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Multivariate and abstract approximation theory for Banach space valued functions

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Abstract: Here we study quantitatively the high degree of approximation of sequences of linear operators acting on Banach space valued Fréchet differentiable functions to the unit operator, as well as other basic approximations including those under convexity. These operators are bounded by real positive linear companion operators. The Banach spaces considered here are general and no positivity assumption is made on the initial linear operators for which we study their approximation properties. We derive pointwise and uniform estimates, which imply the approximation of these operators to the unit assuming Fréchet differentiability of functions, and then we continue with basic approximations. At the end we study the special case where the approximated function fulfills a convexity condition resulting into sharp estimates. We give applications to Bernstein operators.

Keywords: Continuous Banach space valued functions, Banach space valued Fréchet differentiable functions, positive linear operator, convexity, modulus of continuity, rate of convergence

MSC: 41A17, 41A25, 41A36

1 Motivation

Let $\left(X, \|\cdot\|_{\beta}\right)$ be a general Banach space. Consider $g \in C\left([0,1]^m\right)$, $m \in \mathbb{N}$, and the classic multivariate Bernstein polynomials: let $n_1, \dots, n_m \in \mathbb{N}$, we define

$$\left(\widetilde{B}_{n_1,\dots,n_m}(g)\right)(x_1,\dots,x_m) := \sum_{\substack{0 \le k_j \le n_j \\ j \in \{1,\dots,m\}}} g\left(\frac{k_1}{n_1},\dots\frac{k_m}{n_m}\right) \prod_{j=1}^{m} \left(\binom{n_j}{k_j} x_j^{k_j} \left(1-x_j\right)^{n_j-k_j} \right), \tag{1}$$

$\forall (x_1,\dots,x_m) \in [0,1]^m$.

For $g \in C\left([0,1]^m\right)$ we have that $\widetilde{B}_{n_1,\dots,n_m}(g)$ converge uniformly to g, as $n_1, \dots n_m \to \infty$.

Let also $f \in C\left([0,1]^m, X\right)$ and define the vector valued in X multivariate Bernstein linear operators

$$\left(B_{n_1,\dots,n_m}(f)\right)(x_1,\dots,x_m) := \sum_{\substack{0 \le k_j \le n_j \\ j \in \{1,\dots,m\}}} f\left(\frac{k_1}{n_1},\dots\frac{k_m}{n_m}\right) \prod_{j=1}^{m} \left(\binom{n_j}{k_j} x_j^{k_j} \left(1-x_j\right)^{n_j-k_j} \right), \tag{2}$$

$\forall (x_1,\dots,x_m) \in [0,1]^m$.

That is $\left(B_{n_1,\dots,n_m}(f)\right)(x_1,\dots,x_m) \in X$.

Clearly, here $\|f\|_{\beta} \in C\left([0,1]^m\right)$.

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We notice that

$$\|(B_{n_1,\dots,n_m}(f))(x_1,\dots,x_m)\|_\beta \leq \sum_{\substack{0\leq k_j\leq n_j \\ j\in\{1,\dots,m\}}} \left\| f\left(\frac{k_1}{n_1},\dots\frac{k_m}{n_m}\right)\right\|_\beta \prod_{j=1}^{m}\left(\binom{n_j}{k_j} x_j^{k_j}\left(1-x_j\right)^{n_j-k_j}\right) \tag{3}$$

$$= \left(\widetilde{B}_{n_1,\dots,n_m}\left(\|f\|_\beta\right)\right)(x_1,\dots,x_m).$$

That is

$$\|(B_{n_1,\dots,n_m}(f))(x_1,\dots,x_m)\|_\beta \leq \left(\widetilde{B}_{n_1,\dots,n_m}\left(\|f\|_\beta\right)\right)(x_1,\dots,x_m), \tag{4}$$

$\forall (x_1,\dots,x_m)\in[0,1]^m, \forall f\in\left([0,1]^m, X\right)$.

The last property (4) is shared by almost all summation/integration similar operators and motivates our work here.

If $f(x) = c \in X$ the constant function, then

$$B_{n_1,\dots,n_m}(c) = c. \tag{5}$$

If $g \in C\left([0,1]^m\right)$ and $c \in X$, then $cg \in C\left([0,1]^m, X\right)$ and

$$B_{n_1,\dots,n_m}(cg) = c\widetilde{B}_{n_1,\dots,n_m}(g). \tag{6}$$

Again (5), (6) are fulfilled by many summation/integration operators.

In fact here (6) implies (5), when $g \equiv 1$.

