x_{\sigma_i(1)}^{\star}, x_{\sigma_i(2)}^{\star}, \ldots, x_{\sigma_i(k)}^{\star}), F(z_{\sigma_i(1)}^{n-1}, z_{\sigma_i(2)}^{n-1}, \ldots, z_{\sigma_i(k)}^{n-1}))) \\
&\leq \theta(\max_{j \in \Lambda_k} \{d(x_{\sigma_i(j)}^{\star}, z_{\sigma_i(j)}^{n-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{\star}, z_{\sigma_i(j)}^{n-1})\}),
\end{aligned}
\tag{4.27}
$$

for all $i \in \Lambda_k$ and $n \geq 1$. Recall that $\sigma_i(\Lambda_k) \subseteq \Lambda_k$. Hence

$$
\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{\star}, z_{\sigma_i(j)}^{n-1})\} \leq \mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n-1})
$$

for all $n \geq 1$. Taking into account the inequality $\psi(x) \geq \theta(x) - \varphi(x) \; x \geq 0$, we get

$$
\theta(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{\star}, z_{\sigma_i(j)}^{n-1})\}) - \varphi(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{\star}, z_{\sigma_i(j)}^{n-1})\})
$$

$$
\leq \psi(\max_{j \in \Lambda_k}\{d(x_{\sigma_i(j)}^{\star}, z_{\sigma_i(j)}^{n-1})\}) \leq \psi(\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n-1}))
$$

for all $i \in \Lambda_k$ and $n \geq 1$. This implies

$$
\max_{i \in \Lambda_k}\{\psi(d(x_i^{\star}, z_i^n))\} \leq \psi(\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n-1}))
$$

for $n \geq 1$. Since $\psi$ is an altering distance function we have

$$
\psi(\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n)) \leq \psi(\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n-1}))
\tag{4.28}
$$

for all $n \geq 1$. Now it is obvious that if $\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n_0}) = 0$ for some $n_0$ then $\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) = 0$ for all $n \geq n_0$ and this gives the proof of (4.26). Next, assume $\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) > 0$ for all $n \geq 1$. Using the same manner as in proof of (4.9) we can show $\mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) < \mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^{n-1})$. Hence there exists $r \geq 0$ such that $\lim_{n \to \infty} \mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) = r$. A straightforward review of the reasoning given in the proof of (4.10) shows that $r$ cannot be positive. Therefore

$$
\lim_{n \to \infty} \mathbf{d}_k(\mathbf{x}^{\star}, \mathbf{z}^n) = 0.
$$

Moreover, in a similar way we can prove

$$\lim_{n \to \infty} \mathbf{d}_k(\mathbf{y}^*, \mathbf{z}^n) = 0. \tag{4.29}$$

On the other hand, we have

$$\mathbf{d}_k(\mathbf{x}^*, \mathbf{y}^*) \le \mathbf{d}_k(x^*, \mathbf{z}^n) + \mathbf{d}_k(\mathbf{z}^n, \mathbf{y}^*).$$

Taking the limit as $n \to \infty$, we obtain

$$\mathbf{d}_k(\mathbf{x}^*, \mathbf{y}^*) = 0.$$

Therefore $\mathbf{x}^* = \mathbf{y}^*$. This proves the uniqueness of a $Y$-fixed point and completes the proof of Theorem 4.1. □

*Remark* 4.2. Note that condition (*i*) in Theorem 4.1 is equivalent to the following:

(*i*) *there exists an altering distance function $\psi$ and a lower semi-continuous function $\varphi$ : $[0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(F(x_1, x_2, \dots, x_k), F(y_1, y_2, \dots, y_k))) \le \psi(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\mathbf{d}_k(\mathbf{x}, \mathbf{y}))$$

*for all $\mathbf{x} = (x_1, x_2, \dots, x_k), \mathbf{y} = (y_1, y_2, \dots, y_k)$ with $\mathbf{x} \preceq_k \mathbf{y}$, where $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$.*

In the sequel, we present some consequences of Theorem 4.1.

