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A dimensional restriction for a class of contact manifolds

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Abstract: In this work we consider a class of contact manifolds (M, η) with an associated almost contact metric structure (ϕ, ξ, η, g). This class contains, for example, nearly cosymplectic manifolds and the manifolds in the class $C_9 \oplus C_{10}$ defined by Chinea and Gonzalez. All manifolds in the class considered turn out to have dimension $4n + 1$. Under the assumption that the sectional curvature of the horizontal 2-planes is constant at one point, we obtain that these manifolds must have dimension 5.

Keywords: almost contact metric structure, contact manifold, Chinea-Gonzalez classification

MSC: 53D15, 53D25, 53C15, 53D10.

1 Introduction

A *contact manifold* is a $\mathcal{C}^\infty$ odd-dimensional manifold M^{2n+1} together with a 1–form η, usually called a *contact form* on M, such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M; the *contact distribution* D is the vector subbundle of TM defined by

$$D := \ker \eta.$$

We shall denote by D_p the fiber of D at a point p; moreover if $X \in \mathfrak{X}(M)$ is a vector field, we shall write $X \in D$ to indicate that X is a section of D. It is known that $\mathrm{d}\eta|_{D_p \times D_p}$ is non degenerate and

$$T_pM = D_p \oplus \ker \mathrm{d}\eta_p$$

for each $p \in M$.

In [1] Chern showed that the existence of a contact form η on a manifold M^{2n+1} implies that the structural group of the tangent bundle TM can be reduced to the unitary group $U(n)\times 1$. Such a reduction of the structural group of the tangent bundle of a manifold M^{2n+1} is called an *almost contact structure*. In term of structure tensors we say that an *almost contact structure* on a manifold M^{2n+1} is a triple (ϕ, ξ, η) consisting of a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1–form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

see [2, p. 43]. It then follows directly from the definition of almost contact structure that $\phi\xi = 0, \eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2n$. If, in addition, M is endowed with a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then (ϕ, ξ, η, g) is said to be an *almost contact metric structure* on M. Thus, setting $Y = \xi$, we have immediately that

$$\eta(X) = g(X, \xi).$$

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Every contact manifold (M^{2n+1}, η) admits an almost contact metric structure (ϕ, ξ, η, g) such that

$$\mathrm{d}\eta(X, Y) = g(X, \phi Y).$$

In this case g is an *associated metric* and we speak of a *contact metric structure*; the vector field ξ is the Reeb vector field of M^{2n+1} [2]. Of course, it is possible to have a contact manifold (M^{2n+1}, η) with Reeb vector field ξ and an almost contact metric structure (ϕ, ξ, η, g) on M without $\mathrm{d}\eta(X, Y) = g(X, \phi Y)$.

One can also observe that every contact manifold with an almost contact metric structure (ϕ, ξ, η, g) satisfying $(\nabla_X \phi)X = 0$, or equivalently $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0$, i.e., with a *nearly cosymplectic structure*, satisfies the following condition

$$\phi \circ \nabla\xi + \nabla\xi \circ \phi = 0 \tag{*}$$

and of course does not satisfy the contact metric condition $\mathrm{d}\eta(X, Y) = g(X, \phi Y)$. Here ∇ denotes the Levi-Civita connection of g and $\nabla\xi$ is the bundle endomorphism of TM defined by $X \mapsto \nabla_X \xi$. A well-known example of this situation is given by the five-dimensional sphere S^5. This is a consequence of the following theorem [2, Theorem 6.14]:

Theorem. *Let $i : M^{2n+1} \to \tilde{M}^{2n+2}$ be a hypersurface of a nearly Kähler manifold $(\tilde{M}^{2n+2}, J, \tilde{g})$. Then the induced almost contact structure (ϕ, ξ, η, g) satisfies $(\nabla_X \phi)X = 0$ if and only if the second fundamental form σ is proportional to $(\eta \otimes \eta)Ji_\star\xi$.*

If we consider S^5 as a totally geodesic hypersurface of S^6, we have that the nearly Kähler structure $(J, \tilde{g})$ on S^6, defined as in Example 4.5.3 of [2], induces an almost contact metric structure (ϕ, ξ, η, g) on S^5 satisfying $(\nabla_X \phi)X = 0$.

