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Common fixed point theorems via generalized condition (B) in quasi-partial metric space and applications

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Abstract: The aim of this paper is to introduce generalized condition (B) in a quasi-partial metric space acknowledging the notion of Künzi et al. [Künzi H.-P. A., Pajoohesh H., Schellekens M. P., Partial quasi-metrics, Theoret. Comput. Sci., 2006, 365, 237-246] and Karapinar et al. [Karapinar E., Erhan M., Öztürk A., Fixed point theorems on quasi-partial metric spaces, Math. Comput. Modelling, 2013, 57, 2442-2448] and to establish coincidence and common fixed point theorems for two weakly compatible pairs of self mappings. In the sequel we also answer affirmatively two open problems posed by Abbas, Babu and Alemayehu [Abbas M., Babu G. V. R., Alemayehu G. N., On common fixed points of weakly compatible mappings satisfying generalized condition (B), Filomat, 2011, 25(2), 9-19]. Further in the setting of a quasi-partial metric space, the results obtained are utilized to establish the existence and uniqueness of a solution of the integral equation and the functional equation arising in dynamic programming. Our results are also justified by explanatory examples supported with pictographic validations to demonstrate the authenticity of the postulates.

Keywords: Common fixed point, weakly compatible, generalized condition (B), partial-metric space, quasi-partial metric space.

MSC: 47H10, 54H25.

1 Introduction

In 1906, the French mathematician Frèchet [1] initiated the idea of a metric space, which is one of the key notions of mathematics as well as numerous quantitative sciences that necessitate the use of analysis. Internet search engines, image classification, protein classification are some examples in which metric spaces have been significantly used to solve problems. Due to its significance and possible applications, this concept has been extended, improved and generalized in different directions. One such generalization, called a partial quasi metric space, was introduced Künzi et al. [2] by dropping the symmetry condition in the definition of a partial metric. Karapinar et al. [3] called it a quasi-partial metric space and gave the first fixed point result in a quasi-partial metric space.

In the present paper, we introduce the generalized condition (B) in a quasi-partial metric space to obtain coincidence and common fixed points. In the sequel we also answer affirmatively two open problems posed by

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Abbas et al. [4]. Our results generalize, extend and improve many results existing in the literature ([4–8, 10] and so on) illustrating the importance of the generalized condition (B) for quadruple of mappings in a quasi-partial metric space. Two examples are given to illustrate this work. Further, to demonstrate the applicability of the results obtained, applications to the integral equation and the functional equation arising in dynamic programming problem are also given.

2 Preliminaries

Firstly, we recall some definitions and properties, concerning quasi-partial metric spaces.

Definition 2.1. *[10, 11] Let $X \neq \phi$. A partial metric is a function $p : X \times X \to \mathbb{R}^+$ satisfying*

1. $p(x, y) = p(y, x)$ *(symmetry);*
2. *if* $0 \leq p(x, x) = p(x, y) = p(y, y)$, *then* $x = y$ *(non-negativity and indistancy implies equality);*
3. $p(x, x) \leq p(x, y)$ *(small self-distances);*
4. $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ *(triangularity);*
   for all $x, y, z \in X$. *The pair* (X, p) *is called a partial metric space.*

Definition 2.2. *[2] A quasi-partial metric is a function $q : X \times X \to \mathbb{R}^+$ satisfying*

1. $q(x, x) \leq q(y, x)$ *(small self-distances);*
2. $q(x, x) \leq q(x, y)$ *(small self-distances);*
3. $x = y$ *iff* $q(x, x) = q(x, y)$ *and* $q(y, y) = q(y, x)$ *(indistancy implies equality and vice versa);*
4. $q(x, z) + q(y, y) \leq q(x, y) + q(y, z)$ *(triangularity);*
   for all $x, y, z \in X$. *The pair* (X, q) *is called a quasi-partial metric space.*

Karapinar et al. [3] have taken
$(3')$ if $0 \leq q(x, x) = q(x, y) = q(y, y)$, then $x = y$ (equality), instead of (3).
If q satisfies all these conditions except possibly (1), then q is called a lopsided partial quasi-metric [2]. It is interesting to see that for $q(x, y) = q(y, x)$, (X, q) becomes a partial metric space. Also for a quasi-partial metric q on X, the function $d_q : X \times X \to \mathbb{R}^+$ defined by

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y)$$

is a (usual) metric on X.

Example 2.1. *[2] The pair $(\mathbb{R}^+, q)$ with*

1. $q(x, y) = |x - y| + |x|$;
2. $q(x, y) = \max\{y - x, 0\} + x$;

is a quasi-partial metric space.

Definition 2.3. *[3] Let (X, q) be a quasi-partial metric space.*

1. *A sequence* $\{x_n\} \subset X$ *in a quasi-partial metric space converges to a point* $x \in X$ *iff* $q(x, x) = \lim q(x, x_n) = \lim q(x_n, x)$.
2. *A subset E of a quasi-partial metric space* (X, q) *is closed if whenever* $\{x_n\}$ *is a sequence in E such that* $\{x_n\}$ *converges to some* $x \in X$, *then* $x \in E$.

Lemma 2.1. *[3] Let (X, q) be a quasi-partial metric space. Then the following hold*

1. *If* $q(x, y) = 0$, *then* $x = y$;
2. *If* $x \neq y$, *then* $q(x, y) > 0$ *and* $q(y, x) > 0$.

Definition 2.4. *[6] A self mapping S of a metric space (X, d) satisfies condition (B) if there exist $\delta \in [0, 1)$ and $L \geq 0$ and for all $x, y \in X$ we have*

$$d(Sx, Sy) \leq \delta d(x, y) + L \min(d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)).$$

Following Babu et al. [6], Abbas et al. [4] and Abbas and Ilic [12] independently extended the concept of condition (B) to a pair of mappings. Abbas et al. [4] called it generalized condition (B) and Abbas and Ilic [12] called it generalized almost A-contraction.

Definition 2.5. *[4] Let A and S be two self mappings of a metric space (X, d). The mapping S satisfies generalized condition (B) associated with A if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that*

$$d(Sx, Sy) \leq \delta(M(x, y)) + L \min\{d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx)\},$$

where $M(x, y) = \max\{d(Ax, Ay), d(Ax, Sx), d(Ay, Sy), \frac{d(Ax,Sy)+d(Ay,Sx)}{2}\}$.

Clearly condition (B) implies generalized condition (B). However, the converse need not be true. In fact, for $A = I$ generalized condition (B) reduces to condition (B). It is worth mentioning here that any Banach contraction [13], Kannan contraction [14], Chatterjea contraction [8] and Zamfirescu contraction [15], as well as a large class of quasi-contractions $0 \leq \delta < 1$ (Ĉiriĉ [16]), are all included in the generalized condition (B) and play a significant role in the existence of coincidence and common fixed points.

Definition 2.6. *Let A and S be self mappings on a set X. A point $x \in X$ is called a coincidence point of A and S if $Ax = Sx = w$, where w is called a point of coincidence of A and S.*

Definition 2.7. *[9] Let X be a non-empty set. Two mappings $A, S : X \to X$ are said to be weakly compatible if they commute at their coincidence point, i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.*

3 Main Result

Definition 3.1. *Let A and S be two self mappings of a quasi-partial metric space (X, q). The mapping S satisfies generalized condition (B) associated with A (S is a generalized almost A-contraction) if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in X$ we have*

$$\begin{aligned} q(Sx, Sy) \leq \delta \max\{&q(Ax, Ay), q(Ax, Sx), q(Ay, Sy), \frac{1}{2}(q(Sx, Ay) + q(Ax, Sy))\}+ \\ &L \min\{q(Ax, Sx), q(Ay, Sy), q(Ax, Sy), q(Ay, Sx)\}. \end{aligned} \tag{3.1}$$

If $A = id_X$, then S satisfies generalized condition (B) in a quasi-partial metric space.