*Remark* 4.3. Theorem 4.1 generalizes Theorem 3.6 in [14].

Indeed, consider the partition $\mathcal{A} = \{1\}$, $\mathcal{B} = \{2\}$ of $\Lambda_2$ and define $Y = (\sigma_1, \sigma_2)$ as follows

$$Y = \left( \begin{array}{cc} \sigma_1(1) & \sigma_1(2) \\ \sigma_2(1) & \sigma_2(2) \end{array} \right) = \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right). \tag{4.30}$$

The assertion of Theorem 3.6 in [14] now implies from Theorem 4.1. We need the following corollary.

**Corollary 4.4.** *Let $(X, d, \preceq)$ be a complete partially ordered metric space and $Y = (\sigma_1, \sigma_2, \dots, \sigma_k)$ be a $k$-tuple mapping verifying $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$. Assume $F : X^k \to X$ satisfies hypotheses (ii)–(iv) of Theorem 4.1. Moreover suppose*

(*i*) *there exists a constant $\delta \in [0, 1)$ such that*

$$d(F(x_1, x_2, \dots, x_k), F(y_1, y_2, \dots, y_k)) \le \delta \mathbf{d}_k(\mathbf{x}, \mathbf{y})$$

*for all $\mathbf{x} = (x_1, x_2, \dots, x_k), \mathbf{y} = (y_1, y_2, \dots, y_k)$ with $\mathbf{x} \preceq_k \mathbf{y}$. Then $F$ has a $Y$-fixed point. Moreover*

(*v*) *if for any $\mathbf{x} = (x_1, x_2, \dots, x_k), \mathbf{y} = (y_1, y_2, \dots, y_k) \in X^k$ there exists a $\mathbf{z} = (z_1, z_2, \dots, z_k) \in X^k$ such that $\mathbf{x} \preceq_k \mathbf{z}$ and $\mathbf{y} \preceq_k \mathbf{z}$, then $F$ has a unique $Y$-fixed point.*

*Proof.* Taking the functions $\psi(x) := x$, $\theta(x) := \delta x$, $\delta \in [0, 1)$ and $\varphi(x) := 0$ and applying Theorem 4.1, we get the proof of Corollary 4.4. □

*Remark* 4.5. Notice that Theorems 2.1 and 2.2 in [15] are consequences of Corollary 4.4.

Recall that, in [15], $Y = (\sigma_1, \sigma_2)$ is chosen as (4.30) and the contraction condition is

$$d(F(x_1, x_2), F(y_1, y_2)) \le \frac{\delta}{2}(d(x_1, y_1) + d(x_2, y_2))$$

for any $\mathbf{x}, \mathbf{y} \in X^2$ such that $\mathbf{x} \preceq_2 \mathbf{y}$. It implies

$$d(F(x_1, x_2), F(y_1, y_2)) \le \frac{\delta}{2}(d(x_1, y_1) + d(x_2, y_2)) \le \delta \mathbf{d}_2(\mathbf{x}, \mathbf{y}).$$

Therefore, applying Corollary 4.4, we obtain the desired result.

*Remark* 4.6. Corollary 4.4 generalizes the main tripled fixed point result of the work [6].

Indeed, in [6] the partition of $\Lambda_3$ is chosen as $\mathcal{A} = \{1, 3\}$, $\mathcal{B} = \{2\}$ and $Y = (\sigma_1, \sigma_2, \sigma_3)$ is given as the following form

$$Y = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) \\ \sigma_2(1) & \sigma_2(2) & \sigma_2(3) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}. \tag{4.31}$$

The contraction condition of [6] is

$$d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \le \delta_1 d(x_1, y_1) + \delta_2 d(x_2, y_2) + \delta_3 d(x_3, y_3),$$

for any $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in X^3$ such that $\mathbf{x} \preceq_3 \mathbf{y}$, where $\delta_1, \delta_2, \delta_3 \ge 0$ and $\delta_1 + \delta_2 + \delta_3 < 1$. It is obvious

$$d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \le (\delta_1 + \delta_2 + \delta_3)\mathbf{d}_3(\mathbf{x}, \mathbf{y}).$$

Therefore, applying Corollary 4.4, we obtain the desired result.

