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L^p-potentials on infinite networks

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Abstract: Based on the existence of discrete L^p– subharmonic functions, a classification of infinite networks is carried out.

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MSC: 31C20, 31C05

1 Introduction

A network X is an infinite graph, connected and locally finite with a set of transition (between vertices) indices that need not be symmetric. A real-valued function $u(x)$ on X is subharmonic if the Laplacian $\Delta u(x) \geq 0$; it is superharmonic if $\Delta u(x) \leq 0$. A subharmonic function $u(x)$ is known as an L^p– subharmonic function ($p \geq 1$) if $\sum_{x \in X} |u(x)|^p < \infty$. Based on the existence of L^p– subharmonic functions, a classification of networks is presented in this note.

Any L^p–superharmonic function is a potential on X. However, in a network with potentials it is possible that there is no non-zero L^p– superharmonic function. Recall that it is known that in a symmetric network there exists a non-zero L^1– superharmonic function if and only if the Poisson equation $-\Delta p(x) = 1$ has a positive solution. It is proved here that an L^p– superharmonic function can be represented as the sum of a convergent series of L^p– potentials; consequently, an L^p– superharmonic function is vertex-wise increasing limit of a sequence of L^p– potentials. Finally, it is shown also that if v is an L^p– superharmonic function defined outside a finite set, then $v = v_1 - v_2$, where v_1, v_2 are L^p– superharmonic functions on X, with v_2 being harmonic outside a finite set.

2 Preliminaries

In a graph, two vertices x, y are said to be neighbours, written $x \sim y$, if and only if there is an edge $[x, y]$ joining x, y. A network X is a countably infinite graph that is connected (that is, any two vertices can be connected by a path), locally finite (that is, any vertex has only a finite number of neighbours) and without self-loops (that is, $x \sim x$ is not valid for any vertex x); also it is provided with a set of transition indices $\{t(x, y)\}$ such that $t(x, y) \geq 0$ for any two vertices, $t(x, y) > 0$ if and only if $x \sim y$, $t(x, y)$ and $t(y, x)$ need not be the same.

If $u(x)$ is a real-valued function on X, then the Laplacian is $\Delta u(x) = \sum_{y \in X} t(x, y)[u(y) - u(x)] = \sum_{y \sim x} t(x, y)[u(y) - u(x)]$. The function $u(x)$ is said to be subharmonic at a vertex x if $\Delta u(x) \geq 0$ and superharmonic at x if $\Delta u(x) \leq 0$; the function $u(x)$ is subharmonic, superharmonic on X if it is so at every vertex in X. A non-negative superharmonic function $s(x)$ on X is said to be a potential [1] if $h(x)$ is a harmonic function on X such that $0 \leq h(x) \leq s(x)$, then $h = 0$. The Riesz representation theorem states that any non-negative

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superharmonic function $s(x)$ is the unique sum of a potential $p(x)$ and a non-negative harmonic function $h(x)$.

It is, however, possible that there is no positive potential on X, in which case we say that X is a parabolic network; if there are potentials $q > 0$ on X, then X is referred to as a hyperbolic network. In the context of a random walk X with $t(x, y)$ representing the transition probability from the state x to the state y, the terms recurrent and transient are used instead of parabolic and hyperbolic.

Finally, a subharmonic function $u(x)$ on X is referred to as an L^p–subharmonic function if $\sum_{x\in X} |u(x)|^p < \infty$. Clearly, if u is an L^p–subharmonic function on X, then $v = -u$ is an L^p– superharmonic function.

3 L^p– subharmonic functions

For a real-valued function $u(x)$ on X, if $\Delta u(x) \geq 0$ then $u(x) \leq \sum_{y\sim x} \frac{t(x,y)}{t(x)} u(y)$ where $t(x) = \sum_{y\sim x} t(x, y)$. Hence if $\varphi(x)$ is an increasing convex function on $\mathbf{R}$, then we have $\varphi[u(x)] \leq \sum_{y\sim x} \frac{t(x,y)}{t(x)} \varphi[u(y)]$. In particular, if $u(x)$ is a subharmonic function on X and $v = u^+$, then $v^p(x)$ is subharmonic on X for any p, $1 \leq p < \infty$; and if $s(x)$ is subharmonic on X, so is $e^{s(x)}$.