The above can be generalized from $[0,1]^m$ to $\prod_{j=1}^m\left[a_j,b_j\right]$ or on M a convex and compact subset of $\left(\mathbb{R}^k, \|\cdot\|_p\right)$, $p\in[1,\infty]$, $k\in\mathbb{N}$. All this discussion motivates us to consider the following situation.

Let $L_N : C(M,X) \to C(M,X)$, $N \in \mathbb{N}$, be linear operators. Let also $\widetilde{L}_N : C(M) \to C(M)$, a sequence of positive linear operators, $\forall N \in \mathbb{N}$.

We assume that

$$\|(L_N(f))(x_0)\|_\beta \leq \left(\widetilde{L}_N\left(\|f\|_\beta\right)\right)(x_0), \tag{7}$$

$\forall N\in\mathbb{N}, \forall x_0\in M, \forall f\in C(M,X)$.

When $g \in C(M)$, $c \in X$, we assume that

$$L_N(cg) = c\widetilde{L}_N(g). \tag{8}$$

The special case of

$$\widetilde{L}_N(1) = 1, \tag{9}$$

implies

$$L_N(c) = c, \quad \forall c \in X. \tag{10}$$

We call $\widetilde{L}_N$ the companion operator of L_N.

Based on the above fundamental properties we study the high order approximation properties of $\{L_N\}_{N\in\mathbb{N}}$, to the unit operator, as well as other basic approximation properties including approximation under convexity. No kind of positivity property of $\{L_N\}_{N\in\mathbb{N}}$ is assumed. For the high order approximation of $\{L_N\}_{N\in\mathbb{N}}$ we assume Fréchet differentiability of functions under approximation. Other important motivation comes from [1, 2].

2 Background

We make

Remark 2.1. Let $\left(\mathbb{R}^k, \|\cdot\|_p\right)$, $k \in \mathbb{N}$, where $\|\cdot\|_p$ is the l_p-norm (Minkowski-norm), $1 \le p \le \infty$. $\mathbb{R}^k$ is a Banach space, and $\left(\mathbb{R}^k\right)^j$ denotes the j-fold product space $\mathbb{R}^k \times \ldots \times \mathbb{R}^k$ endowed with the max-norm $\|x\|_{(\mathbb{R}^k)^j} := \max_{1\le\lambda\le j} \|x_\lambda\|_p$, where $x := (x_1, \ldots, x_j) \in \left(\mathbb{R}^k\right)^j$.

Let $\left(X, \|\cdot\|_\beta\right)$ be a general Banach space. Then the space $L_j := L_j\left(\left(\mathbb{R}^k\right)^j; X\right)$ of all j-multilinear continuous maps $g : \left(\mathbb{R}^k\right)^j \to X$, $j = 1, \ldots, n$, is a Banach space with the norm

$$\|g\| := \|g\|_{L_j} := \sup_{\|x\|_{(\mathbb{R}^k)^j}=1} \|g(x)\|_\beta = \sup_{\|x\|_{(\mathbb{R}^k)^j}=1} \frac{\|g(x)\|_\beta}{\|x_1\|_p \cdots \|x_j\|_p}. \tag{11}$$

Let M be a non-empty convex and compact subset of $\mathbb{R}^k$ and $x_0 \in M$ is fixed.

Let O be an open subset of $\mathbb{R}^k : M \subset O$. Let $f : O \to X$ be a continuous function whose Fréchet derivatives (see [4]) $f^{(j)} : O \to L_j = L_j\left(\left(\mathbb{R}^k\right)^j; X\right)$ exist and are continuous for $1 \le j \le n$, $n \in \mathbb{N}$. Call $(x - x_0)^j := (x - x_0, \ldots, x - x_0) \in \left(\mathbb{R}^k\right)^j$, $x \in M$.

Here we deal with $f|_M$.

Then by Taylor's formula ([3]), ([4, p. 124]), we get

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_n(x, x_0), \text{ all } x \in M, \tag{12}$$

where the remainder is the Riemann integral

$$R_n(x, x_0) := \int_0^1 \frac{(1-u)^{n-1}}{(n-1)!} \left(f^{(n)}(x_0 + u(x - x_0)) - f^{(n)}(x_0)\right)(x - x_0)^n du, \tag{13}$$

here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

Considering

$$w := \omega_1\left(f^{(n)}, h\right) := \sup_{\substack{x,y\in M:\\ \|x-y\|_p\le h}} \left\|f^{(n)}(x) - f^{(n)}(y)\right\|, \tag{14}$$

we obtain

$$\left\|\left(f^{(n)}(x_0 + u(x - x_0)) - f^{(n)}(x_0)\right)(x - x_0)^n\right\|_\beta \le$$

$$\left\|f^{(n)}(x_0 + u(x - x_0)) - f^{(n)}(x_0)\right\| \cdot \|x - x_0\|_p^n \le w \cdot \|x - x_0\|_p^n \cdot \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil, \tag{15}$$

by [1, p. 208, Lemma 7.1.1] $\lceil\cdot\rceil$ is the ceiling.