*Remark* 4.7. Corollary 4.4 generalizes the main multidimensional fixed point theorem of [16].

Indeed, in this work the partition of $\Lambda_k$ is $\mathcal{A} = \{1, 2, ..., m\}$ and $\mathcal{B} = \{m + 1, m + 2, ..., k\}$ and the contraction condition is

$$d(F(x_1, x_2, ..., x_k), F(y_1, y_2, ..., y_k)) \le \sum_{i=1}^{k} \delta_i d(x_i, y_i), \tag{4.32}$$

where $\delta_i \in [0, 1)$ and $\delta = \sum_{i=1}^{k} \delta_i < 1$. The inequality

$$\sum_{i=1}^{k} \delta_i d(x_i, y_i) \le \delta \mathbf{d}_k(\mathbf{x}, \mathbf{y})$$

and Corollary 4.4 imply the desired result.

## 4.2 $Y$-coincidence point theorem without $O$-compatibility

Below we use the following notations. Let $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$ and $g : X \to X$. For simplicity, we denote from now on

$$\left[g(X)\right]^k := \underbrace{g(X) \times g(X) \times \cdots \times g(X)}_{k}$$

and consider the mapping $g_{\mathbf{x}} : X^k \to [g(X)]^k$, defined as $g_{\mathbf{x}} = (g(x_1), g(x_2), \ldots, g(x_k))$, where $k \in \mathbb{N}$. The following is our second main theorem.

**Theorem 4.8.** *Let $(X, d, \preceq)$ be a complete partially ordered metric space and $Y = (\sigma_1, \sigma_2, ..., \sigma_k)$ be a $k$-tuple mapping verifying $\sigma_i \in \Omega_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A}, \mathcal{B}}$ if $i \in \mathcal{B}$. Let $F : X^k \to X$ and $g : X \to X$ be two mappings satisfying the following conditions:*

*(i') $g(X)$ is complete, $g$ is continuous and increasing;*
*(ii') $F(X^k) \subset g(X)$;*
*(iii')there exists an altering distance function $\psi$, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\theta(0) = \varphi(0) = 0$ and $\psi(x) - \theta(x) + \varphi(x) > 0$ for $x > 0$, verifying*

$$\psi(d(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k))) \le \theta(\mathbf{d}_k(g_{\mathbf{x}}, g_{\mathbf{y}})) - \varphi(\mathbf{d}_k(g_{\mathbf{x}}, g_{\mathbf{y}}))$$

*for all $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ for which $g_{\mathbf{x}} \preceq_k g_{\mathbf{y}}$;*
*(iv')there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that $g(x_i^0) \preceq_i F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ for all $i \in \Lambda_k$;*
*(v') $F$ has the mixed $g$-monotone property w.r.t. $\{\mathcal{A}, \mathcal{B}\}$;*

*(vi')(a) F is continuous or*
  *(b) $(X, d, \preceq)$ is regular.*

*Then F and g have a Y-coincidence point. Moreover*
*(vii')if for any $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ there exists a $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$, such that*
  *$\mathbf{x} \preceq_k \mathbf{z}$ and $\mathbf{y} \preceq_k \mathbf{z}$, then F and g have a unique Y-coincidence point.*