Proposition 1. *Let $u \geq 0$ be subharmonic on X. Then either $u = 0$ or $\sum_{x\in X} u(x) = \infty$.*

Proof. Suppose $\sum_{x\in X} u(x) = M < \infty$. Then for any $\varepsilon > 0$, there is a finite set A of X such that $\sum_{x\in X\setminus A} u(x) < \varepsilon$. Since $u(x) \geq 0$, we conclude that $u(x) < \varepsilon$ if $x \in X\setminus A$. Then by the Maximum Principle for subharmonic functions, we have $u(x) \leq \varepsilon$ if $x \in A$. Consequently, $u(x) \leq \varepsilon$ on X, leading to the conclusion $u = 0$ on X. □

Corollary 2. *If $u \geq 0$ is a non-zero subharmonic function on X and $p \geq 1$, then $\sum_{x\in X} u^p(x) = \infty$; similarly, if $s(x)$ is any subharmonic function on X, then $\sum_{x\in X} e^{s(x)} = \infty$.*

Proposition 3. *If $u(x)$ is an L^p– subharmonic function, then $u \leq 0$ and $lim_{x\to\infty} u(x) = 0$ (the limit in the sense that given $\varepsilon > 0$, there exists a finite set A such that $|u(x)| < \varepsilon$ if $x \notin A$). Consequently, if h is L^p–harmonic on X, then $h = 0$.*

Proof. Let $v = u^+$. Then $s(x) = v^p(x) \geq 0$ is a subharmonic function such that $\sum_x s(x) = \sum_x v^p(x) \leq \sum_x |u(x)|^p < \infty$. Hence, by the above Proposition 1, $s = 0$ which shows that $u^+ = 0$; that is $u \leq 0$.
Moreover, for any $\varepsilon > 0$, there exists a finite set A such that $\sum_{x\in X\setminus A} |u(x)|^p < \varepsilon^p$. In particular, $|u(x)| < \varepsilon$ if $x \in X\setminus A$; hence $lim_{x\to\infty} u(x) = 0$. □

Now we give an example of an L^1–subharmonic function.

Example

Let $X = \{0, 1, 2, 3, \ldots.\}$; $t(n, n+1) = \frac{3}{4}$ *if* $n \geq 0$, $t(n, n-1) = \frac{1}{4}$ *if* $n \geq 1$.

Consider the function $u(n) = -3^{-n}$ *if* $n \geq 0$. We have $\Delta u(0) = \frac{1}{2}$ and $\Delta u(n) = 0$ if $n \geq 1$. Hence, $u(n)$ is a subharmonic function on X with harmonic support at the vertex 0 (that is, $u(n)$ is harmonic at every vertex other than the vertex 0); $\sum_{n\geq 0} |u(n)| < \infty$; thus $u(n)$ is a negative L^1–subharmonic function which tends to 0 when $n \to \infty$.

Remark In this context, the paper Rigoli, Salvatori and Vignati [2] is of interest, wherein G is an infinite connected graph with uniformly bounded vertex degree and symmetric unit transition functions. Place certain asymptotic growth conditions on the cardinality of balls in G so that G behaves like a discrete version of a complete Riemannian manifold whose geometry is controlled in terms of volume, avoiding curvature assumptions. Then they prove certain Liouville type theorems for subharmonic functions on G when they are of logarithmic or of small polynomial growth. Incidentally they prove also some properties of L^p– subharmonic functions on G when $p \geq 2$.

4 L^p– Superharmonic functions

If a real-valued function $v(x)$ is L^p– superharmonic on X, then $v \geq 0$ (from Proposition 3). Consequently, if X is a parabolic network, then 0 is the only L^p– superharmonic function on X. If $v(x)$ is an L^p–superharmonic function on X, then $v(x)$ is a non-negative superharmonic function on the parabolic network X, hence a constant c. Then necessarily $c = 0$. Thus, if there is a non-zero L^p– superharmonic function on X, then X has to be a hyperbolic network.

In fact, if $v(x)$ is a non-zero L^p– superharmonic function on a network X, then $v(x)$ is a potential on X. For the superharmonic function, $v(x)$ being non-negative is the sum of a potential and a non-negative harmonic function $h(x)$. Since $h \leq v$, h also is an L^p– harmonic function, so that $h = 0$ (Proposition 3). Thus $v(x)$ is a potential on X.