Example 3.1. *Let $X = [0, \infty)$ be endowed with the quasi-partial metric $q(x, y) = |x - y| + |x|$.*

Let A and S be two self mappings such that

$$Sx = \begin{cases} \frac{x}{10}, & x \in [0, 1] \\ \frac{1}{2}, & x > 1, \end{cases} \quad Ax = \begin{cases} x, & 0 \leq x \leq 1 \\ 5, & x > 1. \end{cases}$$

Clearly, if $x, y \in [0, 1]$, we have

$$q(Sx, Sy) = |\frac{x}{10} - \frac{y}{10}| + |\frac{x}{10}| \leq \frac{1}{10}\{|x - y| + |x|\}.$$

If $x \in [0, 1]$ and $y > 1$, we have

$$q(Sx, Sy) = |\frac{x}{10} - \frac{1}{2}| + |\frac{x}{10}| \leq \frac{1}{10}\{|5 - \frac{1}{2}| + |5|\}.$$

If $x > 1$ and $y \in [0, 1]$, we have

$$q(Sx, Sy) = |\frac{1}{2} - \frac{y}{10}| + |\frac{1}{2}| \le \frac{1}{10}\{|5 - y| + |5|\}.$$

If $x, y > 1$, we have

$$q(Sx, Sy) = \frac{1}{2} \le \frac{5}{10}.$$

Consequently, S satisfies generalized condition (B) associated with A, for $\delta = \frac{1}{10}$ and $L = 0$.

Definition 3.2. *Let A, B, S and T be four self mappings of a quasi-partial metric space (X, q). The pair of mappings (A, S) satisfies generalized condition (B) associated with (B, T) ((A, S) is a generalized almost (B, T)-contraction) if there exist $\delta \in (0, 1)$ and $L \ge 0$, such that for all $x, y \in X$ we have*

$$q(Sx, Ty) \le \delta \max\{q(Ax, By), q(Ax, Sx), q(By, Ty), \frac{1}{2}(q(Sx, By) + q(Ax, Ty))\} + L \min\{q(Ax, Sx), q(By, Ty), q(Ax, Ty), q(By, Sx)\}. \quad (3.2)$$

Theorem 3.1. *Let A, B, S and T be self mappings of a quasi-partial metric space (X, q). If the pair of mappings (A, S) satisfies generalized condition (B) associated with (B, T) for all $x, y \in X$, and we have*

1. *$TX \subset AX$ and $SX \subset BX$,*
2. *AX or BX is closed,*
3. *$(\delta + 2L) < 1$,*

then the pairs (A, S) and (B, T) have a coincidence point. Further, A, B, S and T have a unique common fixed point, provided that the pairs (A, S) and (B, T) are weakly compatible.

Proof. Let $x_0 \in X$. Since $SX \subset BX$, there exists a point $x_1 \in X$, such that $y_1 = Bx_1 = Sx_0$. Suppose there exists a point $y_2 \in Tx_1$ corresponding to this point y_1. Also since $TX \subset AX$, there exists $x_2 \in X$, such that $y_2 = Ax_2 = Tx_1$. Continuing in this manner, we can define a sequence $\{y_n\}$ in X as follows

$$\begin{cases} y_{2n+1} = Bx_{2n+1} = Sx_{2n}, \\ y_{2n+2} = Ax_{2n+2} = Tx_{2n+1}. \end{cases}$$

Now

$$\begin{aligned} q(y_{2n+1}, y_{2n+2}) = q(Sx_{2n}, Tx_{2n+1}) &\le \delta \max\{q(Ax_{2n}, Bx_{2n+1}), q(Ax_{2n}, Sx_{2n}), q(Bx_{2n+1}, Tx_{2n+1}), \\ &\frac{1}{2}(q(Sx_{2n}, Bx_{2n+1}) + q(Ax_{2n}, Tx_{2n+1}))\} + L \min\{q(Ax_{2n}, Sx_{2n}), q(Bx_{2n+1}, Tx_{2n+1}), \\ &q(Ax_{2n}, Tx_{2n+1}), q(Bx_{2n+1}, Sx_{2n})\} \\ &\le \delta \max\{q(y_{2n}, y_{2n+1}), q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2}), \frac{1}{2}(q(y_{2n+1}, y_{2n+1}) + q(y_{2n}, y_{2n+2}))\} + \\ &L \min\{q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2}), q(y_{2n}, y_{2n+2}), q(y_{2n+1}, y_{2n+1})\} \\ &= \delta \max\{q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2})\} + L \min\{q(y_{2n}, y_{2n+2}), q(y_{2n+1}, y_{2n+1})\}. \end{aligned}$$

Now the following four cases arise:

Case I: When

$$\max\{q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2})\} = q(y_{2n}, y_{2n+1})$$

and

$$\min\{q(y_{2n}, y_{2n+2}), q(y_{2n+1}, y_{2n+1})\} = q(y_{2n}, y_{2n+2}),$$

then

$$q(y_{2n+1}, y_{2n+2}) \le \delta q(y_{2n}, y_{2n+1}) + L q(y_{2n}, y_{2n+2})$$

$$\leq \delta q(y_{2n}, y_{2n+1}) + L\{q(y_{2n}, y_{2n+1}) + q(y_{2n+1}, y_{2n+2}) - q(y_{2n+1}, y_{2n+1})\}$$
$$\leq (\delta + L)q(y_{2n}, y_{2n+1}) + Lq(y_{2n+1}, y_{2n+2})\}$$

i.e.,

$$(1 - L)q(y_{2n+1}, y_{2n+2}) \leq (\delta + L)q(y_{2n}, y_{2n+1})$$

i.e.,

$$q(y_{2n+1}, y_{2n+2}) \leq \frac{(\delta + L)}{(1 - L)} q(y_{2n}, y_{2n+1}).$$

Now let $k_1 = \frac{(\delta+L)}{(1-L)}$. Since $(\delta + 2L) < 1$ and $L \geq 0$, then $k_1 < 1$. Therefore

$$q(y_{2n+1}, y_{2n+2}) \leq k_1 q(y_{2n}, y_{2n+1}).$$

Case II: When

$$\max\{q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2})\} = q(y_{2n}, y_{2n+1})$$

and

$$\min\{q(y_{2n}, y_{2n+2}), q(y_{2n+1}, y_{2n+1})\} = q(y_{2n+1}, y_{2n+1}).$$

Then

$$q(y_{2n+1}, y_{2n+2}) \leq \delta q(y_{2n}, y_{2n+1}) + Lq(y_{2n+1}, y_{2n+1})$$

i.e.,

$$q(y_{2n+1}, y_{2n+2}) \leq \delta q(y_{2n}, y_{2n+1}) + Lq(y_{2n}, y_{2n+1})$$

i.e.,

$$q(y_{2n+1}, y_{2n+2}) \leq (\delta + L)q(y_{2n}, y_{2n+1}).$$

Now let $k_2 = (\delta + L)$. Since $(\delta + 2L) < 1$, then $k_2 < 1$. Therefore

$$q(y_{2n+1}, y_{2n+2}) \leq k_2 q(y_{2n}, y_{2n+1}).$$

Case III: When

$$\max\{q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2})\} = q(y_{2n+1}, y_{2n+2})$$

and

$$\min\{q(y_{2n}, y_{2n+2}), q(y_{2n+1}, y_{2n+1})\} = q(y_{2n}, y_{2n+2}),$$

then

$$q(y_{2n+1}, y_{2n+2}) \leq \delta q(y_{2n+1}, y_{2n+2}) + Lq(y_{2n}, y_{2n+2})$$

