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An alternative and easy approach to fixed point results via simulation functions

Abstract: We discuss, extend, improve and enrich results on simulation functions established by several authors. Furthermore, by using Lemma 2.1 of Radenović et al. [Bull. Iran. Math. Soc., 2012, 38, 625], we get much shorter and nicer proofs than the corresponding ones in the existing literature.

1 Introduction and Preliminaries

In 2015 Khojasteh *et al.* [1] introduced the concept of $\mathcal{Z}$-contraction. Indeed, a $\mathcal{Z}$-contraction is a new type of nonlinear contraction defined by mean of a specific family of functions called simulation functions. Of course, Khojasteh *et al.* proved the existence and uniqueness of fixed points for the class of $\mathcal{Z}$-contraction mappings. The advantage of this notion is in providing a unique point of view for several fixed point problems (for more details, we refer the reader to [2, 3] and the references therein). In this paper we discuss, improve and enrich results on simulation functions established by several authors.

The notion of simulation function was introduced in [1] as follows:

Definition 1. A mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:
(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim\limits_{n \to +\infty} t_n = \lim\limits_{n \to +\infty} s_n > 0$, then $\limsup\limits_{n \to +\infty} \zeta(t_n, s_n) < 0$.

We stress that some authors revised the above definition slightly. More precisely, they withdrew the condition (ζ_1). Furthermore, in [3] the authors revised condition (ζ_3) by taking $t_n < s_n$; see also [2]. Hence, we can say that a mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is a simulation function if it satisfies the following conditions:
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim\limits_{n \to +\infty} t_n = \lim\limits_{n \to +\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup\limits_{n \to +\infty} \zeta(t_n, s_n) < 0$.

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Here, as well as in [1], we denote the set of all simulation functions by $\mathcal{Z}$. The following are typical examples of simulation functions:

(S_1) $\zeta_1(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, +\infty)$, where $\phi, \psi : [0, +\infty) \to [0, +\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \le \phi(t)$ for all $t > 0$.

(S_2) $\zeta_2(t, s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0, +\infty)$, where $f, g : [0, +\infty) \to [0, +\infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.

(S_3) $\zeta_3(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

(S_4) $\zeta_4(t, s) = \frac{s}{s+1} - t$ for all $t, s \in [0, +\infty)$.

(S_5) $\zeta_5(t, s) = \lambda s - t$ for all $t, s \in [0, +\infty)$, where $\lambda \in [0, 1)$.

(S_6) $\zeta_6(t, s) = s\varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, 1)$ is a mapping such that $\limsup_{t \to r^+} \varphi(t) < 1$, for all $r > 0$.

(S_7) $\zeta_7(t, s) = \eta(s) - t$ for all $t, s \in [0, +\infty)$, where $\eta : [0, +\infty) \to [0, +\infty)$ is an upper semi continuous mapping such that $\eta(s) < s$ for all $s > 0$ and $\eta(0) = 0$.

(S_8) $\zeta_8(t, s) = s - \int_0^t \phi(u)\, du$ for all $t, s \in [0, +\infty)$, where $\phi : [0, +\infty) \to [0, 1)$ is a function such that $\int_0^t \phi(u)\, du$ exists and $\int_0^\varepsilon \phi(u)\, du > \varepsilon$, for each $\varepsilon > 0$.

(S_9) Let $h : [0, +\infty)^2 \to [0, +\infty)$ be a function such that $h(t, s) < 1$ for all $t, s > 0$ and $\limsup_{n \to +\infty} h(t_n, s_n) < 1$ provided that $\{t_n\}$ and $\{s_n\} \subset (0, +\infty)$ are two sequences such that $\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} s_n > 0$. Let

$$\zeta_9(t, s) = sh(t, s) - t \quad \text{for all } t, s \in [0, +\infty),$$

then ζ_9 is a simulation function.

We note that the examples (S_1)-(S_8) are in [1] while example (S_9) is in [4]. We also refer the reader to [1, Examples 2.2, 2.9], [5, Example 2.3], [2, Examples 2.6, 2.7, 4.5], [3, Examples 3.3-3.11, 5.11, 5.12] and [6, Examples 2.1, 3.1-3.4].