Example of a hyperbolic network on which any L^p– superharmonic function ($p \geq 1$) can only be the zero function:

Lemma 4. *Let $X = \{0, 1, 2, 3, \ldots.\}$ be a network with the symmetric transition index $\frac{1}{2}$ on each edge. Let $h(n)$ be a non-negative bounded function that is harmonic at every vertex $n \neq 0$. Then $h(n)$ is a constant, $h(n) = h(0)$ for all n.*

Proof. Let $h(0) = \lambda$ and $h(1) = a$. Then $h(n) = na - (n-1)\lambda$ for all $n \geq 0$. Since $h(n) \geq 0$, then $a \geq \frac{n-1}{n}\lambda$. Hence allowing $n \to \infty$, we note $a \geq \lambda$. Suppose $a = \lambda + \varepsilon$ where $\varepsilon \geq 0$. Then $h(n) = n(\lambda + \varepsilon) - (n-1)\lambda = \lambda + n\varepsilon$. But $h(n)$ is bounded, so that $\varepsilon = 0$. So $h(n) = \lambda$ for all n. □

Now consider the example of the network $X = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots,\}$ with transition indices $t(n, n+1) = \frac{1}{2} = t(n+1, n)$ if $n \leq -1$; and $t(n, n+1) = \frac{3}{4}$, $t(n+1, n) = \frac{1}{4}$ if $n \geq 0$. Take the function $Q(n) = 1$ if $n \leq 0$ and $Q(n) = 3^{-n}$ if $n \geq 1$. Then $Q(n)$ is harmonic at every vertex $n \neq 0$; and $\Delta Q(0) = t(0,1)[Q(1) - Q(0)] + t(0,-1)[Q(-1) - Q(0)] = -\frac{1}{2}$. Hence, $Q(n)$ is a positive superharmonic function with harmonic support at the vertex $n = 0$.

In fact, $Q(n)$ is a positive potential. Let $h(n) \geq 0$ be a harmonic function such that $h(n) \leq Q(n)$. Let $h(0) = \lambda$. Then by the above Lemma 4, the bounded harmonic function $h(n) = \lambda$ for all $n \leq 0$. Now $0 = \Delta h(0) = \frac{3}{4}[h(1) - \lambda] + \frac{1}{2}[\lambda - \lambda]$, hence $h(1) = \lambda$. Similar calculation at successive vertex shows that $h(n) = \lambda$ for $n \geq 1$. Consequently, $\lambda = h(n) \leq Q(n)$ for all n; since $Q(n) \to 0$ when $n \to \infty$, we have $\lambda = 0$, hence $Q(n)$ is a potential with vertex harmonic support at a single vertex. Moreover for any $p \geq 1$, $\sum_n [Q(n)]^p = \infty$. Hence by the following Proposition 5 (c) there are no L^p–superharmonic functions on X. □

Theorem 5. *On a hyperbolic network X, the following are equivalent:*

a. *There is a non-zero L^p– superharmonic function on X, for some $p \geq 1$.*
b. *There is a superharmonic function $s > 0$ on X such that $\sum_x [s(x)]^p < \infty$.*
c. *Any potential $Q(x)$ with finite harmonic support on X is an L^p– superharmonic function.*
d. *For any vertex z, if $G_z(x)$ is the Green potential with harmonic support at z, then $\sum_x G_z^p(x) < \infty$.*

Proof. (a)⇒ (b) : If $s(x)$ is a non-zero L^p– superharmonic function, then $s > 0$ (Proposition 3).
(b)⇒ (c) : Let $v(x)$ be a potential with finite harmonic support. Then for some $\alpha > 0$, $v(x) \leq \alpha s(x)$ on the harmonic support of $v(x)$. By the Domination Principle [3, Theorem 3.3.6], $v(x) \leq \alpha s(x)$ for all $x \in X$. Consequently, $\sum_x v^p(x) < \infty$.
(c)⇒ (d): For $G_z(x)$ is a potential with harmonic support at the vertex z.
(d)⇒ (a): Evident since $s(x) = G_z(x)$ is a superharmonic function and $\sum_x [s(x)]^p < \infty$ by the assumption. □

Corollary 6. *If there is a non-zero L^p– superharmonic function on X, then every potential $v(x)$ with finite harmonic support in X tends to 0 at infinity.*