or

$$(1 - \delta)q(y_{2n+1}, y_{2n+2}) \leq L\{q(y_{2n}, y_{2n+1}) + q(y_{2n+1}, y_{2n+2}) - q(y_{2n+1}, y_{2n+1})\}$$

i.e.,

$$(1 - \delta - L)q(y_{2n+1}, y_{2n+2}) \leq Lq(y_{2n}, y_{2n+1})$$

i.e.,

$$q(y_{2n+1}, y_{2n+2}) \leq \frac{L}{1 - (\delta + L)} q(y_{2n}, y_{2n+1}).$$

Let $k_3 = \frac{L}{1-(\delta+L)}$. Since $(\delta + 2L) < 1$, then $k_3 < 1$. Therefore

$$q(y_{2n+1}, y_{2n+2}) \leq k_3 q(y_{2n}, y_{2n+1}).$$

Case IV: When

$$\max\{q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2})\} = q(y_{2n+1}, y_{2n+2})$$

and

$$\min\{q(y_{2n}, y_{2n+2}), q(y_{2n+1}, y_{2n+1})\} = q(y_{2n+1}, y_{2n+1}),$$

then

$$q(y_{2n+1}, y_{2n+2}) \le \delta q(y_{2n+1}, y_{2n+2}) + Lq(y_{2n+1}, y_{2n+1})$$

i.e.,

$$(1-\delta)q(y_{2n+1}, y_{2n+2}) \le Lq(y_{2n}, y_{2n+1})$$

i.e.,

$$q(y_{2n+1}, y_{2n+2}) \le \frac{L}{1-\delta} q(y_{2n}, y_{2n+1}).$$

Let $k_4 = \frac{L}{1-\delta}$. Since $(\delta + 2L) < 1$, then $k_4 < 1$. Therefore

$$q(y_{2n+1}, y_{2n+2}) \le k_4 q(y_{2n+1}, y_{2n+2}).$$

Choose $k = \max\{k_1, k_2, k_3, k_4\}$. Therefore $0 < k < 1$ and we get

$$q(y_{2n+1}, y_{2n+2}) \le kq(y_{2n}, y_{2n+1}) \le k^2 q(y_{2n-1}, y_{2n}) \le \ldots \le k^{2n+1} q(y_0, y_1).$$

So by induction we get

$$q(y_n, y_{n+1}) \le k^n q(y_0, y_1),$$

which tends to 0 as n tends to ∞.
So $\{y_n\}$ is convergent and hence its subsequence $\{y_{2n+2}\} = \{Ax_{2n+2}\}$ is also convergent to z. Let AX be closed. So $z \in AX$, i.e., there exists $u \in X$ such that $z = Au$. We claim $z = Su$. If not, by using (3.2), we get

$$q(Su, Tx_{2n+1}) \le \delta \max\{q(Au, Bx_{2n+1}), q(Au, Su), q(Bx_{2n+1}, Tx_{2n+1}),$$
$$\frac{1}{2}(q(Su, Bx_{2n+1}) + q(Au, Tx_{2n+1}))\} + L\min\{q(Au, Su), q(Bx_{2n+1}, Tx_{2n+1}),$$
$$q(Au, Tx_{2n+1}), q(Bx_{2n+1}, Su)\}.$$

Letting $n \to \infty$, then

$$q(Su, z) \le \delta \max\{q(Au, z), q(Au, Su), q(z, z), \frac{1}{2}(q(Su, z) + q(Au, z))\}+$$
$$L\min\{q(Au, Su), q(z, z), q(Au, z), q(z, Su)\}$$

i.e.,

$$q(Su, z) \le (\delta + L)q(Su, z),$$

a contradiction to (3). Hence, $q(Su, z) = 0$, i.e., $Su = z$.
So $Au = Su$, i.e., A and S have a coincidence point. Since $SX \subset BX$, there exists $v \in X$ such that $z = Su = Bv$. We claim that $Tv = z$. If not, by using (3.2) we get

$$q(Su, Tv) \le \delta \max\{q(Au, Bv), q(Au, Su), q(Bv, Tv), \frac{1}{2}(q(Su, Bv) + q(Au, Tv))\}+$$
$$L\min\{q(Au, Su), q(Bv, Tv), q(Au, Tv), q(Bv, Su)\}$$

i.e.,

$$q(z, Tv) \le \delta \max\{q(z, z), q(z, z), q(z, Tv), \frac{1}{2}(q(z, z) + q(z, Tv))\}+$$
$$L\min\{q(z, z), q(z, Tv), q(z, Tv), q(z, z)\}$$

i.e.,

$$q(z, Tv) \le \delta q(z, Tv) + Lq(z, Tv)$$

i.e.,

$$q(z, Tv) \le (\delta + L)q(z, Tv),$$

a contradiction to (3). Hence, $q(z, Tv) = 0$, i.e. $Tv = z$. So $Bv = Tv$, i.e., B and T have a coincidence point.
If we assume that BX is closed, then an argument analogous to the previous argument establishes that the

pairs (A, S) and (B, T) have a coincidence point. Hence, $Au = Su = Bv = Tv = z$.
Since (A, S) and (B, T) are weakly compatible,

$$Az = ASu = SAu = Sz,$$

and

$$Bz = BTv = TBv = Tz.$$

Now we will show that $z = Az$. If not, by using (3.2) we get

$$q(Sz, Tv) \le \delta \max\{q(Az, Bv), q(Az, Sz), q(Bv, Tv),$$

$$\frac{1}{2}(q(Sz, Bv) + q(Az, Tv))\} + L \min\{q(Az, Sz), q(Bv, Tv), q(Az, Tz), q(Bv, Sz)\},$$

$$q(Az, z) \le \delta \max\{q(Az, z), d(z, z), \frac{1}{2}(q(Az, z) + q(Az, z)\} + L \min\{q(Sz, Sz), q(z, z), q(Az, z), q(z, Az)\},$$

i.e.

$$q(Az, z) \le \delta q(Az, z) + Lq(Az, z),$$

i.e.

$$q(Az, z) \le (\delta + L)q(Az, z),$$

a contradiction to (3). So $q(Az, z) = 0$, then $z = Az$. Similarly we can prove that $z = Bz$. Hence, $z = Az = Bz = Sz = Tz$, i.e., z is a common fixed point for A, B, S and T. Uniqueness of the fixed point is an easy consequence of (3.2). □

Theorem 3.1 is an extension of Theorem 2.1 and 2.2 to two pairs of self mappings using a more natural condition of closedness of the range space in [4] to a quasi-partial metric space. Also it generalizes and extends Theorem 3.2 of Abbas et al. [5], Theorem 2.3 in Babu et al. [6] and Theorem 3.4 of Berinde [7] and many others, existing in the literature.

Example 3.2. *Let $X = [0, 2]$ be a set endowed with quasi-partial metric $q(x, y) = |x - y| + |x|$. Let A, B, S and T be self mappings defined by*

$$Ax = \begin{cases} \frac{x}{4}, & 0 \le x \le 1 \\ \frac{5}{8}, & 1 < x \le 2, \end{cases} \qquad Bx = \begin{cases} \frac{3x}{4}, & 0 \le x \le 1 \\ \frac{3}{4}, & 1 < x \le 2, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{12}, & 0 \le x \le 1 \\ \frac{1}{4}, & 1 < x \le 2, \end{cases} \qquad Tx = \begin{cases} \frac{x}{8}, & 0 \le x \le 1 \\ \frac{1}{8}, & 1 < x \le 2. \end{cases}$$

Here $AX = [0, \frac{1}{4}] \cup \{\frac{5}{8}\}$ and $BX = [0, \frac{3}{4}]$. So $TX = [0, \frac{1}{8}] \subset AX$ and $SX = [0, \frac{1}{12}] \cup \{\frac{1}{4}\} \subset BX$.
The point 0 is a coincidence point of the four mappings. Further $AS0 = SA0 = 0$ and $TB0 = BT0 = 0$, i.e., the two pairs (A, S) and (B, T) are weakly compatible.