In the next section we will treat contact manifolds with an almost contact metric structure satisfying condition (*). Such manifolds will result of dimension $4n+1$, $n \geqslant 1$. If we suppose that ϕ is η-parallel and the sectional curvature of the horizontal 2-planes is constant at one point, then we obtain that these manifolds have dimension 5 (Theorem 1).

It is well known that the contact condition imposes strong restrictions on the Riemannian curvature of an associated metric. For example Z. Olszak in [3] proves that if an associated metric has constant curvature, then $c = 1$ and g must be a Sasakian metric; earlier D.E. Blair in [4] showed that in dimension $\geqslant 5$ there are no flat associated metrics. We obtain that this is sometimes true also in the case of non associated metrics; for example when g is the metric of a nearly cosymplectic structure, see Theorem 3 in Section 3.

2 A class of contact manifolds

Let (ϕ, ξ, η, g) be an almost contact metric structure on a contact manifold (M, η). We denote by A the vector bundle endomorphism $\nabla\xi : TM \to TM$. Let $B : D \to D$ be the skew-symmetric part of $A|_D$, i.e.,

$$B = \frac{1}{2}(A|_D - A^\star)$$

where $A^\star$ is the adjoint of $A|_D$ with respect to $g|_{D\times D}$. Then, for all $X, Y \in D$, we have

$$\mathrm{d}\eta(X, Y) = -\frac{1}{2}\eta([X, Y]) = -\frac{1}{2}g([X, Y], \xi) = g(BX, Y). \tag{1}$$

Even if η is a contact form, ξ in general is not the Reeb vector field of η.

Proposition 1. *Let (ϕ, ξ, η, g) be an almost contact metric structure on a contact manifold (M, η) such that*

$$d\eta(\phi X, \phi Y) = -d\eta(X, Y), \text{ for all } X, Y \in D$$

or equivalently

$$B\phi + \phi B = 0 \text{ on } D.$$

Then $\dim M = 4n + 1$, $n \geqslant 1$ *and $B : D \to D$ is a bundle automorphism.*

Proof. We know that if (M,η) is a contact manifold then $d\eta|_{D\times D}$ is non degenerate. Thus equation (1) implies that B is an automorphism. The fact that $\dim M = 4n+1$ is an application of Lemma 1, point 2. □

Lemma 1. *Let* $<,>$ *be an Hermitian scalar product on a complex vector space* (D,J). *If* $A : D \to D$ *is a nonzero linear operator such that* $AJ + JA = 0$, *then*

1. *there exist* $Y, Z \in D$ *such that* Y, JY, AY *are linearly independent,* $Z \in \text{span}\{Y, JY, AY\}^{\perp}$ *and* $< Z, JAY >\neq 0$*;*
2. *if* A *is non singular and skew-symmetric then* $\dim D \equiv 0 \pmod 4$.

Proof. Let $X_1,..,X_n \in D$ be vectors such that $\{X_1, JX_1,..,X_n, JX_n\}$ is a basis of D. We begin by proving the existence of a vector $Y \in D$ such that Y, JY, AY are linearly independent. If by contradiction $AY \in \text{span}\{Y, JY\}$ for all $Y \in D$, then

$$AX_i \in \text{span}\{X_i, JX_i\},$$
$$AJX_i = -JAX_i \in \text{span}\{X_i, JX_i\},$$

and hence A is represented with respect to our basis by a block-diagonal matrix of the form

$$\begin{pmatrix} \begin{matrix} a_1 & b_1 \\ b_1 & -a_1 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{matrix} a_2 & b_2 \\ b_2 & -a_2 \end{matrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \begin{matrix} a_n & b_n \\ b_n & -a_n \end{matrix} \end{pmatrix}$$

where $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $a_i, b_i \in \mathbb{R}$, $i \in \{1,..,n\}$. Since

$$A(X_i + X_j) \in \text{span}\{X_i + X_j, JX_i + JX_j\},$$

we have $a_i = a_j$ and $b_i = b_j$. Thus

$$A \equiv \begin{pmatrix} \begin{matrix} a_1 & b_1 \\ b_1 & -a_1 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{matrix} a_1 & b_1 \\ b_1 & -a_1 \end{matrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \begin{matrix} a_1 & b_1 \\ b_1 & -a_1 \end{matrix} \end{pmatrix}.$$

Now we consider $JX_1 + X_2$. Since

$$A(JX_1 + X_2) \in \text{span}\{JX_1 + X_2, -X_1 + JX_2\}$$

it follows $a_1 = b_1 = 0$. This contradicts the hypothesis $A \neq 0$.