Proof. By the above Theorem 5, $v(x)$ is an L^p–-superharmonic function. Then the corollary follows from Proposition 3. □

Corollary 7. *(Yamasaki [4]) In a symmetric network (that is, the transition indices are symmetric), there exists a non-zero L^1– superharmonic function if and only if the Poisson equation $\Delta u = -1$ has a positive solution.*

Proof. If there is a non-zero L^1– superharmonic function, then by the above Theorem 5 (d), $\sum_x G_z(x) < \infty$. Hence, for a fixed z, by symmetry assumption, $\sum_x G_x(z) < \infty$. Written differently, if $Q(x) = \sum_y G_y(x)$, then $Q(x)$ is finite at the vertex z; hence $Q(x)$ is a potential and $\Delta Q(x) = -1$.

Conversely, suppose $u > 0$ is a solution of $\Delta u = -1$. Then u is a positive superharmonic function, hence the sum of a potential $Q(x)$ and a non-negative harmonic function. That shows $\Delta Q(x) = -1$. Now the potential $Q(x)$ has the representation $Q(x) = \sum_y (-\Delta Q(y))G_y(x) = \sum_y G_y(x) = \sum_y G_x(y)$. In particular, $G_z(x)$ is an L^1– superharmonic function on X. □

Remark In a non-symmetric network X, if $p(x) = \sum_{y\in X} G_y(x)$ is finite for one vertex, then $p(x)$ is a potential on X and $-\Delta p(x) = 1$ on X. The classification of non-symmetric networks on which $-\Delta p = 1$ has a bounded or at least a positive solution has not been considered extensively, see [5]. In the symmetric case, Yamasaki [4, Example 4.3] constructs a symmetric network that has a positive, but not bounded, solution for the equation $-\Delta p = 1$.

Example of the homogeneous tree on which there is no L^1– potential but L^p–potentials exist for $p > 1$: Let T be a homogeneous tree of order $(q+1)$, $q \geq 2$. Fix a vertex e in T and measuring distances from e, let $|x|$ denote the distance of the vertex x from e. Then the Green function on T with singularity at e is $G_e(s) = \frac{q}{q-1} \times \frac{1}{q^n}$ if $|s| = n$ (Cartier [6]). Now there are $q^{n-1}(q+1)$ vertices at a distance n from e. Hence $\sum_{s\in X} [G_e(s)]^p = \left(\frac{q}{q-1}\right)^p \left[1 + \sum_1^\infty \frac{q^{n-1}(q+1)}{q^{np}}\right]$ is finite if $p > 1$ and infinite if $p = 1$. Consequently, by Theorem 5 there is no L^1–superharmonic function on T, whereas $G_e(s)$ is an L^p–superharmonic function for any $p > 1$.

Lemma 8. *Let $s > 0$ be superharmonic on X, and $0 < \alpha < 1$. Then $s^\alpha(x)$ is also superharmonic.*

Proof. Take $f(\mu) = \mu^\alpha - \alpha\mu - 1 + \alpha$, for $\mu \geq 0$. Then $f'(\mu) = \alpha\mu^{\alpha-1} - \alpha$.
Hence in $(0, 1)$, $f(\mu)$ increases from $-1 + \alpha$ to $f(1) = 0$; and in $(1, \infty)$ decreases. Hence $f(\mu) \leq 0$.
That is $\mu^\alpha - 1 \leq \alpha\mu - \alpha$.
For any vertex x and $y \sim x$, take now $\mu = \frac{s(y)}{s(x)}$. Then,

$$\sum_{y\sim x} t(x,y)\left[\frac{s^\alpha(y)}{s^\alpha(x)} - 1\right] \leq \alpha \sum_{y\sim x} t(x,y)\left[\frac{s(y)}{s(x)} - 1\right]$$

$$\sum_{y\sim x} t(x,y)\left[s^\alpha(y) - s^\alpha(x)\right] \leq \alpha[s(x)]^{\alpha-1} \sum_{y\sim x} t(x,y)\left[s(y) - s(x)\right]$$

$$\Delta s^\alpha(x) \leq \alpha[s(x)]^{\alpha-1}\Delta s(x).$$

Since $\Delta s(x) \leq 0$, we conclude that $s^\alpha(x)$ is superharmonic on X. □

Proposition 9. *Let v be an L^1–superharmonic function on X. Then, for $0 < \alpha < 1$, $v^\alpha(x)$ is a potential on X.*

