DE GRUYTER OPEN Demonstr. Math. 2017; 50:267–277

Research Article Open Access

Michael Gil'*

Norm estimates for functions of non-selfadjoint operators nonregular on the convex hull of the spectrum

https://doi.org/10.1515/dema-2017-0026
Received March 21, 2017; accepted September 6, 2017

Abstract: We consider a bounded linear operator A in a Hilbert space with a Hilbert-Schmidt Hermitian component $(A - A^*)/2i$. A sharp norm estimate is established for functions of A nonregular on the convex hull of the spectrum. The logarithm, fractional powers and meromorphic functions of operators are examples of such functions. Our results are based on the existence of a sequence A_n $(n = 1, 2, \ldots)$ of finite dimensional operators strongly converging to A, whose spectra belongs to the spectrum of A. Besides, it is shown that the resolvents and holomorphic functions of A_n strongly converge to the resolvent and corresponding function of A.

Keywords: functions of non-selfadjoint operators, logarithm, fractional power, meromorphic function

MSC: 47A56, 47A60, 47B10, 47A63, 15A45, 15A60

1 Introduction and statement of the main result

In the book [1], I. M. Gel'fand and G. E. Shilov have established an estimate for the norm of a regular matrix-valued function in connection with their investigations of partial differential equations. However that estimate is not sharp, it is not attained for any matrix. In the paper [2] the author has derived a sharp estimate for matrix-valued functions regular on the convex hull of the spectrum. That estimate is attained for normal matrices. The results of the paper [2] were generalized to various operators [3]-[5]. Obviously, functions having singular points can be nonregular on the convex hull of the spectrum. But such functions, in particular, the logarithm, fractional powers and meromorphic functions of operators, arise in many applications, cf. [6]-[11] and references given therein.

In the paper [12] the author has obtained a norm estimate for functions of finite matrices which are nonregular on the convex hull of the spectrum, but the results from [12] do not admit an extension to infinite dimensional operators. In the present paper we establish a sharp norm estimate for a function of a non-selfadjoint operator nonregular on the convex hull of the spectrum. Besides, in the finite dimensional case we improve the main result from [12].

Let $\mathcal{H}$ be a separable Hilbert space with the scalar product $(\cdot, \cdot)$ and unit operator I; $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators in $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A)$ is the spectrum and $R_\lambda(A) = (A - \lambda I)^{-1}$ $(\lambda \notin \sigma(A))$ is the resolvent, A^* is the operator adjoint to A. SN_p $(p \in [1, \infty))$ denotes the Schatten-von Neumann ideal of compact operators K in $\mathcal{H}$ with the finite norm $N_p(K) = (\text{trace } (KK^*)^{p/2})^{1/p}$.

***Corresponding Author: Michael Gil':** Department of Mathematics, Ben Gurion University of the Negev, P.0. Box 653, Beer-Sheva 84105, Israel, E-mail: gilmi@bezeqint.net

It is assumed that the spectrum of A is the union of two sets σ_1 and σ_2, separated by means of open disjoint simply-connected sets M_1 and M_2:

$$\sigma(A) = \sigma_1 \cup \sigma_2, \sigma_j \subset M_j \ (j = 1, 2) \text{ and } M_1 \cap M_2 = \emptyset. \tag{1.1}$$

Note that our our arguments can be easily extended to the case $\sigma(A) = \cup_{j=1}^m \sigma_j$ $(2 \le m < \infty)$ with $\sigma_j \cap \sigma_k = \emptyset$ $(j \neq k)$.

Let $f(z)$ be a scalar function regular on $M = M_1 \cup M_2$. Then

$$f(A) := -\frac{1}{2\pi i} \sum_{j=1}^{2} \int_{L_j} f(\lambda) R_\lambda(A) d\lambda, \tag{1.2}$$

where $L_j \subset M_j$ are closed Jordan contours surrounding σ_j and the integration is performed in the positive direction. It is also assumed that

$$\Im A = (A - A^*)/2i \in SN_2. \tag{1.3}$$

Put

$$\delta := \ distance\ (\sigma_1, \sigma_2), d_t := \sum_{k=0}^{t} \frac{t!}{((t-k)!k!)^{3/2}} \quad (t = 1, 2, ...)$$

and

$$\xi(A) := \left(1 + \sum_{k=0}^{\infty} \frac{d_k (\sqrt{2} N_2(\Im A))^{k+1}}{\delta^{k+1}}\right)^2.$$

Observe that $\frac{t!}{(t-k)!k!} \le 2^t$ and consequently,

$$d_t = \frac{1}{(t!)^{1/2}} \sum_{k=0}^{t} \frac{(t!)^{3/2}}{((t-k)!k!)^{3/2}} \le \frac{2^{t/2}}{(t!)^{1/2}} \sum_{k=0}^{t} \frac{t!}{(t-k)!k!} = \frac{2^{3t/2}}{(t!)^{1/2}} \quad (t = 1, 2, ...).$$

So

$$\xi(A) \le \left(1 + \sum_{k=0}^{\infty} \frac{2^{2k+1/2} N_2^{k+1}(\Im A)}{(k!)^{1/2} \delta^{k+1}}\right)^2$$

and therefore, the series in the definition of $\xi(A)$ converges. Moreover, by the Schwarz inequality

$$\left(\sum_{k=0}^{\infty} \frac{N_2^k(\Im A)}{(k!)^{1/2} \delta^{k+1}}\right)^2 = \left(\sum_{k=0}^{\infty} \frac{2^{3k} N_2^k(\Im A)}{2^k (k!)^{1/2} \delta^k}\right)^2$$

$$\le \sum_{k=0}^{\infty} \frac{2^{6k} N_2^{2k}(\Im A)}{k! \delta^{2k}} \sum_{j=0}^{\infty} \frac{1}{2^{2j}} = \exp\left[\frac{64 N_2^2(\Im A)}{\delta^2}\right] \frac{4}{3}.$$

Thus,

$$\xi(A) \le \left(1 + \frac{2\sqrt{2} N_2(\Im A))}{\sqrt{3}\delta} \exp\left[\frac{32 N_2^2(\Im A)}{\delta^2}\right]\right)^2. \tag{1.4}$$

Let $co(\sigma_j)$ be the closed convex hull of σ_j $(j = 1, 2)$, and $co(A)$ be the closed convex hull of $\sigma(A)$.