Now, we recall the notion of $\mathcal{Z}$-contraction.

Definition 2. Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. A mapping $T : X \to X$ is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta(d(Tx, Ty), d(x, y)) \ge 0 \quad \text{for all } x, y \in X. \tag{1}$$

According to the previous definition, it is clear that $\zeta(t, t) < 0$ when $t > 0$. Furthermore, (1) implies that $d(Tx, Ty) < d(x, y)$ when $x \ne y$ for all $x, y \in X$. This assures that each $\mathcal{Z}$-contraction is a contractive mapping and hence it is continuous. We recall for convenience of the reader some results of [1].

Theorem 1. [1, Theorem 2.8] *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.*

Let $X \ne \emptyset$, $T : X \to X$ and $x_0 \in X$. We recall that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is called the Picard sequence generated by T with initial point x_0. We underline that, in order to prove Theorem 1, F. Khojasteh *et al.* used the following auxiliary results.

Lemma 1. [1, Lemma 2.5] *Let (X, d) be a metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the fixed point of T in X is unique, provided that it exists.*

Lemma 2. [1, Lemma 2.6] *Let (X, d) be a metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then T is asymptotically regular at every $x \in X$ (i.e., $\lim_{n \to +\infty} d(T^n x, T^{n+1} x) = 0$).*

Lemma 3. [1, Lemma 2.7] *Let (X, d) be a metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the Picard sequence $\{x_n\}$ generated by T with initial value $x_0 \in X$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, is a bounded sequence.*

Recently, Karapinar in [7] introduced the notion of α-admissible $\mathcal{Z}$-contraction with respect to a given simulation function.

Definition 3. Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0, +\infty)$ such that

$$\zeta\left(\alpha\left(x, y\right) d\left(Tx, Ty\right), d\left(x, y\right)\right) \geq 0 \quad \text{for all } x, y \in X$$

then we say that T is an α-admissible $\mathcal{Z}$-contraction with respect to ζ.

Furthermore, Karapinar in [7] proved the following fixed point result.

Theorem 2. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible $\mathcal{Z}$-contraction with respect to ζ. Suppose that*
(i)   T is triangular α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha\left(x_0, Tx_0\right) \geq 1$;
(iii) T is continuous.

Then there exists $u \in X$ such that $Tu = u$.

The reader is referred to [7] and [8] for more details on α-admissible, triangular α-admissible and α-orbital admissible mappings.

Remark 1. We stress that Theorem 2 remains true if we replace (*iii*) by
(*iv*) if $\{x_n\}$ is a sequence in X such that $\alpha\left(x_n, x_{n+1}\right) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n_k}, x\right) \geq 1$ for all $k \in \mathbb{N}$.

2 Main results

In this section we discuss, extend and improve some recent results on simulation functions established by several authors. Indeed, by using Lemma 2.1 of [9], we get much shorter and nicer proofs than the corresponding ones in the literature. In particular, we stress that such lemma was used in various papers to establish the proofs of several fixed point results. Here, we formulate and prove a new version of Lemma 2.1 of [9] and, furthermore, we generalize it slightly.

Lemma 4. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n \to +\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{2}$$

If $\{x_n\}$ is not a Cauchy sequence in (X, d), then there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $n_k > m_k > k$ and such that the following sequences

$$\{d(x_{m_k}, x_{n_k})\}, \ \{d(x_{m_k}, x_{n_k+1})\}, \ \{d(x_{m_k-1}, x_{n_k})\}, \ \{d(x_{m_k-1}, x_{n_k+1})\}, \ \{d(x_{m_k+1}, x_{n_k+1})\} \tag{3}$$

tend to ε as $k \to +\infty$.