Case I. *For $x, y \in [0, 1]$, we have*

$$q(Sx, Ty) = \{|\frac{x}{12} - \frac{y}{8}| + |\frac{x}{12}|\} \le \frac{4}{5}\{|\frac{x}{4} - \frac{3y}{4}| + |\frac{x}{4}|\}.$$

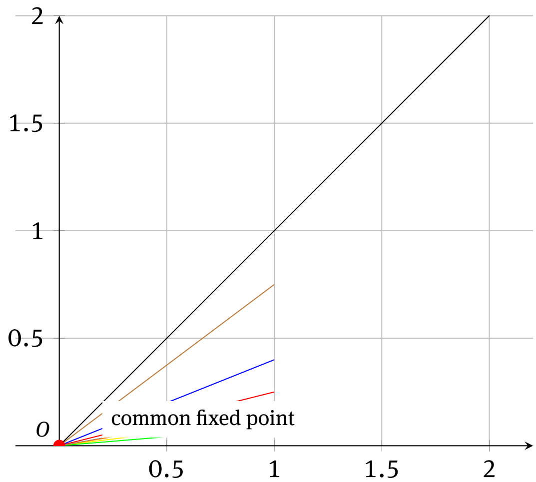

Figure 1: Case-I, (2D-View).

– *In Figure 1: Case-I, (2D-view), the red line denotes Ax, the yellow line denotes Ty, the green line denotes Sx, the brown line denotes By, the black line denotes $y = x$, the blue line denotes the right hand-side of the function and the orange line denotes the left hand-side of the function. Clearly, the functions A, B, S and T intersect on the line $y = x$ only at $x = 0$, i.e., $x = 0$ is the unique common fixed point of A, B, S and T.*

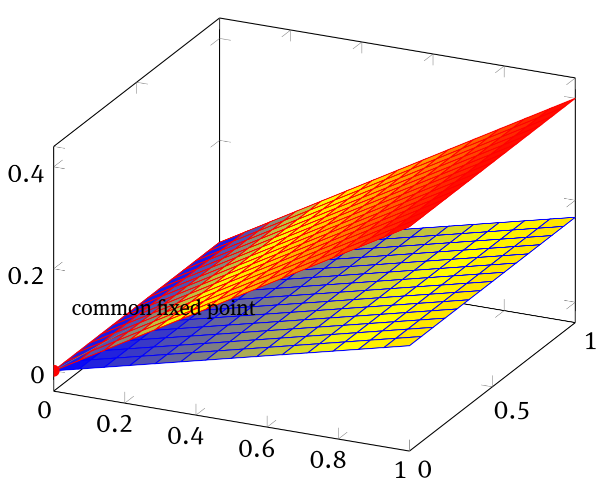

Figure 2: Case-I, (3D-view).

– *In Figure 2: Case-I, (3D-view), the plane in blue colour denotes the left hand-side of the inequality and the plane in red colour denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.2) is satisfied for $x, y \in [0, 1]$.*

Case II. *For $x \in [0, 1]$ and $y \in (1, 2]$, we have*

$$q(Sx, Ty) = \{|\frac{x}{12} - \frac{1}{8}| + |\frac{x}{12}|\} \leq \frac{4}{5}\{|\frac{3}{4} - \frac{1}{8}| + |\frac{3}{4}|\}.$$

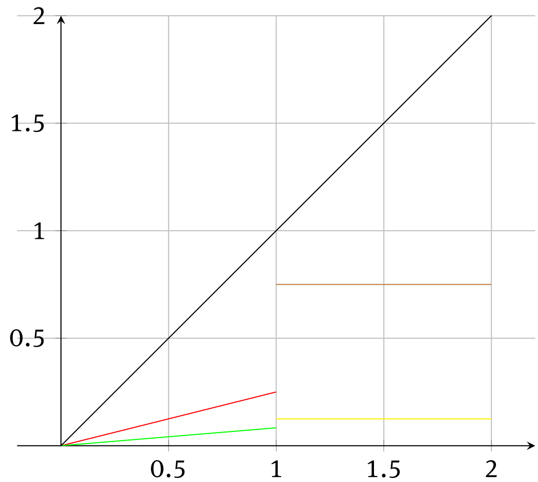

Figure 3: Case-II, (2D-View).

– *In Figure 3: Case-II, (2D-view), the red line denotes* Ax*, the yellow line denotes* Ty*, the green line denotes* Sx*, the brown line denotes* By *and the black line denotes* $y = x$*.*

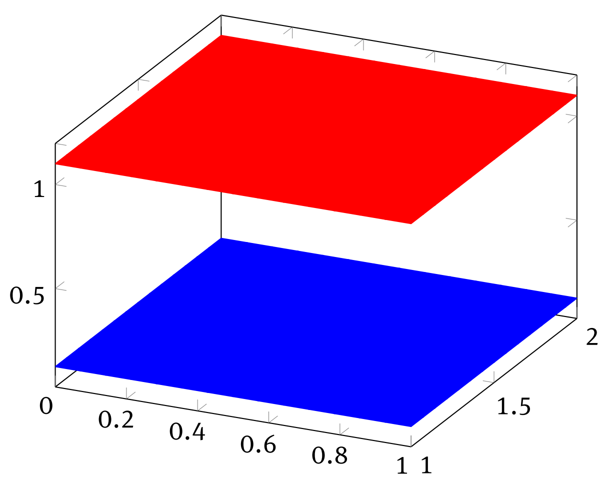

Figure 4: Case-II, (3D-view).

– *In Figure 4: Case-II, (3D-view), the plane in blue colour denotes the left hand-side of the inequality and the plane in red colour denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.2) is satisfied for* $x \in [0, 1]$ *and* $y \in (1, 2]$*.*

Case III. *For* $x \in (1, 2]$ *and* $y \in [0, 1]$*, we have*

$$q(Sx, Ty) = \{|\frac{1}{4} - \frac{y}{8}| + |\frac{1}{4}|\} \le \frac{4}{5}\{|\frac{5}{8} - \frac{3y}{4}| + |\frac{5}{8}|\}.$$

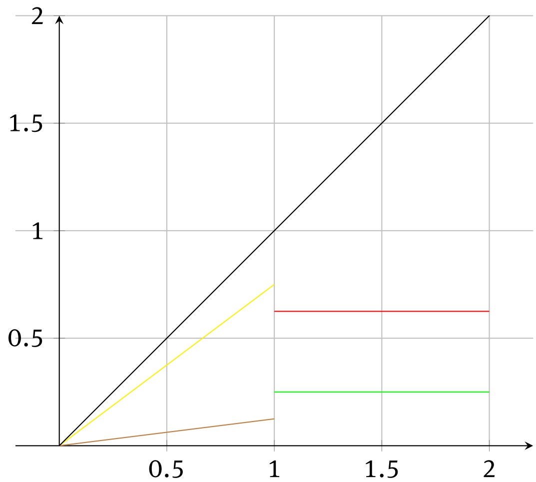

Figure 5: Case-III, (2D-View).