Theorem 1.1. *Let conditions (1.1) and (1.3) hold. Let $f(z)$ be regular on a neighborhood of $co(\sigma_1) \cup co(\sigma_2)$. Then*

$$\|f(A)\| \le \xi(A) \max_{j=1,2} \left(\sup_{s \in \sigma_j} |f(s)| + \sum_{k=1}^{\infty} \sup_{s \in co(\sigma_j)} |f^{(k)}(s)| \frac{(\sqrt{2} N_2(\Im A))^k}{(k!)^{3/2}}\right).$$

The proof of this theorem is presented in the sequel sections.

The series in Theorem 1.1 converges. Indeed, by the Cauchy formula

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_L \frac{f(s)ds}{(s-z)^{k+1}} \ (z \in co(\sigma_j))$$

where L is a closed Jordan contour surrounding $co(\sigma_j)$, we have

$$|f^{(k)}(z)| \le k! m_0 v_0^{-k-1} \ (z \in co(\sigma_j)), \text{ where } m_0 = \frac{1}{2\pi}\int_L |f(s)||ds|$$

and $v_0 = \inf_{s\in L, z\in co(\sigma_j)} |s-z|$. Since

$$\sum_{k=1}^{\infty} \frac{(\sqrt{2}N_2(\Im A))^k}{v_0^k (k!)^{1/2}} < \infty,$$

the series in Theorem 1.1 really converges.

Theorem 1.1 is sharp: if A is selfadjoint, then $\xi(A) = 1$ and we obtain the equality $\|f\| = \sup_{s\in\sigma(A)} |f(s)|$.

Example 1.2. *Let*

$$\sigma(A) = \sigma_1 \cup \sigma_2, \ \textit{with } \sigma_1 \subseteq [-b,-a], \sigma_2 \subseteq [a,b] \tag{1.5}$$

$(0 < a < b)$, and

$$\ln A := -\frac{1}{2\pi i}\sum_{j=1}^{2}\int_{L_j} \ln z \, R_z(A) dz,$$

where the principal branch of $\ln z$ *is used, L_j is a closed Jordan contour surrounding σ_j that does not surround $z = 0 \cup \sigma_k$, $(k \ne j; j, k = 1, 2)$, and where $L_1 \cap L_2 = \emptyset$.*

Clearly, $\ln z$ is regular on $co(\sigma_1) \cup co(\sigma_2)$, but nonregular on $co(A)$. We have $\delta = dist(\sigma_1, \sigma_2) \ge 2a$,

$$\xi(A) \le \xi_1(A) := \left(1 + \sum_{k=0}^{\infty} \frac{d_k(\sqrt{2}N_2(\Im A))^{k+1}}{(2a)^{k+1}}\right)^2. \tag{1.6}$$

In addition,

$$\sup_{s\in\sigma_j} |\ln\ s| \le [\ln^2 b + \pi^2]^{1/2} \text{ and } \sup_{s\in\sigma_j} |(\ln\ s)^{(k)}| \le (k-1)!(2a)^{-k} \ (j = 1, 2; k = 1, 2, \dots).$$

Now Theorem 1.1 implies

$$\|\ln\ A\| \le \xi_1(A)\left([\ln^2 b + \pi^2]^{1/2} + \sum_{k=1}^{\infty} \frac{(\sqrt{2}N_2(\Im A))^k}{k(k!)^{1/2}(2a)^k}\right).$$

Example 1.3. *Under condition (1.5), let*

$$A^{\alpha} := -\frac{1}{2\pi i}\sum_{j=1}^{2}\int_{L_j} z^{\alpha} R_z(A) dz \quad (0 < \alpha < 1),$$

where the contours L_j are the same as in the previous example and the principal branch of z^{α} is used. Clearly, z^{α} is regular on $co(\sigma_1) \cup co(\sigma_2)$. As above, $\delta = dist(\sigma_1, \sigma_2) > 2a$ and (1.6) holds. We have

$$\sup_{s\in\sigma_j} |s^{\alpha}| \le b^{\alpha} = e^{\alpha \ln\ b} \text{ and } \sup_{s\in\sigma_j} |(s^{\alpha})^{(k)}| \le \alpha(1-\alpha)\dots(k-\alpha+1)(2a)^{\alpha-k} \quad (j = 1, 2, \dots).$$

Now Theorem 1.1 implies

$$\|A^{\alpha}\| \le \xi_1(A)\left(b^{\alpha} + \sum_{k=1}^{\infty} \frac{(\sqrt{2}N_2(\Im A))^k}{(k!)^{3/2}} \alpha(1-\alpha)\dots(k-\alpha+1)(2a)^{\alpha-k}\right).$$

2 Maximal chains

For two orthogonal projections P_1, P_2 in $\mathcal{H}$ we write $P_1 < P_2$ if $P_1\mathcal{H} \subset P_2\mathcal{H}$. A set $\mathcal{P}$ of orthogonal projections in $\mathcal{H}$ containing at least two orthogonal projections is called *a chain* if, from $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$, it follows that either $P_1 < P_2$ or $P_1 > P_2$. For two chains $\mathcal{P}_1, \mathcal{P}_2$ we write $\mathcal{P}_1 < \mathcal{P}_2$ if from $P \in \mathcal{P}_1$ it follows that $P \in \mathcal{P}_2$. In this case we say that $\mathcal{P}_1$ precedes $\mathcal{P}_2$. The chain that precedes only itself is called *a maximal chain*.