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$n_k > m_k > k, \ d\left(x_{m_k}, x_{n_k-1}\right) < \varepsilon, \ d\left(x_{m_k}, x_{n_k}\right) \geq \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Hence, we have

$$
\begin{aligned}
\varepsilon \;\leq\; & d(x_{m_k}, x_{n_k}) \leq d\left(x_{m_k}, x_{n_k-1}\right) + d\left(x_{n_k-1}, x_{n_k}\right) \\
< \;& \varepsilon + d\left(x_{n_k-1}, x_{n_k}\right).
\end{aligned}
$$

Now, by using (2), we conclude that

$$
\lim_{k \to +\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{4}
$$

We notice that

$$
d(x_{m_k}, x_{n_k}) \leq d\left(x_{m_k}, x_{n_k+1}\right) + d\left(x_{n_k+1}, x_{n_k}\right)
$$

as well as

$$
d\left(x_{m_k}, x_{n_k+1}\right) \leq d(x_{m_k}, x_{n_k}) + d\left(x_{n_k}, x_{n_k+1}\right).
$$

So, passing to the limit as $k \to +\infty$, we obtain by (2) and (4) that

$$
\lim_{k \to +\infty} d\left(x_{m_k}, x_{n_k+1}\right) = \varepsilon. \tag{5}
$$

Also, we observe that

$$
d\left(x_{m_k+1}, x_{n_k+1}\right) \leq d\left(x_{m_k+1}, x_{m_k}\right) + d\left(x_{m_k}, x_{n_k+1}\right) \tag{6}
$$

and

$$
d\left(x_{n_k+1}, x_{m_k}\right) \leq d\left(x_{n_k+1}, x_{m_k+1}\right) + d\left(x_{m_k+1}, x_{m_k}\right). \tag{7}
$$

So, by the previous inequalities, passing to the limit as $k \to +\infty$ we get

$$
\lim_{k \to +\infty} d\left(x_{m_k+1}, x_{n_k+1}\right) = \varepsilon.
$$

In a similar way one can prove that also the sequences $\{d(x_{m_k-1}, x_{n_k})\}$ and $\{d(x_{m_k-1}, x_{n_k+1})\}$ tend to ε as $k \to +\infty$. $\qquad\square$

Now, by using Lemma 4, we prove the next result.

Lemma 5. *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then the Picard sequence $\{x_n\}$ generated by T with initial value at $x_0 \in X$ is a Cauchy sequence.*

Proof. We notice that, by Lemma 2, the Picard sequence $\{x_n\}$ generated by T, with initial value at $x_0 \in X$, is such that $\lim\limits_{n \to +\infty} d(x_n, x_{n+1}) = 0$.

If $\{x_n\}$ is not a Cauchy sequence in (X, d), then by Lemma 4 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and

$$
\lim_{k \to +\infty} d(x_{m_k}, x_{n_k}) = \lim_{k \to +\infty} d\left(x_{m_k+1}, x_{n_k+1}\right) = \varepsilon.
$$

Putting $x = x_{m_k}$ and $y = x_{n_k}$ in (1), we get

$$
\begin{aligned}
0 \;\leq\; & \zeta\left(d\left(Tx_{m_k}, Tx_{n_k}\right), d(x_{m_k}, x_{n_k})\right) = \zeta\left(d\left(x_{m_k+1}, x_{n_k+1}\right), d(x_{m_k}, x_{n_k})\right) \\
< \;& d(x_{m_k}, x_{n_k}) - d\left(x_{m_k+1}, x_{n_k+1}\right) \to \varepsilon - \varepsilon = 0 \text{ as } k \to +\infty.
\end{aligned}
$$

Taking $t_k = d\left(x_{m_k+1}, x_{n_k+1}\right) > 0$ and $s_k = d(x_{m_k}, x_{n_k}) > 0$, we obtain that

$$
0 \leq \zeta(t_k, s_k) < s_k - t_k \tag{8}
$$

and so $t_k < s_k$ for all $k \in \mathbb{N}$. Now, by using (ζ_3), we have

$$
0 \leq \lim_{k \to \infty} \zeta(t_k, s_k) = \limsup_{k \to \infty} \zeta(t_k, s_k) < 0,
$$

which is a contradiction. This assures that $\{x_n\}$ is a Cauchy sequence. $\qquad\square$

Remark 2. Taking into account that each $\mathcal{Z}$-contraction $T : X \to X$ is a contractive mapping, we can conclude that T has a unique fixed point in X. Hence, Lemma 3 [1, Lemma 2.7] is an immediate consequence of Lemma 5. We stress that by using Lemma 5 we can improve and generalize Theorem 2.8 of [1] and, further, we can give for it a shorter proof. We also underline that our method together with Lemma 4 greatly improves Lemmas 3.5 and 3.6 of [10] and Lemma 3.1 of [2]. As a consequence, the condition that the Picard sequence is bounded is now superfluous.