– *In Figure 5: Case-III, (2D-view), the red line denotes Ax, the brown line denotes Ty, the green line denotes Sx, the yellow line denotes By and the black line denotes $y = x$.*

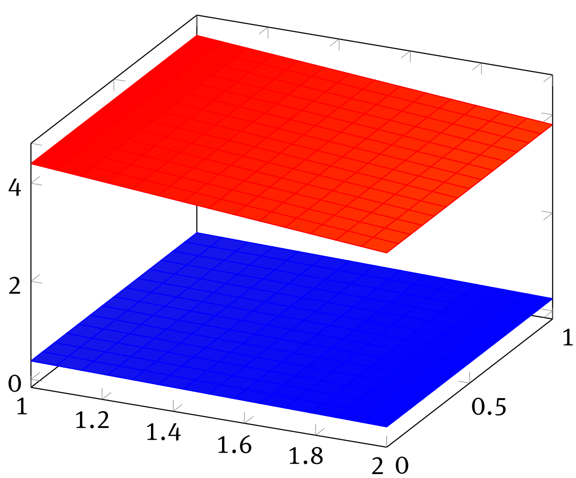

Figure 6: Case-III, (3D-View).

– *In Figure 6: Case-III, (3D-view), the plane with blue surface denotes the left hand-side of the inequality and the plane with red surface denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.2) is satisfied for $x \in (1, 2]$ and $y \in [0, 1]$.*

Case IV. *For $x, y \in (1, 2]$, we have*

$$q(Sx, Ty) = \{|\frac{1}{4} - \frac{1}{8}| + |\frac{1}{4}|\} \leq \frac{4}{5}\{|\frac{3}{4} - \frac{1}{8}| + |\frac{3}{4}|\}.$$

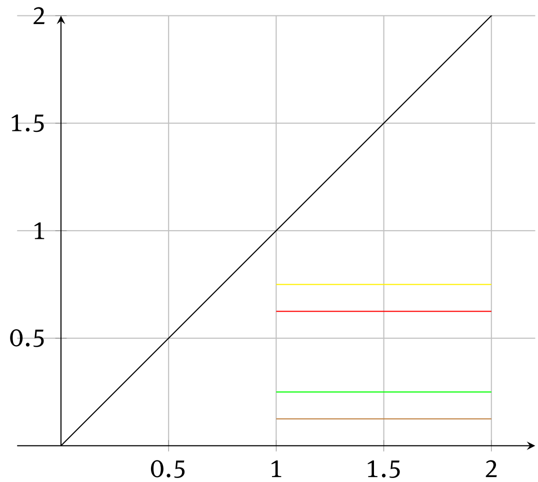

Figure 7: Case-IV, (2D-view).

– *In Figure 7: Case-IV, (2D-view), the red line denotes Ax, the brown line denotes Ty, the green line denotes Sx, the yellow line denotes By and the black line denotes $y = x$.*

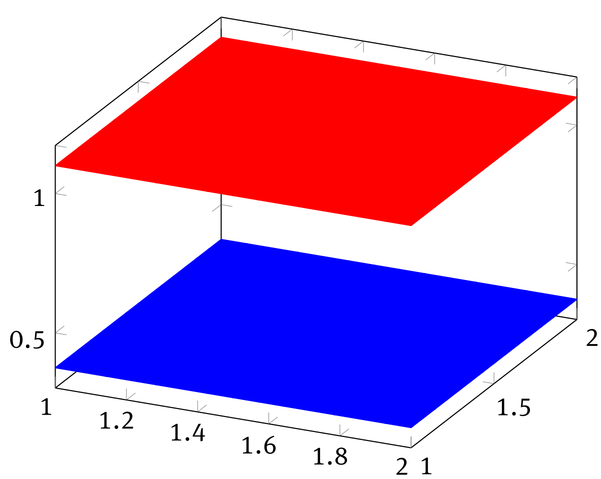

Figure 8: Case-IV, (3D-view).

– *In Figure 8: Case-IV, (3D-view), the plane with blue surface denotes the left hand-side of the inequality and the plane with red surface denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.2) is satisfied for x and $y \in (1, 2]$.*

Consequently, all hypotheses of Theorem 3.1 are satisfied (for $\delta = \frac{4}{5}$ and $L = 0$) and 0 is the unique common fixed point of A, B, S and T.

If $A = B$ and $S = T$, we get the following Corollary

Corollary 3.1. *Let A and T be self mappings of a quasi-partial metric space (X, q). If A satisfies generalized condition (B) associated with T for all $x, y \in X$ and we have*

1. *$TX \subset AX$,*
2. *AX is closed,*
3. *$(\delta + 2L) < 1$,*

then A and T have a coincidence point. Further, A and T have a unique common fixed point, provided that the pair (A, T) is weakly compatible.

It is worth mentioning here that Corollary 3.1 extends Theorem 2.1, Theorem 2.2 and Corollary 2.3 in [4] in the setting of a quasi-partial metric space.

Corollary 3.2. *Let A, B, S and T be self mappings of a quasi-partial metric space (X, q). If the pairs of mappings (A, S) and (B, T) satisfy*

$$q(Sx, Ty) \leq \delta \max\{q(Ax, By), q(Ax, Sx), q(By, Ty), \frac{1}{2}(q(Sx, By) + q(Ax, Ty))\}$$

for all $x, y \in X$ and we have
1. *$TX \subset AX$ and $SX \subset BX$,*
2. *AX or BX is closed,*

then the pairs (A, S) and (B, T) have a coincidence point. Further, A, B, S and T have a unique common fixed point, provided that the pairs (A, S) and (B, T) are weakly compatible.

Proof. The Proof follows similar lines to the proof of Theorem 3.1, using $L = 0$. □

Corollary 3.3. *Let A and T be self mappings of a quasi-partial metric space (X, q). If the pair of mappings (A, T) satisfies*

$$q(Tx, Ty) \leq \delta \max\{q(Ax, Ay), q(Ax, Tx), q(Ay, Ty), \frac{1}{2}(q(Tx, Ay) + q(Ax, Ty))\}$$

for all $x, y \in X$ and we have
1. *$TX \subset AX$,*
2. *AX is closed,*

then the pair (A, T) has a coincidence point. Further, A and T have a unique common fixed point, provided that the pair (A, T) is weakly compatible.

Proof. The Proof follows similar lines to the proof of Theorem 3.1, using $L = 0$, $A = B$ and $S = T$. □

Corollary 3.4. *Let A and T be self mappings of a quasi-partial metric space (X, q). If the pair of mappings (A, T) satisfies*

$$q(Tx, Ty) \leq \delta q(Ax, Ay)$$

for all $x, y \in X$ and we have
1. *$TX \subset AX$,*
2. *AX is closed,*

then the pair (A, T) has a coincidence point. Further, A and T have a unique common fixed point, provided that the pair (A, T) is weakly compatible.

Proof. The Proof follows similar lines to the proof of Theorem 3.1. □

The result is slightly more interesting when the closure of the range space TX or SX is considered.

Theorem 3.2. *Let A, B, S and T be self mappings of a quasi-partial metric space (X, q). If there exist $\delta \in (0, 1)$ and $L \geq 0$, such that for all $x, y \in X$, the pairs of mappings (A, S) and (B, T) satisfy*

$$q(Sx, Ty) \leq \delta \max\{q(Ax, By), q(Ax, Sx), q(By, Ty), q(Ax, Ty), q(Sx, By)\} + L \min\{q(Ax, Sx), q(By, Ty), q(Ax, Ty), q(By, Sx)\} \tag{3.3}$$

and we have

1. $\overline{TX} \subset AX$ or $\overline{SX} \subset BX$,
2. $(\delta + 2L) < 1$,

then the pairs (A, S) and (B, T) have a coincidence point. Further, A, B, S and T have a unique common fixed point, provided that the pairs (A, S) and (B, T) are weakly compatible.