Let $P^-, P^+ \in \mathcal{P}$, and $P^- < P^+$. If for every $P \in \mathcal{P}$ we have either $P < P^-$ or $P > P^+$, then the pair (P^+, P^-) is called a gap of $\mathcal{P}$. Besides, dim $(P_+\mathcal{H}) \ominus (P_-\mathcal{H})$ is the dimension of the gap.

An orthogonal projection P in $\mathcal{H}$ is called a limit projection of a chain $\mathcal{P}$ if there exists a sequence $P_k \in \mathcal{P}$ $(k = 1, 2, \ldots)$ which strongly converges to P. A chain is said to be closed if it contains all its limit projections.

Recall the following result proved in [13, Proposition XX.4.1, p. 478], [14, Theorem II.14.1]: a chain is maximal if and only if it is closed, contains 0 and I, and all its gaps (if they exist) are one dimensional.

We will say that *a maximal chain $\mathcal{P}$ is invariant for $A \in \mathcal{B}(\mathcal{H})$, or A has a maximal invariant chain $\mathcal{P}$*, if $PAP = AP$ for any $P \in \mathcal{P}$.

Any compact operator has a maximal invariant chain [15, Theorem I.3.1].

Let $\sigma_d(A)$ be the discrete spectrum of A, that is, the set of all eigenvalues of A with finite algebraic multiplicities and which are isolated points of $\sigma(A)$. The essential spectrum $\sigma_{ess}(A)$ of A is defined as the complement of $\sigma_d(A)$ in $\sigma(A)$.

Definition 2.1. *Let*

$$A = D + V, \tag{2.1}$$

where $D \in \mathcal{B}(\mathcal{H})$ is a normal operator and V is a compact quasi-nilpotent operator in $\mathcal{H}$, i.e. $\sigma(V) = \{0\}$. Let V have a maximal invariant chain $\mathcal{P}$ and $PD = DP$ for all $P \in \mathcal{P}$. In addition, let $\sigma_{ess}(A)$ lie on an unclosed Jordan curve. Then A will be called a $\mathcal{P}$-triangular operator, equality (2.1) is its triangular representation, D and V are the diagonal and nilpotent parts of A, respectively.

Let us explain why we require that $\sigma_{ess}(A)$ belongs to an unclosed Jordan curve. To apply the integral representation for analytic functions we need to show that the resolvent $R_\lambda(A)$ has the invariant subspaces for all regular λ, but the equality $PR_\lambda(A)P = R_\lambda(A)P$ for sufficiently large λ is due to the equality $PAP = AP$ and Neumann series

$$R_\lambda(A) = -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}} \quad (|\lambda| > \|A\|).$$

If the set of regular points of A is simply connected, by the resolvent identity one can extend the equality $PR_\lambda(A)P = R_\lambda(A)P$ to all regular λ of A, but if $\sigma_{ess}(A)$ forms at least one closed curve, we could not extend that equality inside the curve. For more details see [16, pp. 32-33].

Lemma 2.2. *Let A be $\mathcal{P}$-triangular. Then $\sigma(A) = \sigma(D)$, where D is the diagonal part of A.*

For the proof see Lemma 11.2.9 from [17].

Let $\psi(P)$ be a scalar valued function of $P \in \mathcal{P}$. If for some $J_1 \in \mathcal{B}(\mathcal{H})$ and any $\epsilon > 0$, there is a partitioning $\mathcal{P}_n$ $(n < \infty)$ of $\mathcal{P}$ of the form

$$0 = P_0 < P_1 < P_2 < \ldots < P_n = I \quad (P_k \in \mathcal{P}, k = 1, \ldots, n),$$

such that

$$\|J_1 - \sum_{k=1}^{n} \psi(P_k)\Delta P_k\| < \epsilon \quad (P_k \in \mathcal{P}_n, \Delta P_k = P_k - P_{k-1}),$$

then J_1 is called an *integral in the Shatunovsky sense*. We write

$$J_1 = \int_{\mathcal{P}} \psi(P)dP.$$

The theory of such integrals can be found in [18], [13, Chapters XX and XI] and the references therein.

Lemma 2.3. *If the condition*

$$\Im A \in SN_p \quad (1 \le p < \infty) \tag{2.2}$$

holds and $\sigma(A)$ is real, then A has a maximal invariant chain $\mathcal{P}$. Moreover, there are a real nondecreasing function $a(P)$ defined on $\mathcal{P}$ (i.e. $a(P) \le a(P_1)$ if $P < P_1$), and a compact quasi-nilpotent operator in $\mathcal{H}$, such that

$$A = \int_{\mathcal{P}} a(P)dP + V, \tag{2.3}$$

and $\mathcal{P}$ is invariant with respect to V.

For the proof see [17, Corollary 11.2.3]. Note that this result is based on the well-known papers [19] and [18, Theorem 3.2]. In particular, (2.3) shows that under the hypothesis of Lemma 3, A is $\mathcal{P}$-triangular.

Lemma 2.4. *Let A be $\mathcal{P}$-trianagular and condition (2.2) hold. Then the nilpotent part V of A is in SN_p.*

Proof. Due to the Weyl inequalities, cf. [20, Theorem II.5.1], we have $N_p(\Im D) \le N_p(\Im A)$. Consequently, $N_p(\Im V) \le N_p(\Im A) + N_p(\Im D) \le 2N_p(\Im A)$. Hence, Theorem III.6.1 from [15] implies $V \in SN_p$, as claimed. □

Now consider operators with non-real spectra. According to Theorem I.5.2 from [20], if $\Im A$ is compact, then the nonreal spectrum of A consists of no more than a countable set of points which are normal (i.e. isolated and having finite multiplicities) eigenvalues.