*Proof.* Consider the mapping $G : [g(X)]^k \to X$ defined by

$$G(g(x_1), g(x_2), \ldots, g(x_k)) = F(x_1, x_2, \ldots, x_k),\qquad(4.33)$$

for all $g(x_1), g(x_2), \ldots, g(x_k) \in g(X)$. Note that $G$ is well defined on $[g(X)]^k$, since $g$ is increasing. The main object of the proof is to show that the function $G$ satisfies all conditions of Theorem 4.1. It is obvious that $(g(X), d, \preceq)$ is a partially ordered metric space. By condition $(iii')$ and relation (4.33) we have

$$\psi(d(G(g(x_1), g(x_2), \ldots, g(x_k)), G(g(y_1), g(y_2), \ldots, g(y_k)))) \leq \theta(\mathbf{d}_k(g_{\mathbf{x}}, g_{\mathbf{y}})) - \varphi(\mathbf{d}_k(g_{\mathbf{x}}, g_{\mathbf{y}}))$$

for all $g_{\mathbf{x}}, g_{\mathbf{y}} \in [g(X)]^k$ for which $g_{\mathbf{x}} \preceq_k g_{\mathbf{y}}$. By $(iv')$ there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ as well as $g_{\mathbf{x}^0} = (g(x_1^0), g(x_2^0), \ldots, g(x_k^0)) \in [g(X)]^k$ such that

$$g(x_i^0) \preceq_i G(g(x_{\sigma_i(1)}^0), g(x_{\sigma_i(2)}^0), \ldots, g(x_{\sigma_i(k)}^0))$$

for all $i \in \Lambda_k$. By condition $(v')$ it implies that if

$$g(y) \preceq g(z) \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k) \preceq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_k),$$

then

$$G(g(x_1), \ldots, g(x_{i-1}), g(y), g(x_{i+1}), \ldots, g(x_k)) \preceq_i G(g(x_1), \ldots, g(x_{i-1}), g(z), g(x_{i+1}), \ldots, g(x_k))$$

for all $x_1, x_2 \ldots, x_k, y, z \in X$ and $i \in \Lambda_k$. Thus $G$ has the mixed monotone property. From item $(a)$ of $(vi')$ it follows that $G$ is continuous, since $F$ is continuous. Next suppose $(b)$ holds, i.e. $(X, d, \preceq)$ is regular. Since $g$ is continuous and increasing $(g(X), d, \preceq)$ is regular. Therefore $G$ satisfies the conditions $(i) - (iv)$ of Theorem 4.1. Hence $G$ has a $Y$-fixed point on $[g(X)]^k$. Moreover, since $g$ is increasing it can be easily shown that for any $g_{\mathbf{x}} = (g(x_1), g(x_2), \ldots, g(x_k))$, $g_{\mathbf{y}} = (g(y_1), g(y_2), \ldots, g(y_k)) \in [g(X)]^k$ there exists a $g_{\mathbf{z}} = (g(z_1), g(z_2), \ldots, g(z_k)) \in [g(X)]^k$, such that $g_{\mathbf{x}} \preceq_k g_{\mathbf{z}}$ and $g_{\mathbf{y}} \preceq_k g_{\mathbf{z}}$. Therefore $G$ satisfies condition $(v)$ of Theorem 4.1, thus it has a unique $Y$-fixed point; that is, there exists $\mathbf{y}^{\star} = (y_1^{\star}, y_2^{\star}, \ldots, y_k^{\star}) \in [g(X)]^k$ such that

$$G(y_{\sigma_i(1)}^{\star}, y_{\sigma_i(2)}^{\star}, \ldots, y_{\sigma_i(k)}^{\star}) = y_i^{\star}, \quad i \in \Lambda_k.\qquad(4.34)$$

Now we show that $F$ and $g$ have a unique $Y$-coincident point. Since $y_i^{\star} \in g(X)$ and $g$ is increasing, there is a unique $x_i^{\star} \in X$ such that $y_i^{\star} = g(x_i^{\star})$ for all $1 \leq i \leq k$. This and relation (4.34) implies

$$G(g(x_{\sigma_i(1)}^{\star}), g(x_{\sigma_i(2)}^{\star}), \ldots, g(x_{\sigma_i(k)}^{\star})) = g(x_i^{\star}), \quad i \in \Lambda_k.\qquad(4.35)$$

On the other hand by (4.33) we have

$$F(x_{\sigma_i(1)}^{\star}, x_{\sigma_i(2)}^{\star}, \ldots, x_{\sigma_i(k)}^{\star}) = g(x_i^{\star}), \quad i \in \Lambda_k.\qquad(4.36)$$

Thus, Theorem 4.8 is completely proved. □

In the sequel we present some consequences of Theorem 4.8.