Proof. It can be proved following similar arguments to those given in the proof of Theorem 3.1. □

Example 3.3. *Let $X = [0, \infty)$ be endowed with the quasi-partial metric : $q(x, y) = |x - y| + |x|$ and let A, B, S and T be mappings defined by*

$$Ax = \begin{cases} x, & 0 \leq x \leq 1 \\ 2, & x > 1, \end{cases} \quad Bx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ 1, & x > 1, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{10}, & 0 \leq x \leq 1 \\ 1, & x > 1, \end{cases} \quad Tx = \begin{cases} \frac{x}{5}, & 0 \leq x \leq 1 \\ \frac{1}{2}, & x > 1. \end{cases}$$

Here we have

$$\overline{TX} = [0, \frac{1}{5}] \cup \{\frac{1}{2}\} \subset [0, 1] \cup \{2\} = AX,$$

$$\overline{SX} = [0, \frac{1}{10}] \cup \{1\} \subset [0, \frac{1}{2}] \cup \{1\} = BX.$$

The point 0 is a coincidence point of the four mappings. Further, $AS0 = SA0 = 0$ and $TB0 = BT0 = 0$, i.e., both the pairs (A, S) and (B, T) are weakly compatible.

Case I. *For $x, y \in [0, 1]$, we have*

$$q(Sx, Ty) = |\frac{x}{10} - \frac{y}{5}| + |\frac{x}{10}| = \frac{1}{10}\{|\ 2x - y\ | + |\ x\ |\} \leq \frac{5}{9}\{|\ x - \frac{y}{2}\ | + |\ x\ |\}.$$

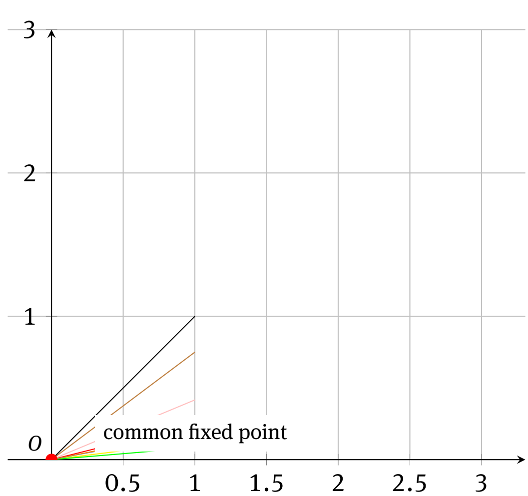

Figure 9: Case-I, (2D-view).

– *In Figure 9: In Case-I, (2D-view), the red line denotes Ax, the yellow line denotes Ty, the green line denotes Sx, the brown line denotes By, the pink line denotes the right hand-side of the function, the orange line denotes the left hand-side of the function and the black line denotes $y = x$. Clearly, the functions A, B, S and T intersect on the line $y = x$ only at $x = 0$, i.e., $x = 0$ is the unique common fixed point of A, B, S and T.*

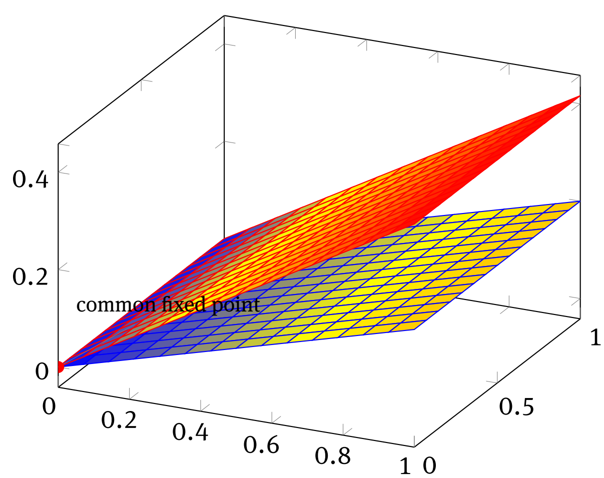

Figure 10: Case-I, (3D-view).

– *In Figure 10: Case-I, (3D-view), the plane in blue colour denotes the left hand-side of the inequality and the plane in red colour denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.3) is satisfied for $x, y \in [0, 1]$.*

Case II. *For $x \in [0, 1]$ and $y > 1$, we have*

$$q(Sx, Ty) = |\frac{x}{10} - \frac{1}{2}| + |\frac{x}{10}| \leq \frac{5}{9}\{|1 - \frac{1}{2}| + |1|\}.$$

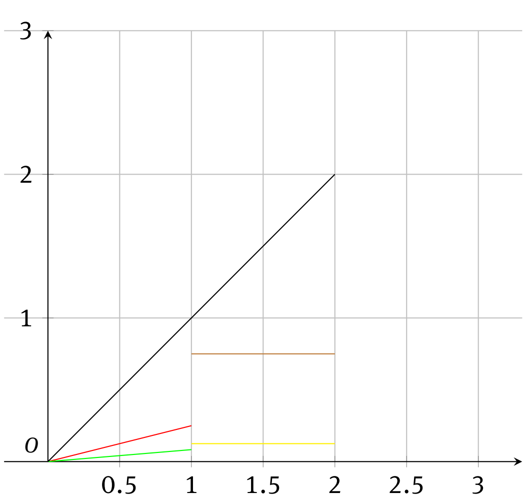

Figure 11: Case-II, (2D-view).

– *In Figure 11: In Case-II, (2D-view), the red line denotes Ax, the yellow line denotes Ty, the green line denotes Sx, the brown line denotes By and the black line denotes $y = x$.*

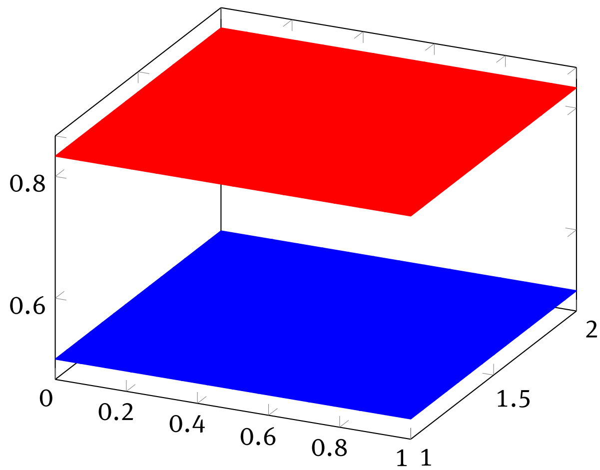

Figure 12: Case-II, (3D-view).

– *In Figure 12: Case-II, (3D-view), the plane in blue colour denotes the left hand-side of the inequality and the plane in red colour denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.3) is satisfied for $x \in [0, 1]$ and $y > 1$.*

Case III. *For $x > 1$ and $y \in [0, 1]$, we have*

$$q(Sx, Ty) = |1 - \frac{y}{5}| + |1| \leq \frac{5}{9}\{|2 - \frac{y}{5}| + |2|\}.$$

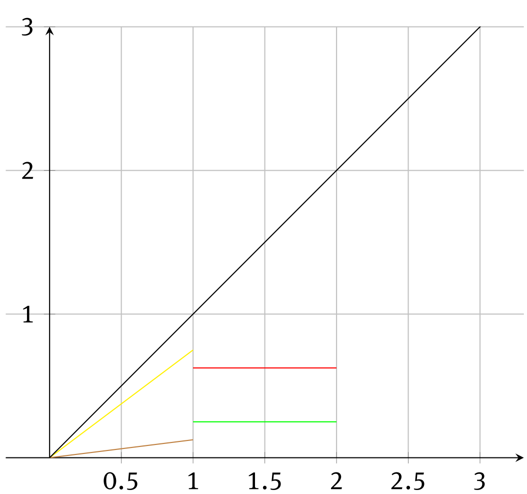

Figure 13: Case-*III*, (2D-view).