Let $\lambda_k(A)$ $(k = 1, 2, \ldots)$ denote the non-real eigenvalues of A taken with their multiplicities. Denote by $\mathcal{E}$ the linear closed hull of all the root vectors of A corresponding to non-real eigenvalues. Choose in each root subspace a Jordan basis. Then we obtain vectors η_k for each of which either $A\eta_k = \lambda_k(A)\eta_k$, or $A\eta_k = \lambda_k(A)\eta_k + \eta_{k+1}$. Orthogonalizing the system $\{\eta_k\}$, we obtain the (orthonormal) Schur basis $\{e_k\}$ of the triangular representation:

$$Ae_k = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{kk}e_k \quad (k = 1, 2, \ldots)$$

with $a_{kk} = \lambda_k(A)$ (see [20, Section II.6]). Besides, $\mathcal{E}$ is an invariant subspace of A. Let $Z_{\mathcal{E}}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{E}$ and $C = AZ_{\mathcal{E}} = Z_{\mathcal{E}}AZ_{\mathcal{E}}$. So $\sigma(C)$ consists of the nonreal spectrum of A. Denote $M = (I - Z_{\mathcal{E}})A(I - Z_{\mathcal{E}})$ and $W = Z_{\mathcal{E}}A(I - Z_{\mathcal{E}})$. Since $(I - Z_{\mathcal{E}})AZ_{\mathcal{E}} = 0$, we have

$$A = (Z_{\mathcal{E}} + (I - Z_{\mathcal{E}}))A(Z_{\mathcal{E}} + (I - Z_{\mathcal{E}})) = C + M + W. \tag{2.4}$$

So on $Z_{\mathcal{E}}\mathcal{H} \oplus (I - Z_{\mathcal{E}})\mathcal{H}$, A is represented by the matrix

$$A = \begin{pmatrix} C & W \\ 0 & M \end{pmatrix}. \tag{2.5}$$

Besides $\sigma(A) = \sigma(M) \cup \sigma(C)$, and $\sigma(M)$ is real. Take into account that

$$Ce_k = Ae_k = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{kk}e_k = (D_C + V_C)e_k, \tag{2.6}$$

where

$$D_Ce_k = a_{kk}e_k \quad (k \ge 1) \text{ and } V_Ce_k = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{k-1,k}e_k \quad (k \ge 2),\ V_Ce_1 = 0.$$

In addition, $M - M^* = (I - Z_{\mathcal{E}})(A - A^*)(I - Z_{\mathcal{E}})$, $C - C^* = Z_{\mathcal{E}}(A - A^*)Z_{\mathcal{E}}$. So $N_p(M - M^*) \le N_p(A - A^*)$ and $N_p(C - C^*) \le N_p(A - A^*)$. Hence, $N_p(W - W^*) \le N_p(A - A^*) + N_p(M - M^*) + N_p(C - C^*) < \infty$. Due to Lemma 2.3 M has in $(I - Z_{\mathcal{E}})\mathcal{H}$ a maximal invariant chain denoted by $\mathcal{P}_M$, and M is $\mathcal{P}_M$-triangular. So $M = D_M + V_M$, where D_M is normal and V_M is compact quasi-nilpotent.

Put $D_A = D_M + D_C$ and $V_A = V_M + V_C + W$. According to (2.6), the chain $\mathcal{P}_C = \{\hat{P}_k\}_{k=1}^{\infty}$, where

$$\hat{P}_k = \sum_{j=1}^{k} (., e_k)e_k \ (k = 1, 2, \ldots), \hat{P}_0 = 0, \hat{P}_\infty = Z_\mathcal{E} \tag{2.7}$$

is the maximal invariant chain of C in subspace $Z_\mathcal{E}\mathcal{H}$. Built the chain $\mathcal{P}_A$ in the following way: any $P \in \mathcal{P}_A$ belongs to $\mathcal{P}_M \oplus \mathcal{P}_C$ and ordered as follows: if $P < Z_\mathcal{E}$, then $P = \hat{P}_k$ for some $\hat{P}_k \in \mathcal{P}_C$. If $P > Z_\mathcal{E}$, then $P = Z_\mathcal{E} + P_M$, where $P_M \in \mathcal{P}_M$. Clearly $\mathcal{P}_A$ is the maximal invariant chain of A.

Since V_M and V_C are quasi-nilpotent and mutually orthogonal, $V_M + V_C$ is quasi-nilpotent. $\mathcal{P}_A$ is invariant for $V_M + V_C$ and for W, and W is quasi-nilpotent. Hence it easily follows that $V_A = V_M + V_C + W$ is quasi-nilpotent and $\mathcal{P}_A$ is its invariant chain. Also, $N_p(\Im V_A) \le N_p(\Im A) + N_p(\Im D_A)$ and Lemma 2.4 implies $V_A \in SN_p$. We thus arrive at:

Theorem 2.5. *Let condition (2.2) hold. Then A is $\mathcal{P}_A$-triangular, its nilpotent part $V_A \in SN_p$ and its diagonal part is representable as*

$$D_A = \int_{\mathcal{P}_M} a(P)dP + \sum_{k=1}^{\infty} \lambda_k(A)\Delta\hat{P}_k \quad (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1}; \hat{P}_k \in \mathcal{P}_C, k = 1, 2, \ldots),$$

where $\lambda_k(A)$ are the nonreal eigenvalues with their multiplicities and $a(P)$ is a nondecreasing function of $P \in \mathcal{P}_M$.

3 Basic lemma

The symbol $A_n \xrightarrow{s} A$ means that $\lim_{n\to\infty} A_n = A$ in the strong topology. It is well known that the spectrum is not continuous with respect to the strong topology in general, cf. [21, Section VIII.1, p. 427]. That is, from $A_n \xrightarrow{s} A$ the relation

$$\lim_{n\to\infty} \sigma(A_n) \subseteq \sigma(A)$$

does not follow in the general case. In this section we point the sequence A_n, for which the just pointed limit is valid.