Proof. By the above Lemma 8, $v^\alpha(x)$ is a non-negative superharmonic function. Let $h(x)$ be a harmonic function such that $0 \leq h(x) \leq v^\alpha(x)$. Take $p = \frac{1}{\alpha}$. Then $h^p(x) \leq v(x)$ so that $\sum_x h^p(x) < \infty$. Since h is an L^p–harmonic function, $h = 0$ (Proposition 3). Hence $v^\alpha(x)$ is a potential on X. □

5 Representation of L^p– superharmonic functions

On a hyperbolic network X, write $G_z(x)$ as the Green potential with vertex harmonic support z. If $v(x)$ is an L^p– superharmonic function on X, then we have seen (Theorem 5(d)) that $G_z(x)$ also is an L^p– superharmonic function. In this section, we obtain a unique representation of $v(x)$ by means of certain variants of $G_z(x)$; that will show that an L^p– superharmonic function $v(x)$ is the sum of a convergent series of L^p– potentials, hence $v(x)$ is the vertex-wise increasing limit of a sequence of L^p– potentials.

Lemma 10. *A real-valued function $v(x)$ on X is a potential if and only if $v(x)$ is of the form $v(x) = \sum_z \alpha(z) G_z(x)$ where $\alpha(z) \geq 0$.*

Proof. Suppose $v(x) = \sum_z \alpha(z) G_z(x)$, $\alpha(z) \geq 0$. Since the real-valued function $v(x)$ is the sum of a convergent series of potentials, it is a potential.
Conversely, suppose $v(x)$ is a potential. Let $\{E_m\}$ be a collection of increasing finite sets such that $X = \cup E_m$. Then $h_m(x) = v(x) - \sum_{z\in E_m} [-\Delta v(x)] G_z(x)$ is harmonic at every vertex in E_m, and h_m is decreasing in m; moreover, since$-\Delta v(x) \leq 0$ at every vertex in X, $h_m(x)$ is a superharmonic function on X such that $-h_m(x) \leq \sum_{z\in E_m} [-\Delta v(z)] G_z(x)$ on X so that the subharmonic function $-h_m \leq 0$ on X. Consequently, $h(x) = lim_m h_m(x) = lim_m \left[v(x) - \sum_{z\in E_m} \{-\Delta v(z)\} G_z(x)\right] \leq v(x)$. Since $v(x)$ is a potential and $h(x) \geq 0$ is harmonic we conclude from $h(x) \leq v(x)$ that $h(x) = 0$. Thus, $v(x) = lim_m \sum_{z\in E_m} [-\Delta v(z)] G_z(x) = \sum_{z\in X} [-\Delta v(z)] G_z(x)$. □

Let C_p represent the cone of positive L^p– superharmonic functions on X.
For $u \in C_p$, write $\left[\|u\|_p\right]^p = \sum_{x\in X} u^p(x)$.
Write $B = \left\{u \in C_p : \|u\|_p = 1\right\}$. When $C_p \neq \phi$, $G_z \in C_p$ as remarked above.
Write $G'_{z,p}(x) = \frac{G_z(x)}{\|G_z\|_p}$. Then $G'_{z,p} \in B$.
Write $\mathcal{E}_p = \left\{G'_{z,p} : z \in X\right\}$.

Theorem 11. *If $v(x)$ is an L^p– superharmonic function on X, then there exists a unique measure μ supported by $\mathcal{E}_p$ such that $v(x) = \sum_{z\in X} \mu\left(G'_{z,p}\right) G'_{z,p}(x)$, for $x \in X$.*

Proof. If $v \in C_p$, and if h is a harmonic function on X such that $0 \leq h \leq v$, then h is an L^p– harmonic function, hence $h = 0$ (Proposition 3). That is $v(x)$ is a potential on X. Hence (Lemma 10) it has a representation $v(x) = \sum_{z\in X} [-\Delta v(z)] G_z(x)$. Write $\mu\left(G'_{z,p}\right) = [-\Delta v(z)] \|G_z\|_p$. Then $v(x) = \sum_z \mu\left(G'_{z,p}\right) G'_{z,p}(x)$ where $\mu\left(G'_{z,p}\right)$ can be considered as a measure supported by $\mathcal{E}_p$. □