Let $Y \in D$ be such that Y, JY, AY are linearly independent. We can observe that

$$JAY \notin \text{span}\{Y, JY, AY\},$$

so that $JAY = W + Z$, with $Z \in \text{span}\{Y, JY, AY\}^{\perp}$, $Z \neq 0$ and $W \in \text{span}\{Y, JY, AY\}$. Thus we found $Z \in D$ orthogonal to Y, JY, AY such that $< Z, JAY >\neq 0$.

Now we assume that A is non singular and skew-symmetric. Let $X \in D$ be an eigenvector of the symmetric linear operator A^2. Since A anti-commutes with J, we have that JX, AX, JAX are also eigenvectors of A^2. Moreover the vectors X, JX, AX, JAX are pairwise orthogonal and hence $\dim D \geqslant 4$.

Assume $\dim D > 4$. By the Spectral Theorem we can choose $Y \in D$ eigenvector of A^2 orthogonal to X, JX, AX, JAX. We have that

$$X, JX, AX, JAX, Y, JY, AY, JAY$$

are eigenvectors of A^2, pairwise orthogonal and hence $\dim D \geqslant 8$. Iterating this argument we obtain the assertion. □

After these preliminaries we can state our main result that involves contact manifolds with an almost contact metric structure satisfying condition (*).

Theorem 1. *Let (ϕ, ξ, η, g) be an almost contact metric structure on a contact manifold (M^{2n+1}, η) such that*

$$A\phi + \phi A = 0 \tag{2}$$

$$g((\nabla_X \phi)Y, Z) = 0 \tag{3}$$

for each $X, Y, Z \in D$. Suppose there exist $p \in M$ and $c \in \mathbb{R}$ such that the sectional curvature $K_p(\pi) = c$, for each 2–plane π of D_p. Then $\dim M = 5$. *Moreover A_p is an isomorphism if and only if $c \neq 0$.*

Proof. For each vector field Z on M, we denote by Z^H and Z^V the components of Z in D and in its orthogonal complement $D^\perp$ respectively. We say that Z^H is the *horizontal part* of Z and Z^V the *vertical part* of Z. Let ∇ be the Levi-Civita connection of g. We define a new linear connection

$$\tilde{\nabla} := \nabla + H$$

on M such that for each $X, Y \in D$

$$H(X, \xi) = -AX, \quad H(X, Y) = g(AX, Y)\xi,$$
$$H(\xi, X) = \frac{1}{2}BX, \quad H(\xi, \xi) = 0.$$

Then for each $X, Y \in D$

$$(\tilde{\nabla}_X \phi)Y = 0,$$

and hence for each $X, Y, Z \in D$ we have that $\tilde{\nabla}_X Y \in D$ and also

$$\begin{aligned} \tilde{R}(X, Y)\phi Z - \phi\tilde{R}(X, Y)Z &= \tilde{\nabla}_X\tilde{\nabla}_Y\phi Z - \tilde{\nabla}_Y\tilde{\nabla}_X\phi Z - \tilde{\nabla}_{[X,Y]}\phi Z \\ &\quad - \phi(\tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z) \\ &= -\tilde{\nabla}_{[X,Y]}\phi Z + \phi\tilde{\nabla}_{[X,Y]}Z \\ &= 2g(BX, Y)(\tilde{\nabla}_\xi \phi)Z \end{aligned} \tag{4}$$

where $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$. On the other hand, for each $X, Y, Z \in D$ we have

$$\begin{aligned} \tilde{R}(X, Y)Z = R(X, Y)Z &- H(X, H(Y, Z)) + H(Y, H(X, Z)) \\ &+ H(H(X, Y), Z) - H(H(Y, X), Z) + (\tilde{\nabla}_X H)(Y, Z) \\ &- (\tilde{\nabla}_Y H)(X, Z) \end{aligned}$$