Below $A/\mathcal{H}_1$ means the restriction of A onto $\mathcal{H}_1 \subset \mathcal{H}$. The following lemma is our main tool in the proof of Theorem 1.1.

Lemma 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let condition (2.2) hold. Then there exist a sequence B_n $(n = 1, 2, \ldots)$ of finite dimensional operators strongly converging to A and a sequence $Z_n \xrightarrow{s} I$ of orthogonal projections, such that*

$$Z_n B_n = B_n Z_n \text{ and } \sigma(B_n/Z_n\mathcal{H}) \subseteq \sigma(A) \ (n = 1, 2, \ldots). \tag{3.1}$$

To prove Lemma 3.1 we need the following result.

Lemma 3.2. *Let P_k $(k = 0, \ldots, n;\ n > 2)$ be a finite chain of orthogonal projections in $\mathcal{H}$: $0 = P_0\mathcal{H} \subset P_1\mathcal{H} \subset \ldots. \subset P_n\mathcal{H} = \mathcal{H}$. Let $B \in \mathcal{B}(\mathcal{H})$ be defined by*

$$B = \sum_{k=1}^{n} \phi_k \Delta P_k + W \ \ (\Delta P_k = P_k - P_{k-1}), \tag{3.2}$$

where ϕ_k $(k = 1, \ldots, n)$ are complex numbers and W is a compact operator satisfying the relations

$$P_{k-1}WP_k = WP_k \ \ (k = 1, \ldots, n). \tag{3.3}$$

Then there is a sequence Q_l $(l = 1, 2, \ldots)$ of finite dimensional orthogonal projections strongly converging to I, such that the operators $T_l = Q_l B Q_l$ $(l = 1, 2, \ldots)$ have the property

$$\sigma(T_l/Q_l\mathcal{H}) \subseteq \ \sigma(B) \ \ (l = 1, 2, \ldots). \tag{3.4}$$

Moreover, the nilpotent part W_l of T_l satisfies the equality $W_l Q_l = Q_l W Q_l$.

Proof. Put

$$S = \sum_{k=1}^{n} \phi_k \Delta P_k.$$

Clearly, the spectrum of S consists of the numbers ϕ_k $(k = 1, \dots, n)$. Due to (3.3) $W^n = 0$. In addition, W and S have joint invariant subspaces. Since $\Delta P_k W \Delta P_k = 0$, we have $\sigma(S) = \sigma(B)$. Consequently, ϕ_k $(k = 1, \dots, n)$ are eigenvalues of B.

Furthermore, let $(\Delta P_k f, g)$ $(f, g \in \Delta P_k \mathcal{H})$ be the scalar product in $\Delta P_k \mathcal{H}$. Recall that $(.,.)$ is the scalar product in $\mathcal{H}$.

Let $\{e_m^{(k)}\}_{m=1}^{\infty}$ be an orthogonal normal basis in $\Delta P_k \mathcal{H}$. That is, $(\Delta P_k e_m^{(k)}, e_j^{(k)}) = 0$ if $j \neq m$ and $(\Delta P_k e_m^{(k)}, e_m^{(k)}) = 1$. Put

$$\Delta Q_l^{(k)} = \sum_{m=1}^{l} (., e_m^{(k)}) e_m^{(k)} \quad \text{and } Q_l^{(j)} = \sum_{k=1}^{j} \Delta Q_l^{(k)} \quad (j = 1, \dots, n; \; l = 1, 2, \dots).$$

Clearly operators $\Delta Q_l^{(k)}$ strongly converge to ΔP_k, and operators $Q_l^{(j)}$ strongly converge to P_j as $l \to \infty$. So $Q_l := Q_l^{(n)}$ strongly converge to I as $l \to \infty$. In addition, Q_l is nl-dimensional,

$$\Delta Q_l^{(k)} \Delta P_k = \Delta P_k \Delta Q_l^{(k)} = \Delta Q_l^{(k)}, \; Q_l^{(j)} P_j = P_j Q_l^{(j)} = Q_l^{(j)},$$

$(j = 1, \dots, n; \; l = 1, 2, \dots)$. The nl-dimensional operators

$$S_l = \sum_{k=1}^{n} \phi_k \Delta Q_l^{(k)} = S Q_l$$

strongly converge to S as $l \to \infty$ and $\sigma(S_l) = \sigma(S_l / Q_l \mathcal{H}) = \{\phi_k\} \subseteq \sigma(B)$. Besides, the multiplicity of ϕ_k as the eigenvalue of S_l is finite, while the multiplicity of ϕ_k as the eigenvalue of S is infinite $(k = 1, \dots, n)$. Due to condition (3.3) we obtain

$$W = \sum_{j=1}^{n} \sum_{k=1}^{n} \Delta P_j W \Delta P_k = \sum_{k=2}^{n} P_{k-1} W \Delta P_k.$$

Introduce the operators

$$W_l := Q_l W Q_l = \sum_{k=2}^{n} Q_l^{(k-1)} W \Delta Q_l^{(k)}.$$

Since $Q_l \xrightarrow{s} I$ as $l \to \infty$, operators $W_l \xrightarrow{s} W$. But W is compact and therefore operators W_l converge to W in the operator norm. Take into account that

$$W_l Q_l^{(j)} = \sum_{k=1}^{j} Q_l^{(k-1)} W \Delta Q_l^{(k)} = Q_l^{(j-1)} \sum_{k=1}^{n} Q_l^{(k-1)} W \Delta Q_l^{(k)}.$$

Hence, $W_l Q_l^{(j)} = Q_l^{(j-1)} W_l Q_l^{(j)}$ $(j = 1, \dots, n)$. So $Q_l^{(j)}$ $(j = 1, \dots, n)$ are invariant projections of W_l and, in addition, W_l is nilpotent. Put $T_l := S_l + W_l = Q_l B Q_l$. Since W_l and S_l have joint invariant subspaces, we obtain $\sigma(T_l) = \sigma(S_l)$ and $\sigma(S_l / Q_l \mathcal{H}) \subseteq \sigma(B)$. The lemma is proved. □