*Remark* 4.9. Theorem 4.8 extends the main quadruple fixed point theorem of [17].

Indeed, in [17], the partition of $\Lambda_4$ is given by $\mathcal{A} = \{1, 3\}$, $\mathcal{B} = \{2, 4\}$, and where $Y = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is given by

$$Y = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) & \sigma_1(4) \\ \sigma_2(1) & \sigma_2(2) & \sigma_2(3) & \sigma_2(4) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) & \sigma_3(4) \\ \sigma_4(1) & \sigma_4(2) & \sigma_4(3) & \sigma_4(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}. \tag{4.37}$$

The contraction condition of [17] is

$$d(F(x_1, x_2, x_3, x_4), F(y_1, y_2, y_3, y_4)) \le \theta\left(\frac{1}{4}\sum_{i=1}^{4} d(g(x_i), g(y_i))\right) \tag{4.38}$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = (y_1, y_2, y_3, y_4) \in X^4$ such that $\mathbf{x} \preceq_4 \mathbf{y}$. It is obvious that

$$\theta\left(\frac{1}{4}\sum_{i=1}^{4} d(g(x_i), g(y_i))\right) \le \theta(\mathbf{d}_4(g_\mathbf{x}, g_\mathbf{y})).$$

Taking $\psi(x) := x$ and $\varphi(x) := 0$ in Theorem 4.8 we extend the main result of [17].

*Remark* 4.10. Theorem 4.8 generalizes the main multidimensional fixed point theorem of the work [18].

Indeed, in [18], the partition of $\Lambda_k$ is given $\mathcal{A} = \{odd\ numbers\ of\ \Lambda_k\}$, $\mathcal{B} = \Lambda_k \setminus \mathcal{A}$ and $Y = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is given as follows

$$Y = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \ldots & \sigma_1(k) \\ \sigma_2(1) & \sigma_2(2) & \ldots & \sigma_2(k) \\ \sigma_3(1) & \sigma_3(2) & \ldots & \sigma_3(k) \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_k(1) & \sigma_k(2) & \ldots & \sigma_k(k) \end{pmatrix} = \begin{pmatrix} 1 & 2 & \ldots & k-2 & k-1 & k \\ 2 & 3 & \ldots & k-1 & k & 1 \\ 3 & 4 & \ldots & k & 1 & 2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ k & 1 & \ldots & k-3 & k-2 & k-1 \end{pmatrix}.$$

The contraction condition of [18] is given as in (4.38). Therefore applying Theorem 4.8 we get the desired result. Note that, the above election has a gap in the case of $k$ is odd. For more information see [19].

# 5 Application to integral equations

In this section we provide an application of Theorem 4.1. More precisely, applying Theorem 4.1 to a nonlinear integral equation we show the existence and uniqueness of a solution. Let $T > 0$ be a real number. Consider the following integral equation on the space of continuous functions $C([0, T])$:

$$x(t) = \int_0^t \mathcal{G}(t, s)\Big[f_1(s, x(s)) + f_2(s, x(s))\Big]\,ds + p(t), \quad t \in [0, T]. \tag{5.1}$$

We assume:
(a) $\mathcal{G} : [0, T] \times [0, T] \to [0, \infty)$ is continuous and

$$\max_{0 \le t \le T} \int_0^T \mathcal{G}(t, s)\,ds \le 1;$$

(b) $p : [0, T] \to [0, T]$ is continuous;
(c) $f_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous;
(d) $f_1(s, \cdot)$ is non-decreasing, $f_2(s, \cdot)$ is non-increasing for all $s \in [0, T]$ and $f(s, t) = 0$ on $D = \{(s, t) : 0 \le s \le T,\ t \in \{0, T\}\}$;