The horizontal part of $\tilde{R}(X, Y)Z$ is given by

$$\begin{aligned} (\tilde{R}(X, Y)Z)^H &= (R(X, Y)Z)^H + g(AY, Z)AX - g(AX, Z)AY \\ &\quad + \frac{1}{2}g(AX, Y)BZ - \frac{1}{2}g(AY, X)BZ \\ &= (R(X, Y)Z)^H + g(AY, Z)AX - g(AX, Z)AY \\ &\quad + g(BX, Y)BZ, \end{aligned}$$

thus

$$\begin{aligned}(\tilde R(X,Y)\phi Z-\phi(\tilde R(X,Y)Z))^H = (R(X,Y)\phi Z)^H &+ g(AY,\phi Z)AX\\ &- g(AX,\phi Z)AY+g(BX,Y)B\phi Z\\ &-\phi((R(X,Y)Z)^H+g(AY,Z)AX\\ &-g(AX,Z)AY+g(BX,Y)BZ).\end{aligned}$$

Comparing this last equation with (4) we have

$$\begin{aligned}2g(BX,Y)((\tilde\nabla_\xi\phi)Z-B\phi Z)^H = (R(X,Y)\phi Z)^H &- \phi(R(X,Y)Z)\\ &+g(AY,\phi Z)AX-g(AX,\phi Z)AY\\ &-g(AY,Z)\phi AX+g(AX,Z)\phi AY.\end{aligned} \tag{5}$$

If $c=0$, i.e., all the sectional curvatures $K_p(\pi)$ with $\pi\subset D_p$ vanish, then for every $X,Y,Z\in D_p$

$$\begin{aligned}2g(BX,Y)((\tilde\nabla_\xi\phi)Z-B\phi Z)^H = g(AY,\phi Z)AX &- g(AX,\phi Z)AY\\ &-g(AY,Z)\phi AX+g(AX,Z)\phi AY.\end{aligned} \tag{6}$$

Consider $Y\in D_p$ such that $AY\neq 0$. Hence if we take $Z=\phi AY$ we have

$$\begin{aligned}g(AY,AY)AX = &-2g(BX,Y)((\tilde\nabla_\xi\phi)\phi AY+BAY)^H\\ &+g(AX,AY)AY+g(AX,\phi AY)\phi AY\end{aligned} \tag{7}$$

for every $X\in D_p$ and thus $A:D_p\to D_p$ has rank $\leqslant 3$. Then there exists $X\in D_p, X\neq 0$ such that $AX=0$. Then, by (6) and (1) we have that

$$d\eta(X,Y)((\tilde\nabla_\xi\phi)Z-B\phi Z)^H=0,$$

for each $Y,Z\in D_p$. Thus, being η a contact form, for each $Z\in D_p$

$$((\tilde\nabla_\xi\phi)Z-B\phi Z)^H=0.$$

In conclusion, the equation (7) becomes

$$g(AY,AY)AX=g(AX,AY)AY+g(AX,\phi AY)\phi AY,$$

for every $X\in D_p$, yielding $\mathrm{rank}(A)\leqslant 2$. Now the contact condition implies that $\dim(\ker A)\leqslant n$. Thus $2n\leqslant 2+n$, namely $n\leqslant 2$ and hence $\dim M\leqslant 5$. On the other hand, observing that (2) also implies that B anti-commutes with ϕ, by Proposition 1, we have that $\dim M\geqslant 5$.

Now suppose $c\neq 0$. Then $A:D_p\to D_p$ is an isomorphism. Indeed, assume $X\in D_p$ such that $AX=0$, and $Y\in D_p$ orthogonal to $X,\phi X,BX$ (for example take $Y=\phi BX$). For $X_1,X_2,X_3\in D$ we set

$$S(X_1,X_2,X_3):=\tilde R(X_1,X_2)\phi X_3-\phi(\tilde R(X_1,X_2)X_3).$$

Then we have

$$S(X,Y,X)=2g(BX,Y)(\tilde\nabla_\xi\phi)X=0;$$

but on the other hand

$$\begin{aligned}(S(X,Y,X))^H &= (R(X,Y)\phi X)^H+g(AY,\phi X)AX-g(AX,\phi X)AY\\ &\quad+g(BX,Y)B\phi X-\phi((R(X,Y)X)^H+g(AY,X)AX\\ &\quad-g(AX,X)AY+g(BX,Y)BX)\\ &=cg(X,X)\phi Y,\end{aligned}$$

so that $X=0$.