6 L^p– Superharmonic functions near infinity

Let A be a subset of the network X. A vertex x is said to be an interior vertex of A if x and all its neighbours are in A. Denote by $\overset{o}{A}$ the set of all interior vertices of A, and $\partial A = A \setminus \overset{o}{A}$. When A is a finite set, if $f(z)$ is a real-valued function on ∂A, then there exists a unique function h on A [7] such that $\Delta h(x) = 0$ if $x \in \overset{o}{A}$ and $h(z) = f(z)$ if $z \in \partial A$; write $h(x) = H_f^A(x)$ on A. A real-valued function $u(x)$ defined outside a finite set $E \subset \overset{o}{A}$ in X, where A also is a finite set, is said to be an L^p– superharmonic function near infinity if $-\Delta u(x) \geq 0$ for every $x \in X \setminus E$ and $\sum_{x\in X\setminus A} |u(x)|^p < \infty$.

Theorem 12. *Let X be network with L^p– superharmonic functions. Let $u(x)$ be an L^p– superharmonic function near infinity. Then $u = s_1 - s_2$ outside a finite set where s_1, s_2 are two L^p– superharmonic functions on X and s_2 has finite harmonic support.*

Proof. Suppose that $u(x)$ is an L^p- superharmonic function on $X \backslash E$, where E is a finite set in X. Let A be a finite set, $\overset{o}{A} \supset E$. Let $v(x)$ be the function on X, such that $v = u$ on $X \backslash \overset{o}{A}$ and $v = H_u^A$ on A.
Let $s(x) = v(x) - \sum_{z \in \partial A} [-\Delta v(z)] G_z(x)$.
Then $[-\Delta s(x)] = 0$ if $x \in A$, and $[-s(x)] \geq 0$ if $z \in X \backslash A$. Hence $s(x)$ is superharmonic on X. Moreover, since $G_z(x)$ is L^p- superharmonic on X (Theorem 5 (d)) and $v(x)$ is an L^p- function on $X \backslash A$, we conclude $\sum_{x \in X \backslash A} |s(x)|^p < \infty$. Consequently, since A is a finite set, $\sum_{x \in X} |s(x)|^p < \infty$.
Write $\partial A = A_1 \cup A_2$, where $[-\Delta v(z)] \geq 0$ on A_1 and $[-\Delta v(z)] < 0$ on A_2. Write
$s_1(x) = s(x) + \sum_{z \in A_1} [-\Delta v(z)] G_z(x)$, and

$$s_2(x) = \sum_{z \in A_2} [\Delta v(z)] G_z(x).$$

Then $v(x) = s_1(x) - s_2(x)$, where $s_1(x)$, $s_2(x)$ are L^p- superharmonic functions on X and $s_2(x)$ is harmonic outside the finite set A. Consequently, near infinity, $u(x) = s_1(x) - s_2(x)$. □

Corollary 13. *On a network with L^p–superharmonic functions, if $u(x)$ is an L^p–superharmonic function near infinity, then $u(x)$ tends to 0 at infinity.*

Proof. Write $u = s_1 - s_2$ outside a finite set. Since s_1, s_2 are L^p- superharmonic functions, they are non-negative and tend to 0 at infinity (Proposition 3). Hence $u(x)$ tends to 0 at infinity. □

Corollary 14. *On a network with L^p- superharmonic functions, let $u(x)$ be a harmonic function defined outside a finite set and tending to 0 at infinity. Then $u(x)$ is the difference of two L^p- potentials with finite harmonic support on X, hence $u(x)$ is an L^p- harmonic function near infinity.*

Proof. Defining the function $v(x)$ as in Theorem 12 above, let us write $s(x) = v(x) - \sum_{z \in \partial A} [-\Delta v(z)] G_z(x)$. Now remark that $-\Delta s(x) = 0$ for all $x \in X$. That is, $s(x)$ is harmonic on X, and moreover $s(x)$ tends to 0 at infinity. Hence $s = 0$. Consequently, outside the finite set A, $u(x) = v(x) = \sum_{z \in \partial A} [-\Delta v(z)] G_z(x)$. Hence $u(x)$ is the difference of two L^p- superharmonic functions on X with finite harmonic support. Now $G_z(x)$ is an L^p- superharmonic function on X, so that $\sum_{x \in X \backslash A} |u(x)|^p < \infty$. □

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