(e)  $\omega_i(s; \delta) < \delta$ for all $\delta > 0$, and $s \in [0, T]$, where $\omega_i(s; \delta)$ is the modulus of continuity of $f_i(s, \cdot)$, that is

$$\omega_i(s; \delta) = \sup_{|t_1 - t_2| < \delta} |f(s, t_1) - f(s, t_2)|.$$

*Remark* 5.1. Since $f_i$ is uniformly continuous, we can easily check that the function $\omega_i(s; \delta)$ is a continuous function of $s$ for any $\delta \geq 0$. Moreover $\omega_i(s; 0) = 0$, $\omega_i(s; \delta)$ is a continuous and non-decreasing function of $\delta$ (see [20]).

**Theorem 5.2.** *Under assumptions (a)-(e), the equation (5.1) has a unique solution in $C[0, T]$.*

*Proof.* To prove this theorem we use Theorem 4.1. First we define some necessary notions. We consider the space $X = C[0, T]$ of continuous real functions defined on $[0, T]$ endowed with the standard metric given by

$$d(u, v) = \max_{0 \leq t \leq T} | u(t) - v(t) | \quad \text{for} \quad u, v \in X.$$

We endow this space with the partial order $\preceq$ given by $x, y \in C[0, T]$

$$x \preceq y \Leftrightarrow x(t) \leq y(t) \quad \text{for all} \quad t \in [0, T].$$

Let $\Lambda_2 = \{1, 2\}$. Consider the partition $\mathcal{A} = \{1\}$ and $\mathcal{B} = \{2\}$ of $\Lambda_2$. Let $Y = (\sigma_1, \sigma_2)$ be the election

$$Y = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) \\ \sigma_2(1) & \sigma_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Next we consider an operator $\mathbb{A} : X^2 \to X$ defined as

$$\mathbb{A}(\mathbf{x}) = \mathbb{A}(x_1, x_2) = \int_0^t \mathcal{G}(t, s)\Big[f_1(s, x_1(s)) + f_2(s, x_2(s))\Big] ds + p(t),$$

where $t \in [0, T]$ and $\mathbf{x} = (x_1, x_2) \in X^2$. Further, we show that $\mathbb{A}$ satisfies all conditions of Theorem 4.1. Take $\mathbf{x} = (x_1, x_2)$, $\mathbf{z} = (z_1, z_2) \in X^2$ and define a metric in $X^2$ as follows

$$\mathbf{d}_2(\mathbf{x}, \mathbf{z}) = \max_{i=1,2}\{d(x_i, z_i)\} = \max_{i=1,2}\{\max_{0 \leq t \leq T} | x_i(t) - z_i(t) |\}.$$

It can easily be seen that $\mathbb{A} : X^2 \to X$ is continuous and has the mixed monotone property w.r.t. $\{\mathcal{A}, \mathcal{B}\}$. Next, we show that $y_i^0 \preceq_i \mathbb{A}(y_{\sigma_i(1)}^0, y_{\sigma_i(2)}^0)$ $i = 1, 2$ for $y_1^0(x) \equiv 0$ and $y_2^0(x) \equiv T$. Indeed, by (b) and (d) we have

$$0 \leq \int_0^t \mathcal{G}(t, s)\Big[f_1(s, 0) + f_2(s, T)\Big] ds + p(t) = \mathbb{A}(0, T)$$

and

$$\int_0^t \mathcal{G}(t, s)\Big[f_1(s, T) + f_2(s, 0)\Big] ds + p(t) = \mathbb{A}(T, 0) \leq T.$$