Now, we give a generalization of Theorem 2.8 of [1]. We stress that the following result improves the corresponding result of [3] (see Theorem 4.8, p. 349).

Theorem 3. *Let (X, d) be a metric space and $T, S : X \to X$ be two given mappings. Assume that there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(d\left(Tx, Ty\right), d\left(Sx, Sy\right)\right) \geq 0 \quad \text{for all } x, y \in X. \tag{9}$$

If $TX \subseteq SX$ and TX or SX is a complete subset of X, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible then T and S have a unique common fixed point in X.

Proof. At first, we prove that if a point of coincidence of T and S exists then it is unique. If ω_1 and ω_2 are two distinct points of coincidence of T and S, then there exist two points $u_1, u_2 \in X$ such that $Tu_1 = Su_1 = \omega_1 \neq \omega_2 = Su_2 = Tu_2$. Hence, by using (9), we obtain that

$$0 \leq \zeta\left(d\left(Tu_1, Tu_2\right), d\left(Su_1, Su_2\right)\right) = \zeta\left(d\left(\omega_1, \omega_2\right), d\left(\omega_1, \omega_2\right)\right) < 0,$$

but this is a contradiction. Thus, we conclude that $\omega_1 = \omega_2$.

Now, let x_0 be an arbitrary point in X. Choose $x_1 \in X$ such that $Tx_0 = Sx_1$. We notice that the point x_1 exists since $TX \subseteq SX$. Continuing this process, choosing x_n in X we obtain x_{n+1} in X such that $Tx_n = Sx_{n+1} = y_n$ (i.e., we have a Jungck sequence generated by x_0, T and S). If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$ then $Sx_{n+1} = y_n = y_{n+1} = Tx_{n+1}$. This implies that x_{n+1} is the (unique) requested point of coincidence and thus the proof is completed. Therefore, we suppose that $y_{n-1} \neq y_n$ for all $n \in \mathbb{N}$. Hence, we have

$$0 \leq \zeta\left(d\left(Tx_n, Tx_{n+1}\right), d\left(Sx_n, Sx_{n+1}\right)\right) = \zeta\left(d\left(y_n, y_{n+1}\right), d\left(y_{n-1}, y_n\right)\right)$$
$$< d\left(y_{n-1}, y_n\right) - d\left(y_n, y_{n+1}\right) \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

This ensures that the sequence $\{d\left(y_{n-1}, y_n\right)\}$ is decreasing. Consequently, there exists $\lim_{n \to +\infty} d\left(y_{n-1}, y_n\right) = D \geq 0$. We affirm that $D = 0$. In fact, if $D > 0$ by using (10) it follows that

$$0 \leq \lim_{n \to +\infty} \zeta\left(d\left(y_n, y_{n+1}\right), d\left(y_{n-1}, y_n\right)\right) = \limsup_{n \to +\infty} \zeta\left(d\left(y_n, y_{n+1}\right), d\left(y_{n-1}, y_n\right)\right) < 0$$

where $t_n = d\left(y_n, y_{n+1}\right) < d\left(y_{n-1}, y_n\right) = s_n$ and $t_n, s_n \to D > 0$. Clearly, this is a contradiction and so $\lim_{n \to +\infty} d\left(y_n, y_{n+1}\right) = 0$.

Now, we prove that $\{y_n\}$ is a Cauchy sequence in (X, d). We suppose, by contradiction, that $\{y_n\}$ is not a Cauchy sequence. Then by Lemma 4 there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that the sequences

$$\{d(y_{m_k}, y_{n_k+1})\}, \{d(y_{m_k-1}, y_{n_k})\}$$

tend to ε as $k \to +\infty$. Notice that we can assume $d(y_{m_k}, y_{n_k+1})$, $d(y_{m_k-1}, y_{n_k}) \neq 0$ for all $k \in \mathbb{N}$. Applying (9), with $x = x_{m_k}$ and $y = x_{n_k+1}$, we get

$$0 \leq \zeta\left(d\left(y_{m_k}, y_{n_k+1}\right), d\left(y_{m_k-1}, y_{n_k}\right)\right) < d\left(y_{m_k-1}, y_{n_k}\right) - d\left(y_{m_k}, y_{n_k+1}\right) \to 0 \text{ as } k \to +\infty. \tag{11}$$