– *In Figure 13: Case-III, (2D-view), the red line denotes Ax, the brown line denotes Ty, the green line denotes Sx, the yellow line denotes By and the black line denotes $y = x$.*

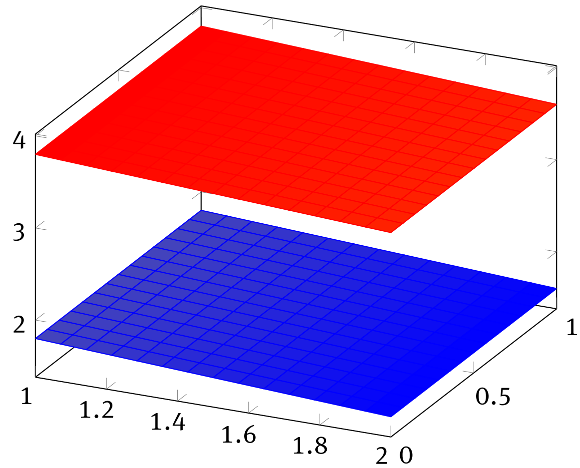

Figure 14: Case-*III*, (3D-view).

– *In Figure 14: Case-III, (3D-view), the plane with the blue surface denotes the left hand-side of the inequality and the plane with the red surface denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.3) is satisfied for* $x > 1$ *and* $y \in [0, 1]$.

Case IV. *For* $x, y > 1$, *we have*

$$q(Sx, Ty) = |1 - \frac{1}{2}| + |1| \leq \frac{15}{9}.$$

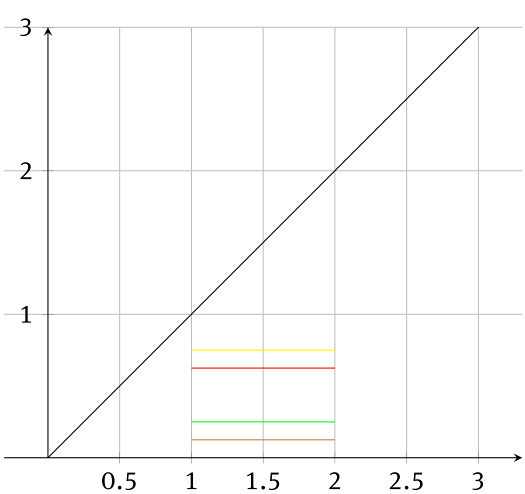

Figure 15: Case-*IV*, (2D-View).

– *In Figure 15: Case-IV, (2D-view), the red line denotes* Ax, *the brown line denotes Ty, the green line denotes Sx, the yellow line denotes By and the black line denotes* $y = x$.

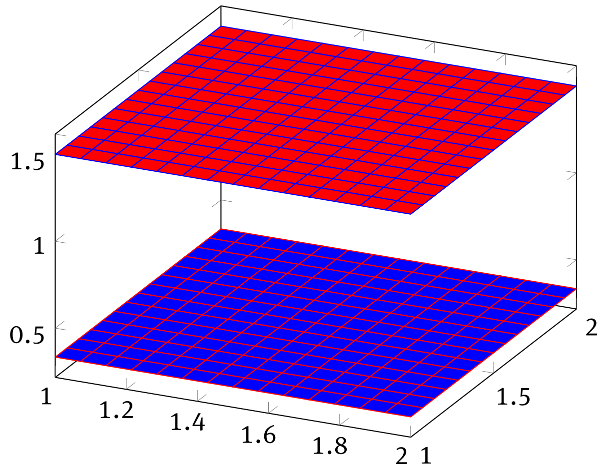

Figure 16: Case-IV, (3D-view).

- *In Figure 16: Case-IV, (3D-view), the plane with the blue surface denotes the left hand-side of the inequality and the plane with the red surface denotes the right hand-side of the inequality. Clearly, the figure verifies that the left hand-side with the blue surface is dominated by the right hand-side with the red surface. Hence, the inequality (3.3) is satisfied for $x, y > 1$.*

Consequently, all hypotheses of Theorem 3.2 are satisfied (for $\delta = \frac{5}{9}$ and $L = 0$) and 0 is the unique common fixed point of A, B, S and T.

For $A = B$ and $S = T$ Theorem 3.2 reduces to following Corollary

Corollary 3.5. *Let A and T be self mappings of a quasi-partial metric space (X, q). If there exist $\delta \in (0, 1)$ and $L \geq 0$, such that for all $x, y \in X$, the pair of mappings (A, T) satisfies*

$$q(Tx, Ty) \leq \max\{q(Ax, Ay), q(Ax, Tx), q(Ay, Ty), q(Ax, Ty), q(Tx, Ay)\} \\ + L\min\{q(Ax, Tx), q(Ay, Ty), q(Ax, Ty), q(Ay, Tx)\}$$

and we have
1. $\overline{TX} \subseteq AX$,
2. $(\delta + 2L) < 1$,

then the pair (A, T) has a coincidence point. Further, A and T have a unique common fixed point, provided that the pair (A, T) is weakly compatible.

Abbas et al. [4] posed two open problems:

Open problem 1. Is Theorem 3.1 [4] valid for $\frac{1}{2} \leq \delta < 1$?
We answer affirmatively in the case of a non-complete quasi-partial metric space, assuming the closures of the range space TX or SX ($\overline{TX} \subset AX$ or $\overline{SX} \subset BX$) and the pairs (A, S) and (B, T) to be weakly compatible. Hence, our Theorem 3.2 extends the results of Berinde [7] to two pairs of self mappings. It is also demonstrated by illustrative Example 3.2 that Theorem 3.2 is valid for $\delta = \frac{5}{9}$.

Open problem 2. Under what additional assumptions either on f and T, or on the domain of f and T, do the mappings f and T have common fixed points?
In a non-complete quasi partial-metric space when the closure of the range space TX is considered ($\overline{TX} \subset fX$),

the weakly compatible pair (f, T) of self mappings has a unique common fixed point (taking $f = A$ in Corollary 3.5).

Remark 3.1. *We have established common fixed point theorems for quadruple of self mappings in a non-complete quasi-partial metric space (X, q), satisfying generalized condition (B), without exploiting the notion of continuity or any of its variants like reciprocal continuity, weak reciprocal continuity, sub-sequential continuity, sequential continuity of type (Af) or (Ag), conditional reciprocal continuity and so on. For details on variants of continuity one may refer to Tomar and Karapinar [17].*

Remark 3.2. *Since (X, q) is not a metric space, generalized condition (B) for quadruple of self mappings does not reduce to any metric condition. Hence, our results do not reduce to the existing fixed point theorems in metric spaces. Our results generalize, extend and improve the results of Abbas [4], Abbas et al. [5], Babu et al. [6], Banach [13], Berinde [7], Chatterjea [8], Čirič [16], Kannan [14], Zamfirescu [15] and so on to quasi-partial metric spaces. A more natural condition of closedness of the range space is assumed to establish a unique common fixed point.*

4 Application To Integral Equations

Consider the following integral equation

$$u(l) = \int_0^L K(l, s, u(s))ds + g(l), \tag{4.1}$$

where $l \in [0, L]$, $L > 0$, $K : [0, L]\times[0, L]\times\mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The aim of this section is to give an existence theorem for a solution to the above integral equation using Corollary 3.4.

Let $X = [0, L]$. Define

$$q : X \times X \to \mathbb{R}^+$$

by

$$q(x, y) = sup_{l\in[0,L]}|x(l) - y(l)| + sup_{l\in[0,L]}|x(l)|.$$

Then (X, q) is a quasi-partial metric space.