Proof of Lemma 3.1: Let $\hat{P}_n$ $(n = 1, 2, \dots)$ be defined as in the previous section and $P_k^{(n)}(M)$ $(k = 0, \dots, n)$ be a partitioning of $\mathcal{P}_M$:

$$0 = P_0^{(n)}(M) < P_1^{(n)}(M) < P_2^{(n)}(M) < \dots < P_n^{(n)}(M) = I \;\; (P_k^{(n)}(M) \in \mathcal{P}_M, k = 1, \dots, n).$$

Since $\hat{P}_n \xrightarrow{s} Q_{\mathcal{E}}$, according to Theorem 2.5, D_A is a strong limit of the operator sums

$$D_n = \sum_{k=1}^{n-1} \lambda_k(A) \Delta \hat{P}_k + \lambda_n(A)(Q_{\mathcal{E}} - \hat{P}_{n-1}) + \sum_{k=1}^{n} a(P_k^{(n)}(M)) \Delta P_k^{(n)}(M),$$

as $n \to \infty$. Define the projections $P_k^{(n)}(A)$ $(k = 1, \dots, 2n)$ by

$$P_k^{(n)}(A) = \hat{P}_k \ (k < n), P_n^{(n)}(A) = Q_{\mathcal{E}} \text{ and } P_k^{(n)}(A) = P_{k-n}^{(n)}(M) + Q_{\mathcal{E}} \ (n < k \le 2n). \tag{3.5}$$

Then we can write

$$D_n = \sum_{k=1}^{2n} c(P_k^{(n)}(A))\Delta P_k^{(n)}(A) \quad (\Delta P_k^{(n)}(A) = P_k^{(n)}(A) - P_{k-1}^{(n)}(A)), \tag{3.6}$$

where $c(P_k^{(n)}(A)) = \lambda_k(A)$ for $k = 1, \dots, n$, and $c(P_{n+j}^{(n)}(A)) = a(P_j^{(n)}(M))$ for $j = 1, \dots, n$. In addition, denote

$$V_n = \sum_{k=1}^{2n} P_{k-1}^{(n)}(A) V \Delta P_k^{(n)}(A),$$

where V is the nilpotent part of A. Put $A_n = D_n + V_n$. Then relations (3.2) and (3.3) hold with $2n$ instead of n, $B = A_n$, $P_k = P_k^{(n)}(A)$, $\phi_k = c(P_k^{(n)}(A))$ and $W = V_n$. Besides,

$$\sigma(A_n) = \sigma(D_n) = \{c(P_k^{(n)}(A))\}_{k=1}^{2n} \subseteq \sigma(A). \tag{3.7}$$

Since V compact quasi-nilpotent, due to [15, Theorem III.4.1], $V_n \to V$ uniformly, and therefore $A_n \xrightarrow{s} A$ as $n \to \infty$. According to Lemma 3.2, for each $n < \infty$ there are finite dimensional projections Q_{ln} $(l = 1, 2, \dots)$ strongly converging to I as $l \to \infty$, such that the operators $T_{ln} := Q_{ln} A_n Q_{ln}$ have the properties

$$\sigma(T_{ln}/Q_{ln}\mathcal{H}) \subseteq \ \sigma(A_n) \subseteq \sigma(A) \ \ (l, n = 1, 2, \dots).$$

Put $Z_n = Q_{nn}$ and $B_n = T_{nn}$. Then $B_n Z_n = Z_n B_n$ and $\sigma(B_n/Z_n\mathcal{H}) \subseteq \sigma(A)$. Since $A_n \xrightarrow{s} A$, we have $B_n \xrightarrow{s} A$. This finishes the proof. □

4 Convergence of resolvents and operator functions

Put $\rho(A, \lambda) := dist\ (\sigma(A), \lambda)\ (\lambda \notin \sigma(A))$. Let B_n be as in Lemma 3.1. Making use of (3.1), we have $\rho(B_n/Z_n\mathcal{H}, \lambda) \ge \rho(A, \lambda)$. For any finite $p \ge 1$ there is an integer $\nu \ge 1$, such that $2\nu \ge p$, and therefore $\Im A \in SN_p$ implies $\Im A \in SN_{2\nu}$. Then by Theorem 7.9.1 from [3],

$$\|(B_n - \lambda Z_n)^{-1}\| \le \sum_{m=0}^{\nu-1}\sum_{k=0}^{\infty} \frac{(c_\nu N_{2\nu}(\Im B_n))^{k\nu+m}}{\sqrt{k!}\rho^{k\nu+m+1}(A, \lambda)},$$

where the constant c_ν depends on ν, only. Since $\Im B_n \xrightarrow{s} \Im A$, with $d_\nu(A) = c_\nu \sup_n N_{2\nu}(\Im B_n) < \infty$ we have

$$\|(B_n - \lambda Z_n)^{-1}\| \le \sum_{m=0}^{\nu-1}\sum_{k=0}^{\infty} \frac{d_\nu^{k\nu+m}(A)}{\sqrt{k!}\rho^{k\nu+m+1}(A, \lambda)}. \tag{4.1}$$

For any $s_0 \in \sigma(A)$ we have $|s_0 - \lambda| \ge \rho(A, \lambda)$. In addition, B_n and $I - Z_n$ are mutually orthogonal. Hence,

$$\|(B_n + s_0(I - Z_n) - \lambda I)^{-1}\|^2 = \|(B_n - \lambda Z_n + (s_0 - \lambda)(I - Z_n))^{-1}\|^2$$

$$\le \max\{\|(B_n - \lambda Z_n)^{-1}\|^2, |s_0 - \lambda|^{-2}\}.$$

Thus, from (4.1) we get

$$b_0 := \|(B_n - \lambda Z_n + s_0(I - Z_n) - \lambda I)^{-1}\| \le \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} \frac{d_\nu^{knu+m}(A)}{\sqrt{k!}\rho^{k\nu+m+1}(A, \lambda)}.$$