Now, by using (11), it is easy to conclude that

$$\limsup_{k \to +\infty} \zeta\left(d\left(y_{m_k}, y_{n_k+1}\right), d\left(y_{m_k-1}, y_{n_k}\right)\right) = 0,$$

but this is a contradiction with (ζ_3). Thus we deduce that $\{y_n\}$ is a Cauchy sequence. Now, taking into account that TX or SX is a complete subset of (X, d), we have that there exists $u \in X$ such that $y_n \to Su$ as $n \to +\infty$. If

there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} = Tu$, then letting $k \to +\infty$ we get $Su = Tu$ and hence we have the claim. So, we suppose that $y_n \neq Tu$ for all $n \in \mathbb{N}$. Since $y_{n-1} \neq y_n$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \neq Su$ for all $k \in \mathbb{N}$. Using (9), with $x = x_{n_{k+1}}$ and $y = u$, we have

$$\zeta(d(Tx_{n_{k+1}}, Tu), d(Sx_{n_{k+1}}, Su)) < d(y_{n_k}, Su) - d(y_{n_{k+1}}, Tu) \quad \text{for all } k \in \mathbb{N}.$$

The previous inequality implies that $y_{n_{k+1}} \to Tu$ and hence $Tu = Su$ is a unique point of coincidence of T and S. Finally, by using the well-known Jungck result, we have that T and S have a unique common fixed point if they are weakly compatible. Hence, we get the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 1. *Let $X = \mathbb{R}$ so that (X, d) is a complete metric space under the usual metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $T, S : X \to X$ be the mappings defined by $Sx = -2x + \frac{7}{2}$ and $Tx = \frac{1}{2}x + 1$ for all $x \in X$. Then T and S satisfy the contractive condition (9) with respect to the simulation function ζ given by $\zeta(t, s) = s - \frac{t+2}{t+1}t$ for all $s, t \in [0, +\infty)$. We notice that*

$$s - \frac{t+2}{t+1}t \geq 0 \quad \Leftrightarrow \quad t^2 + t(2-s) - s \leq 0.$$

Now, let $t = d(Tx, Ty)$ and $s = d(Sx, Sy)$, then

$$t^2 + t(2-s) - s \leq 0$$
$$\Leftrightarrow \quad \frac{1}{4}|x-y|^2 + \frac{1}{2}|x-y|(2 - 2|x-y|) - 2|x-y| \leq 0$$
$$\Leftrightarrow \quad -1 - \frac{3}{4}|x-y| \leq 0.$$

Hence, taking into account that the last inequality is true, we deduce that all the conditions in Theorem 3 are satisfied. Thus T and S have a unique point of coincidence, which is not a common fixed point because of T and S are not weakly compatible.

On the other hand, if we consider the mapping S defined by $Sx = -2x + 6$, then T and S are weakly compatible and so they have a common fixed point.

We recall that a function $\beta : [0, +\infty) \to [0, 1)$ is called
- a Geraghty function if $\{r_n\} \subset (0, +\infty)$ and if $\beta(r_n) \to 1^-$ implies $r_n \to 0^+$;
- a strong Geraghty function if $\{r_n\} \subset (0, +\infty)$ and if $\limsup_{n \to +\infty} \beta(r_n) = 1^-$ implies $r_n \to 0^+$.

A mapping $T : X \to X$ is called a Geraghty contraction (strong Geraghty contraction) if there exists a Geraghty function (strong Geraghty function) β such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \quad \text{for all } x, y \in X.$$

We refer the reader to [11] for more details on Geraghty functions and Geraghty contractions.

Remark 3. Let $\zeta : [0, +\infty)^2 \to \mathbb{R}$ be defined by $\zeta(t, s) = s\beta(s) - t$ for all $s, t \in [0, +\infty)$ where $\beta : [0, +\infty) \to [0, 1)$ is a function such that

$$\limsup_{n \to +\infty} \beta(r_n) = 1^- \text{ implies } r_n \to 0^+ \quad \text{for all } \{r_n\} \subset (0, +\infty).$$

Then ζ is called "simulation function of strong Geraghty-type". In fact, from $\zeta(t, s) = s\beta(s) - t < s - t$ for all $s, t \in (0, +\infty)$ we deduce that (ζ_2) holds.