Now, supposing that (2) holds, we apply Lemma 1; fix $Y,Z\in D_p$ such that $Z\in\mathrm{span}\{Y,\phi Y,AY\}^\perp$ and $g(Z,\phi AY)\neq 0$, then the equation (5) becomes

$$\begin{aligned}g(AY,\phi Z)AX = 2g(BX,Y)((\tilde\nabla_\xi\phi)Z-B\phi Z)^H &+ cg(\phi Z,X)Y\\ -cg(Z,X)\phi Y+g(AX,\phi Z)AY&-g(AX,Z)\phi AY.\end{aligned}$$

This implies that $\mathrm{rank}(A)\leqslant 5$, so that $n\leqslant 2$. As before, we conclude that $\dim M=5$. □

From the above proof, we see that in the case $c = 0$ one can obtain the assertion replacing the condition (2) with the weaker condition

$$\mathrm{d}\eta(\phi X, \phi Y) = -\mathrm{d}\eta(X, Y),$$

i.e. we have the following

Corollary 1. *Let (ϕ, ξ, η, g) be an almost contact metric structure on a contact manifold (M^{2n+1}, η) such that*

$$d\eta(\phi X, \phi Y) = -d\eta(X, Y),$$

$$g((\nabla_X \phi)Y, Z) = 0,$$

for each $X, Y, Z \in D$. We suppose there exists $p \in M$ such that the sectional curvature $K_p(\pi) = 0$, for each 2–plane π of D_p. Then $\dim M = 5$.

Almost contact metric manifolds are classified by Chinea and Gonzalez in [5]. The authors define twelve classes of manifolds $C_1, \dots, C_{12}$. All manifolds in the classes C_i for $i \in \{5, 6, .., 12\}$ satisfy condition (3), and all manifolds in C_9 or C_{10} satisfy (3) and (2). Thus we have the following

Theorem 2. *Every contact manifold (M, η) carrying an almost contact metric structure (ϕ, ξ, η, g) of class $C_9 \oplus C_{10}$ has dimension $4n + 1$, with $n \geqslant 1$.*

If there exist $p \in M$ and $c \in \mathbb{R}$ such that the sectional curvature $K_p(\pi) = c$, for each 2–plane π of D_p, then $\dim M = 5$.

3 Nearly cosymplectic case

In this section we will show that there does not exist a flat nearly cosymplectic manifold (M, ϕ, ξ, η, g) with η a contact form.

Lemma 2. *Let (M, ϕ, ξ, η, g) be a nearly cosymplectic manifold. Then*

(a) $d\eta(X, Y) = g(AX, Y)$ *for every $X, Y \in TM$,*
(b) $d\eta(X, Y) = -d\eta(\phi X, \phi Y)$ *for every $X, Y \in TM$,*
(c) *ξ is the Reeb vector field of (M^{2n+1}, η).*

If moreover η is a contact form, then

(d) *for every $p \in M^{2n+1}$, A_p is an isomorphism that anti-commutes with ϕ,*
(e) $g((\nabla_X \phi)Y, Z) = 0$, *for every $X, Y, Z \in D$,*
(f) $\dim M = 4n + 1$.

Proof. Let ∇ be the Levi-Civita connection of g. Since ξ is Killing, we have

$$\begin{aligned} 2g(AX, Y) &= 2g(\nabla_X \xi, Y) \\ &= X(g(\xi, Y)) + \xi(g(Y, X)) - Y(g(X, \xi)) \\ &\quad + g([X, \xi], Y) - g([\xi, Y], X) + g([Y, X], \xi) \\ &= X(g(\xi, Y)) - Y(g(X, \xi)) + g([Y, X], \xi) \\ &= X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \\ &= 2\mathrm{d}\eta(X, Y) \end{aligned}$$

for every $X, Y \in TM$. By Lemma 3.1 of [6] we have that

$$A\phi + \phi A = 0.$$

Then

$$\mathrm{d}\eta(\phi X, \phi Y) = g(A\phi X, \phi Y) = -g(AX, Y) = -\mathrm{d}\eta(X, Y),$$