Consequently,

$$b_0 := \sup_n \|(B_n - \lambda Z_n + s_0(I - Z_n) - \lambda I)^{-1}\| < \infty \quad (\lambda \notin \sigma(A)).$$

Since $I - Z_n \xrightarrow{s} 0$, $B_n \xrightarrow{s} A$ we have

$$\|R_\lambda(A)x - (B_n - \lambda Z_n + s_0(I - Z_n) - \lambda I)^{-1}x\|$$
$$= \|(B_n - \lambda Z_n + s_0(I - Z_n) - \lambda I)^{-1}(A - B_n - s_0(I - Z_n))R_\lambda(A)x\|$$
$$\le b_0\|(A - B_n - s_0(I - Z_n))R_\lambda(A)x\| \to 0 \quad (x \in \mathcal{H}).$$

But

$$R_\lambda(B_n + s_0(I - Z_n)) = Z_n R_\lambda(B_n) + \frac{1}{s_0 - \lambda}(I - Z_n) \quad (\lambda \notin \sigma(A)) \tag{4.2}$$

and therefore,

$$R_\lambda(B_n + s_0(I - Z_n)) - Z_n R_\lambda(B_n) = \frac{1}{s_0 - \lambda}(I - Z_n) \xrightarrow{s} 0.$$

We thus have proved

Lemma 4.1. *Let condition (2.2) hold. Then there are a sequence B_n $(n = 1, 2, \ldots)$ of finite dimensional operators strongly converging to A, and a sequence $Z_n \xrightarrow{s} I$ of orthogonal projections, such that (3.1) holds. Moreover, $R_\lambda(B_n) \xrightarrow{s} R_\lambda(A)$ for any $\lambda \notin \sigma(A)$.*

By this lemma and the integral representation of holomorphic operator functions, we arrive at

Corollary 4.2. *Let condition (2.2) hold. Then there is a sequence B_n $(n = 1, 2, \ldots)$ of finite dimensional operators strongly converging to A, such that for any f regular on a simply connected open set containing $\sigma(A)$, we have $Z_n f(B_n) = f(B_n)Z_n \xrightarrow{s} f(A)$, where Z_n $(n = 1, 2, \ldots)$ are taken from Lemma 4.1.*

5 Proof of Theorem 1.1

First, assume that A is an n-dimensional operator; $\hat\lambda_j(A)$ $(j = 1, \ldots, n)$ are the eigenvalues of A taken with their algebraic multiplicities. Put

$$g(A) := \left[N_2^2(A) - \sum_{k=1}^{n} |\hat\lambda_k(A)|^2\right]^{1/2}.$$

The quantity $g(A)$ has the following property:

$$g^2(A) \le 2N_2^2(\Im A), \tag{5.1}$$

cf. [3, Section 2.1]. Let P_1 be the invariant orthogonal projection corresponding to σ_1 and $P_2 = I - P_1$. So $\sigma(AP_1) = \sigma_1$, $P_2A = P_2AP_2$ and $\sigma(P_2A) = \sigma_2$. Since $P_2AP_1 = P_2P_1AP_1 = 0$, we have $A = A_1 + A_2 + \hat C$, where $A_1 = AP_1$, $A_2 = P_2A$ and $\hat C = P_1AP_2$. In the block form we can write

$$A = \begin{pmatrix} A_1 & \hat C \\ 0 & A_2 \end{pmatrix}.$$

Furthermore, under condition (1.1), the equation

$$A_1X - XA_2 = -\hat C \quad (j = 1, 2,) \tag{5.2}$$

has a unique solution X and $(I+X)^{-1}A(I+X) = \hat{D}$, where $\hat{D} = A_1 + A_2$, cf. [22]. It is simple to see that the inverse to $I+X$ is the operator $I-X$. Thus,

$$(I-X)A(I+X) = \hat{D}. \tag{5.3}$$

Let $n_j = \operatorname{rank} A_j$ $(j = 1, 2)$. Corollary 6.2 from [23] implies the inequality

$$N_2(X) \le N_2(C) \sum_{t=0}^{n_1+n_2-2} \frac{1}{\delta^{t+1}} \sum_{k=0}^{t} \binom{t}{k} \frac{g^k(A_1) g^{t-k}(A_2)}{\sqrt{(t-k)!k!}}. \tag{5.4}$$

Taking into account that $n_1 + n_2 \le n$, we can write

$$N_2(X) \le N_2(\hat{C}) \sum_{t=0}^{n-2} \frac{d_t \hat{g}^t}{\delta^{t+1}}, \tag{5.5}$$

where $\hat{g} = \max\{g(A_1), g(A_2)\}$. Due to (5.1),

$$g(A_1) \le \sqrt{2} N_2(\Im A_1) = \sqrt{2} N_2(P_1 \Im A P_1) \le \sqrt{2} N_2(\Im A). \text{ Similarly, } g(A_2) \le \sqrt{2} N_2(\Im A). \tag{5.6}$$

Thus,

$$N_2(X) \le N_2(\hat{C}) \sum_{t=0}^{n-2} \frac{d_t (\sqrt{2} N_2(\Im A))^t}{\delta^{t+1}}. \tag{5.7}$$

But $N_2^2(\hat{C}) = N_2^2(A) - N_2^2(A_2) - N_2^2(A_1)$ and

$$N_2^2(A_2) + N_2^2(A_1) \ge \sum_{k=1}^{n} |\hat{\lambda}_k(A)|^2.$$

Consequently, $N_2(\hat{C}) \le g(A) \le \sqrt{2} N_2(\Im A)$ and (5.7) implies

$$N_2(X) \le \sum_{t=0}^{n-2} \frac{d_t (\sqrt{2} N_2(\Im A))^{t+1}}{\delta^{t+1}}. \tag{5.8}$$