We show that (ζ_3) is also satisfied. Let $\{s_n\}, \{t_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} t_n = l > 0$. From $\lim_{n \to +\infty} s_n = l > 0$ it follows that $\limsup_{n \to +\infty} \beta(s_n) < 1$ and so $\limsup_{n \to +\infty} \zeta(t_n, s_n) = l \limsup_{n \to +\infty} \beta(s_n) - l < 0$, that is, (ζ_3) holds.

Remark 4. We stress that every strong Geraghty contraction T is a $\mathcal{Z}$-contraction with respect to a simulation function of strong Geraghty-type. Hence, the fixed point result of Geraghty in [11] can be deduced as a consequence of Theorem 3. It is sufficient to assume $Sx = x$ for all $x \in X$.

Corollary 1 (Geraghty [11])**.** *Every strong Geraghty contraction T from a complete metric space* (X, d) *into itself has a unique fixed point.*

Proof. The claim follows by Theorem 3 according to Remarks 3 and 4. □

Now, let $F : [0, +\infty) \to [0, +\infty)$ be a mapping satisfing the following condition:

$$0 < F(t) \le t \quad \text{for all } t \in (0, +\infty) \quad \text{and} \quad F(0) = 0.$$

The next two theorems complement and extend recent results in the setting of simulation functions. We prove only the first one. Indeed, the other theorem can be established in a similar way.

Theorem 4. *Let* (X, d) *be a metric space and* $T, S : X \to X$ *be two given mappings. Assume that there exist* $\zeta \in \mathcal{Z}$ *and a function* $F : [0, +\infty) \to [0, +\infty)$ *as above such that*

$$\zeta\left(d(Tx, Ty), F(d(Sx, Sy))\right) \ge 0 \quad \text{for all } x, y \in X. \tag{12}$$

If $TX \subseteq SX$ *and TX or SX is a complete subset of X, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible then T and S have a unique common fixed point in X.*

Proof. We notice that if a point of coincidence of T and S exists then it is unique. Indeed, if ω_1 and ω_2 are two distinct points of coincidence of T and S then there exist two points $u_1, u_2 \in X$ such that $Tu_1 = Su_1 = \omega_1 \ne \omega_2 = Su_2 = Tu_2$. Now, by (12) and (ζ_2) it follows that

$$0 \le \zeta(d(Tu_1, Tu_2), F(d(Su_1, Su_2))) < F(d(\omega_1, \omega_2)) - d(\omega_1, \omega_2) \le 0.$$

Clearly, this is a contradiction and so we have $\omega_1 = \omega_2$. Now, let x_0 be an arbitrary point in X. Choose $x_1 \in X$ such that $Tx_0 = Sx_1$ (we recall that $TX \subseteq SX$). Continuing this process, choosing x_n in X we obtain x_{n+1} in X such that $Tx_n = Sx_{n+1} = y_n$ (i.e., we have a Jungck sequence generated by x_0, T and S). If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$ then $Sx_{n+1} = y_n = y_{n+1} = Tx_{n+1}$. This implies that x_{n+1} is the (unique) requested point of coincidence and thus we have the claim. So, we suppose that $y_{n-1} \ne y_n$ for all $n \in \mathbb{N}$. Then we have

$$0 \le \zeta\left(d(Tx_n, Tx_{n+1}), F(d(Sx_n, Sx_{n+1}))\right) = \zeta\left(d(y_n, y_{n+1}), F(d(y_{n-1}, y_n))\right)$$
$$< F(d(y_{n-1}, y_n)) - d(y_n, y_{n+1}) \le d(y_{n-1}, y_n) - d(y_n, y_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