Theorem 4.1. *Let $T, A : [0, L] \to [0, L]$ be self mappings of a quasi-partial metric space (X, q). Suppose the following hypotheses hold:*

(H_1) :

$$Tx(l) = \int_0^L K_1(l, s, x(s))ds + g(l), l \in [0, L],$$

and

$$Ax(l) = \int_0^L K_2(l, s, x(s))ds + g(l), l \in [0, L],$$

where $K_1, K_2 : [0, L] \times [0, L] \times \mathbb{R} \to \mathbb{R}$,

(H_2) :$|K_1(l, s, u(l)) - K_1(l, s, v(l))| \le h(l, s)\{|Au - Av| + |Au|\}$ *for each $u, v \in \mathbb{R}$ and each $l, s \in [0, L]$,*

(H_3) : $\sup_{l\in[0,L]} \int_0^L h(l, s)ds \le \delta$ *for some $\delta \in [0, 1)$,*

(H_4) :$TX \subset AX$ *and AX is closed,*

(H_5) :$ATx = TAx$, *whenever $Ax = Tx$ for some $x \in [0, L]$.*

Then the integral equation (4.1) has a unique solution $u \in [0, L]$.

Proof. Clearly $TX \subset AX$ and AX is closed.
Now, we have

$$q(Tx, Ty) = sup_{l\in[0,L]}|Tx(l) - Ty(l)| + sup_{l\in[0,L]}|Tx(l)|$$

$$= |\int_0^L K_1(l, s, x(s))ds - \int_0^L K_1(l, s, y(s))ds| + |\int_0^L K_1(l, s, x(s))ds|$$

$$\leq \int_0^L |K_1(l, s, x(s)) - K_1(l, s, y(s))|ds + \int_0^L |K_1(l, s, x(s))|ds$$

$$\leq (sup_{l\in[0,L]}|Ax(s) - Ay(s)| + sup_{l\in[0,L]}|Ax(s)|)sup_{l\in[0,L]} \int_0^L h(l, s)ds$$

$$= q(Ax, Ay)sup_{l\in[0,L]} \int_0^L h(l, s)ds.$$

By hypothesis (H_3), there exists $\delta \in [0, 1)$, such that

$$sup_{l\in[0,L]} \int_0^L h(l, s)ds < \delta.$$

Thus, we have $q(Tx, Ty) \leq \delta q(Ax, Ay)$. Hence, all the hypotheses of Corollary 3.4 are satisfied and there exists a unique common fixed point $u \in [0, L]$ of A and T, i.e., there exists a unique solution $u \in X$ to the integral equation (4.1). □

5 Application To Functional Equations Arising In Dynamic Programming Problem

The existence and uniqueness of solutions to functional equations arising in dynamic programming have been studied by various authors (see [18, 19] and references therein). In this section we prove existence and uniqueness of a solution for a class of functional equations in a quasi-partial metric space using Corollary 3.4.

Let U and V be Banach spaces, $W \subset U$ is a state space, $D \subset V$ is a decision space and $\mathbb{R}$ is the field of real numbers. Let $X = B(W)$ denote the set of all closed and bounded real valued functions on W.

Consider the following functional equation

$$p(x) = sup_{y\in D}\{g(x, y) + M(x, y, p(\tau(x, y)))\}, x \in W. \tag{5.1}$$

Let $g : W \times D \to \mathbb{R}$ and $M : W \times D \times \mathbb{R} \to \mathbb{R}$ be bounded functions. $\tau : W \times D \to W$ represents transformation of the process and $p(x)$ represents the optimal return function with initial state x. For an arbitrary $h \in B(W)$ define $\|h\| = \sup |h(x)|$. Also, $(B(W), \|.\|)$ is a Banach space wherein convergence is uniform.

Define $q : X \times X \to \mathbb{R}^+$ by $q(x, y) = |x - y| + |x|$, then (X, q) is a quasi-partial metric space.

Theorem 5.1. *Let $T, A : B(W) \to B(W)$ be self mappings of a quasi-partial metric space $(B(W), q)$. Suppose there exists a $\delta \in [0, 1)$ such that for every $(x, y) \in W \times D$, $Ah_1, Ah_2 \in B(W)$ and $t \in W$:*

1. $|M(x, y, Ah_1\tau(x, y)) - M(x, y, Ah_2\tau(x, y))| \leq \delta\{|Ah_1\tau(x, y) - Ah_2\tau(x, y)| + |Ah_1\tau(x, y)|\}$ *holds;*
2. $g : W \times D \to R$ *and* $M : W \times D \times R \to R$ *are bounded functions;*
3. $ATh = TAh$*, whenever* $Ah = Th$*, for some* $h \in B(W)$.

Then the functional equation

$$Th_i(x) = \sup_{y \in D}\{g(x, y) + M(x, y, Ah_i(\tau(x, y)))\}, x, y \in W \tag{5.2}$$

has a unique bounded solution in $B(W)$.

Proof. By hypothesis (3), the pair (A, T) is weakly compatible. Let λ be an arbitrary positive real number and $Ah_1, Ah_2 \in B(W)$. For $x \in W$, we choose $y_1, y_2 \in D$ so that

$$T(h_1(x)) < g(x, y_1) + M(x, y_1, Ah_1(\tau_1)) + \lambda, \tag{5.3}$$

$$T(h_2(x)) < g(x, y_2) + M(x, y_2, Ah_2(\tau_2)) + \lambda, \tag{5.4}$$

where $\tau_1 = \tau(x, y_1)$ and $\tau_2 = \tau(x, y_2)$.
From the definition of the mapping T, we have

$$T(h_1(x)) \geq g(x, y_2) + M(x, y_2, Ah_1(\tau_2)), \tag{5.5}$$

$$T(h_2(x)) \geq g(x, y_1) + M(x, y_1, Ah_2(\tau_1)). \tag{5.6}$$

Now, from (5.3) and (5.6), we obtain

$$\begin{aligned} T(h_1(x)) - T(h_2(x)) &< M(x, y_1, Ah_1(\tau_1)) - M(x, y_1, Ah_2(\tau_1)) + \lambda \\ &\leq |M(x, y_1, Ah_1(\tau_1)) - M(x, y_1, Ah_2(\tau_1))| + \lambda \\ &\leq \delta\{|Ah_1 - Ah_2| + |Ah_1|\} + \lambda. \end{aligned}$$

Similarly, from (5.4) and (5.5), we obtain

$$T(h_2(x)) - T(h_1(x)) \leq \delta\{|Ah_1 - Ah_2| + |Ah_1|\} + \lambda$$

Hence, we have

$$|T(h_1(x)) - T(h_2(x))| \leq \delta\{|Ah_1 - Ah_2| + |Ah_1|\} + \lambda. \tag{5.7}$$

Since the inequality (5.7) is true for all $x \in W$ and arbitrary $\lambda > 0$, then we have

$$q(Th_1, Th_2) \leq \delta q(Ah_1, Ah_2).$$

Thus, all the conditions of Corollary 3.4 are satisfied and hence the mappings, A and T have a unique common fixed point, i.e., the functional equation (5.2) has a unique bounded solution. □

6 Conclusion

Generalized condition (B) is introduced in a quasi-partial metric space to establish coincidence and common fixed point for two weakly compatible pairs of self-mappings using more natural condition of closedness of the range space. Results are validated with the help of explanatory examples associated with pictographic validations. The motivation behind using a partial quasi-metric space is the fact that the distance from point x to point y may be different to that from y to x and the self-distance of a point need not always be zero. It is interesting to see that although several authors claimed to have introduced some weaker notions of commuting mappings, weak compatibility is still the minimal and the most widely used notion among all weaker variants of commutativity. For brief development of weaker forms of commuting mappings one may refer to Singh and Tomar [20]. Further results obtained are utilised to establish the existence and uniqueness of a solution to the integral equation and the functional equation arising in dynamic programming.

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