Hence,

$$\kappa_X := \|I+X\| \|I-X\| \le \xi(A, n), \tag{5.9}$$

where

$$\xi(A, n) = \left(1 + \sum_{t=0}^{n-2} \frac{d_t (\sqrt{2} \Im A)^{t+1}}{\delta^{t+1}}\right)^2.$$

Since A_j are mutually orthogonal,

$$f(\hat{D}) = \sum_{k=1}^{2} P_j f(A_j) \text{ and } \|f(\hat{D})\| = \max_{j=1,2} \|P_j f(A_j)\|.$$

Formula (5.3) yields

$$A^m = (I+X)\hat{D}^m (I-X) \quad (m = 1, 2, \ldots) \text{ and } f(A) = (I+X) f(\hat{D})(I-X).$$

So

$$\|f(A)\| \le \kappa_X \max_{j=1,2} \|P_j f_j(A_j)\|. \tag{5.10}$$

Due to [4, Theorem 7.2] we obtain

$$\|f(A_j)\| \le \sup_{s \in \sigma_j} |f(s)| + \sum_{k=1}^{n_j-1} \sup_{s \in co(\sigma_j)} |f^{(k)}(s)| \frac{g^k(A_j)}{(k!)^{3/2}}.$$

Hence, (5.6), (5.9) and (5.10) yield:

Lemma 5.1. *Let $A \in \mathbb{C}^{n\times n}$ and condition (1.1) hold. Let f satisfy the hypothesis of Theorem 1.1. Then*

$$\|f(A)\| \le \xi(A,n) \max_{j=1,2} \left(\sup_{s\in\sigma_j} |f(s)| + \sum_{k=1}^{n_j-1} \sup_{s\in co(\sigma_j)} |f^{(k)}(s)| \frac{(\sqrt{2}N_2(\Im A))^k}{(k!)^{3/2}} \right).$$

Proof of Theorem 1.1: Letting $n \to \infty$ in Lemma 5.1 and applying Lemma 3.1, we finish the proof of Theorem 1.1. □

Acknowledgement: I am very grateful to the referees of this paper for their really helpful remarks.

References

[1] Gel'fand I. M., Shilov G. E., Some questions of theory of differential equations, Nauka, Moscow, 1958 (in Russian)
[2] Gil' M. I., Estimates for norm of matrix-valued functions, Linear Multilinear Algebra, 1993, 35, 65-73
[3] Gil' M. I., Operator functions and localization of spectra, Lecture Notes in Math., 1830, Springer-Verlag, Berlin, 2003
[4] Gil' M. I., Estimates for functions of finite and infinite matrices. Perturbations of matrix functions, Int. J. Math. Game Theory Algebra, 2013, 21(4/5), 328-392
[5] Gil' M. I., Estimates for entries of matrix valued functions of infinite matrices, Math. Phys. Anal. Geom., 2008, 11(2), 175-186
[6] Boyadzhiev K. N., Logarithms and imaginary powers of operators on Hilbert spaces, Collect. Math., 1994, 45, 287-300
[7] Chiumiento E., On normal operator logarithms, Linear Algebra Appl., 2013, 439, 455-462
[8] Conway J. B., Morrel B. B., Roots and logarithms of bounded operators on Hilbert space, J. Funct. Anal., 1987, 70(1), 171-193
[9] Haase M., Spectral properties of operator logarithms, Math. Z., 2003, 245, 761-779
[10] Okazawa N., Logarithms and imaginary powers of closed linear operators, Int. Eqs. Oper. Theor., 2000, 38, 458-500
[11] Schmoeger C., On logarithms of linear operators on Hilbert spaces, Demonstr. Math., 2002, 35(2), 375-384
[12] Gil' M. I., Matrix functions nonregular on the convex hull of the spectrum, Linear Multilinear Algebra, 2012, 60(4), 465-473
[13] Gohberg I. C., Goldberg S., Kaashoek M. A., Classes of linear operators, 2, Birkhäuser Verlag, Basel, 1993
[14] Brodskii M. S., Triangular and Jordan representations of linear operators, Transl. Math. Mongr., 32, Amer. Math. Soc., Providence, R. I., 1971
[15] Gohberg I. C., Krein M. G., Theory and applications of Volterra operators in a Hilbert space, Trans. Mathem. Monographs, 24, Amer. Math. Soc., R. I, 1970
[16] Radjavi H., Rosenthal P., Invariant subspaces, Springer-Verlag, Berlin, 1973
[17] Gil' M. I., Bounds for determinants of linear operators and their applications, CRC Press, Taylor & Francis Group, London, 2017
[18] Brodskii V. M., Gohberg I. C., Krein M. G., General theorems on triangular representations of linear operators and multiplicative representations of their characteristic functions, Funct. Anal. Appl., 1969, 3(4), 255-276
[19] Brodskii M. S., Triangular representation of some operators with completely continuous imaginary parts, Dokl. Akad. Nauk SSSR, 1960, 133, 1271-1274 (in Russian). English translation: Soviet Math. Dokl., 1960, 1, 952-955
[20] Gohberg I. C., Krein M. G., Introduction to the theory of linear nonselfadjoint operators, Trans. Mathem. Monographs, 18, Amer. Math. Soc., R. I., 1969
[21] Kato T., Perturbation theory for linear operators, Springer-Verlag, Berlin 1980
[22] Bhatia R., Rosenthal P., How and why to solve the matrix equation $AX - XB = Y$, Bull. London Math. Soc., 1997, 29, 1-21
[23] Gil' M. I., Resolvents of operators on tensor products of Euclidean spaces, Linear Multilinear Algebra, 2015, 64(4), 699-716