This ensures that the sequence $\{d(y_{n-1}, y_n)\}$ is decreasing and hence there exists $\lim_{n \to +\infty} d(y_{n-1}, y_n) = D \ge 0$. We observe that $D = 0$. In fact, if $D > 0$ then by (13) we obtain that $\lim_{n \to +\infty} d(y_{n-1}, y_n) = D$. Now, using (12) and (ζ_3) we get

$$0 \le \lim_{n \to +\infty} \zeta\left(d(y_n, y_{n+1}), F(d(y_{n-1}, y_n))\right) = \limsup_{n \to +\infty} \zeta\left(d(y_n, y_{n+1}), F(d(y_{n-1}, y_n))\right) = 0,$$

where $t_n = d(y_n, y_{n+1}) < F(d(y_{n-1}, y_n)) = s_n$ and $t_n, s_n \to D > 0$. Since this is a contradiction we necessarily have $\lim_{n \to +\infty} d(y_n, y_{n+1}) = 0$.

Now, we show that $\{y_n\}$ is a Cauchy sequence in (X, d). We suppose, by contradiction, that $\{y_n\}$ is not a Cauchy sequence. Hence, by Lemma 4 there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that the sequences

$$\{d(y_{m_k}, y_{n_k})\}, \{d(y_{m_k}, y_{n_k+1})\}, \{d(y_{m_k-1}, y_{n_k})\}, \{d(y_{m_k-1}, y_{n_k+1})\}, \{d(y_{m_k+1}, y_{n_k+1})\}$$

tend to ε as $k \to +\infty$. Applying (12), with $x = x_{m_k}$ and $y = x_{n_k+1}$, we get that

$$0 \le \zeta\left(d\left(y_{m_k}, y_{n_k+1}\right), F(d\left(y_{m_k-1}, y_{n_k}\right))\right)$$
$$< F(d\left(y_{m_k-1}, y_{n_k}\right)) - d\left(y_{m_k}, y_{n_k+1}\right) \to 0 \text{ as } k \to +\infty. \tag{14}$$

So, by using (14), we deduce that

$$\limsup_{k \to +\infty} \zeta\left(d\left(y_{m_k}, y_{n_k+1}\right), d\left(y_{m_k-1}, y_{n_k}\right)\right) = 0.$$

Clearly, this is a contradiction to (ζ_3) and hence we can conclude that $\{y_n\}$ is a Cauchy sequence. Now, taking into account that TX or SX is a complete subset of (X, d), there exists $u \in X$ such that $y_n \to Su$ as $n \to +\infty$. If there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} = Tu$, letting $k \to +\infty$ we obtain $Su = Tu$ and hence we have the claim. Then, we suppose that $y_n \neq Tu$ for all $n \in \mathbb{N}$. Since $y_{n-1} \neq y_n$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \neq Su$ for all $k \in \mathbb{N}$. Using (12), with $x = x_{n_k+1}$ and $y = u$, we have

$$\zeta(d(Tx_{n_k+1}, Tu), F(d(Sx_{n_k+1}, Su))) < d(y_{n_k}, Su) - d(y_{n_k+1}, Tu) \quad \text{for all } k \in \mathbb{N}.$$

The previous inequality assures that $y_{n_k+1} \to Tu$ and hence $Tu = Su$ is the unique point of coincidence of T and S. Now, by using the well-known Jungck result, we get that T and S have a unique common fixed point if they are weakly compatible and thus the claim is proved. $\square$

Theorem 5. *Let (X, d) be a metric space and $T, S : X \to X$ be two given mappings. Assume that there exist $\zeta \in \mathcal{Z}$ and $\lambda \in (0, \frac{1}{2})$ such that*

$$\zeta\left(d\left(Tx, Ty\right), \lambda \max\left\{d\left(Sx, Sy\right), d\left(Sx, Tx\right), d\left(Sy, Ty\right), d\left(Sx, Ty\right)\right\}\right) \geq 0 \tag{15}$$

for all $x, y \in X$. If $TX \subseteq SX$ and TX or SX is a complete subset of X, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible then T and S have a unique common fixed point in X.

Corollary 2. *If in (15) we put $Sx = x$ for all $x \in X$ then the $\mathcal{Z}$-quasicontraction $T : X \to X$, with respect to ζ, has a unique fixed point in (X, d).*

We conclude by posing at the reader the following open problem:

Problem: Is Theorem 5 also true if $\lambda \in [\frac{1}{2}, 1)$?
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