from which it follows that

$$\mathrm{d}\eta(X, \xi) = -\mathrm{d}\eta(\phi X, \phi\xi) = 0.$$

If η is a contact form, as a consequence of (a), we have that A_p is an isomorphism. Finally (e) follows from (d) and the following equation

$$g((\nabla_X\phi)Y, AZ) = \eta(Y)g(A^2X, \phi Z) - \eta(X)g(A^2Y, \phi Z)$$

due to H. Endo [6]. □

Hence, as a consequence of Theorem 1, we can state

Theorem 3. *Let (M^{2n+1}, η) be a contact manifold endowed with a nearly cosymplectic structure (ϕ, ξ, η, g). Suppose there exist $p \in M$ and $c \in \mathbb{R}$ such that for each 2–plane π of D_p, $K_p(\pi) = c$. Then $c \neq 0$ and* $\dim M = 5$.

Remark 1. H. Endo in [6] determines the curvature tensor of a nearly cosymplectic manifold (M, ϕ, ξ, η, g) with pointwise constant ϕ-sectional curvature c

$$\begin{aligned} 4g(R(W,X)Y,Z) = {} & g((\nabla_W\phi)Z, (\nabla_X\phi)Y) - g((\nabla_W\phi)Y, (\nabla_X\phi)Z) \\ & - 2g((\nabla_W\phi)X, (\nabla_Y\phi)Z) + g(\nabla_W\xi, Z)g(\nabla_X\xi, Y) \\ & - g(\nabla_W\xi, Y)g(\nabla_X\xi, Z) - 2g(\nabla_W\xi, X)g(\nabla_Y\xi, Z) \\ & - \eta(W)\eta(Y)g(\nabla_X\xi, \nabla_Z\xi) + \eta(W)\eta(Z)g(\nabla_X\xi, \nabla_Y\xi) \\ & + \eta(X)\eta(Y)g(\nabla_W\xi, \nabla_Z\xi) - \eta(X)\eta(Z)g(\nabla_W\xi, \nabla_Y\xi) \\ & + c\{g(X,Y)g(Z,W) - g(Z,X)g(Y,W) \\ & + \eta(Z)\eta(X)g(Y,W) - \eta(Y)\eta(X)g(Z,W) \\ & + \eta(Y)\eta(W)g(Z,X) - \eta(Z)\eta(W)g(Y,X) \\ & + g(\phi Y, X)g(\phi Z, W) - g(\phi Z, X)g(\phi Y, W) \\ & - 2g(\phi Z, Y)g(\phi X, W)\}. \end{aligned} \tag{8}$$

One can obtain the conclusion of Theorem 3 also using this formula together with Lemma 2. If there exists a point $p \in M$ such that the sectional curvature of all the 2-planes of D_p is constant, then for every $X, Y, W \in D$ we have

$$R(W,X)Y = c(g(Y,X)W - g(Y,W)X);$$

moreover

$$g((\nabla_W\phi)Z, (\nabla_X\phi)Y) = g(\phi Y, AX)g(\phi Z, AW).$$

Thus by equation (8) we obtain

$$\begin{aligned} 3c(g(Y,X)W - g(Y,W)X) = {} & -g(\phi Y, AX)\phi AW + g(\phi Y, AW)\phi AX \\ & + 2g(\phi X, AW)\phi AY + g(AX, Y)AW \\ & - g(AW, Y)AX - 2g(AW, X)AY \\ & + c\{-g(X, \phi Y)\phi W + g(\phi Y, W)\phi X \\ & + 2g(\phi X, W)\phi Y\}. \end{aligned}$$

If in particular $Y = AW$, then

$$\begin{aligned} 3cg(X, AW)W = {} & \{-g(\phi AW, AX) + 2cg(\phi X, W)\}\phi AW + g(AX, AW)AW \\ & + 2g(\phi X, AW)\phi A^2W - g(AW, AW)AX \\ & - 2g(AW, X)A^2W - cg(\phi AW, X)\phi W, \end{aligned}$$

and hence $\mathrm{rank}(A) \leqslant 6$. By Lemma 2 it follows that $\dim M = 5$.

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