Next, we show that $\mathbb{A}$ satisfies the first condition of Theorem 4.1 with

$$\psi(x) = x, \quad \theta(x) = \omega(x) \text{ and } \varphi(x) = 0,$$

where $\omega(x) = \sup\limits_{0 \le s \le T} (\omega_1(s; x) + \omega_2(s; x))$. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{z} = (z_1, z_2) \in X^2$ such that $\mathbf{x} \preceq_2 \mathbf{z}$, then we have

$$d(\mathbb{A}(\mathbf{x}), \mathbb{A}(\mathbf{z})) = \max_{0 \le t \le T} |\mathbb{A}(x_1, x_2)(t) - \mathbb{A}(z_1, z_2)(t)|$$

$$\le \max_{0 \le t \le T} \int_0^T \mathcal{G}(t, s) \Big| \sum_{i=1}^2 f_i(s, x_i(s)) - f_i(s, z_i(s)) \Big| ds$$

$$\le \max_{0 \le t \le T} \int_0^T \mathcal{G}(t, s) \Big[ \omega_1(s; d(x_1, z_1)) + \omega_2(s; d(x_2, z_2)) \Big] ds$$

$$\le \max_{0 \le t \le T} \int_0^T \mathcal{G}(t, s) ds \cdot \omega(\mathbf{d}_2(\mathbf{x}, \mathbf{z})) \le \omega(\mathbf{d}_2(\mathbf{x}, \mathbf{z})).$$

Hence

$$\psi(d(\mathbb{A}(\mathbf{x}), \mathbb{A}(\mathbf{z}))) \le \theta(\mathbf{d}_2(\mathbf{x}, \mathbf{z})) - \varphi(\mathbf{d}_2(\mathbf{x}, \mathbf{z})),$$

where $\psi(x) = x$, $\theta(x) = \omega(x)$ and $\varphi(x) = 0$. Assumption (e) and Remark 5.1 imply that $\psi(x) - \theta(x) + \varphi(x) > 0$ for $x > 0$ and $\psi(0) = \theta(0) = \varphi(0) = 0$. Thus we have shown that the operator $\mathbb{A}$ satisfies the conditions $(i) - (iv)$ of Theorem 4.1. Hence $\mathbb{A}$ has a $Y$-fixed point $\mathbf{x}^\star = (x_1^\star, x_2^\star)$. That is

$$\mathbb{A}(x_1^\star, x_2^\star) = x_1^\star,$$
$$\mathbb{A}(x_2^\star, x_1^\star) = x_2^\star.$$

Moreover, for any $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in X^2$ there exists a $\mathbf{q} = (q_1, q_2) \in X^2$ such that $\mathbf{x} \preceq_2 \mathbf{q}$ and $\mathbf{y} \preceq_2 \mathbf{q}$. Indeed, consider the function $q_i : [0, T] \to \mathbb{R}$ defined as

$$q_i(s) = \max\{x_i(s), y_i(s)\}, \ s \in [0, T].$$

One can see that the function $q_i(s)$ is continuous on $[0, T]$ since $x_i(s)$ and $y_i(s)$ are continuous. Moreover $x_i(s) \le q_i(s), y_i(s) \le q_i(s)$ for $i = 1, 2$. Therefore $\mathbb{A}$ has a unique $Y$-fixed point $\mathbf{x}^\star = (x_1^\star, x_2^\star)$. Next we show $x_1^\star = x_2^\star$. Indeed, if $\mathbf{x}^\star = (x_1^\star, x_2^\star)$ is the $Y$-fixed point of $\mathbb{A}$, then $\mathbf{y}^\star = (x_2^\star, x_1^\star)$ is also a $Y$-fixed point of $\mathbb{A}$. However, $\mathbb{A}$ has the unique $Y$-fixed point. Therefore $\mathbf{x}^\star = \mathbf{y}^\star$ hence $x_1^\star = x_2^\star$. Therefore, there exists a continuous function $x^\star(t)$ such that

$$x^\star(t) = \mathbb{A}(x^\star, x^\star)(t) = \int_0^T \mathcal{G}(t, s) \Big[ \sum_{i=1}^2 f_i(s, x^\star(s)) ds \Big] + p(t).$$

Theorem 5.1 is therefore proved.                                                □

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