Therefore for all $x \in M$ (see [1, pp. 121-122]):

$$\|R_n(x, x_0)\|_\beta \le w\|x - x_0\|_p^n \int_0^1 \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil \frac{(1-u)^{n-1}}{(n-1)!} du = w\Phi_n\left(\|x - x_0\|_p\right) \tag{16}$$

by a change of variable, where

$$\Phi_n(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t| - s)^{n-1}}{(n-1)!} ds = \frac{1}{n!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^n\right), \ \forall\, t \in \mathbb{R}, \tag{17}$$

is a (polynomial) spline function, see [1, pp. 210-211]. Also from there we get

$$\Phi_n(t) \le \left(\frac{|t|^{n+1}}{(n+1)!h} + \frac{|t|^n}{2n!} + \frac{h|t|^{n-1}}{8(n-1)!}\right), \ \ \forall\, t \in \mathbb{R}, \tag{18}$$

with equality true only at $t = 0$. Therefore it holds

$$\|R_n(x, x_0)\|_\beta \le w\left(\frac{\|x - x_0\|_p^{n+1}}{(n+1)!h} + \frac{\|x - x_0\|_p^n}{2n!} + \frac{h\|x - x_0\|_p^{n-1}}{8(n-1)!}\right), \quad \forall\, x \in M. \tag{19}$$

We have found that

$$\left\|f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!}\right\|_\beta \le \omega_1\left(f^{(n)}, h\right)\left(\frac{\|x - x_0\|_p^{n+1}}{(n+1)!h} + \frac{\|x - x_0\|_p^n}{2n!} + \frac{h\|x - x_0\|_p^{n-1}}{8(n-1)!}\right) < \infty, \tag{20}$$

$\forall\, x, x_0 \in M$.

Here $0 < \omega_1\left(f^{(n)}, h\right) < \infty$, by M being compact and $f^{(n)}$ being continuous on M.

One can rewrite (20) as

$$\left\|f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\beta \le \omega_1\left(f^{(n)}, h\right)\left(\frac{\|\cdot - x_0\|_p^{n+1}}{(n+1)!h} + \frac{\|\cdot - x_0\|_p^n}{2n!} + \frac{h\|\cdot - x_0\|_p^{n-1}}{8(n-1)!}\right), \ \forall\, x_0 \in M, \tag{21}$$

a pointwise functional inequality on M.

Here $(\cdot - x_0)^j$ maps M into $\left(\mathbb{R}^k\right)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $\left(\mathbb{R}^k\right)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into X.

Clearly $f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, hence $\left\|f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\beta \in C(M)$.

Let $\left\{\widetilde{L}_N\right\}_{N\in\mathbb{N}}$ be a sequence of positive linear operators mapping $C(M)$ into $C(M)$.

Therefore we obtain

$$\left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\beta\right)\right)(x_0) \le \omega_1\left(f^{(n)}, h\right) \cdot \tag{22}$$

$$\left(\frac{\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)}{(n+1)!h} + \frac{\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n}\right)\right)(x_0)}{2n!} + \frac{h\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n-1}\right)\right)(x_0)}{8(n-1)!}\right) =: (\xi_1),$$

$\forall\, N \in \mathbb{N}, \forall\, x_0 \in M$.

By the basic Riesz representation theorem we have

$$\left(\widetilde{L}_N(F)\right)(x_0) = \int_M F(t)\, d\mu_{Nx_0}(t), \tag{23}$$

$\forall\, F \in C(M)$, where μ_{Nx_0} is a unique positive finite completed Borel measure on M, for any $x_0 \in M$; $\forall\, N \in \mathbb{N}$.

Using (23) and Hölder's inequality for $k = 1, \ldots, n$, we obtain:

$$\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{k}\right)\right)(x_0) \le \left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{k}{n+1}}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{n+1-k}{n+1}}. \tag{24}$$

Hence it holds

$$(\xi_1) \le \omega_1\left(f^{(n)}, h\right)\left(\frac{\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)}{(n+1)!h} + \frac{\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2n!} + \right. \tag{25}$$

$$\left.\frac{h\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n-1}{n+1}}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8(n-1)!}\right) =: (\xi_2).$$

Here we take $\left(\widetilde{L}_N(1)\right)(x_0) > 0$, $\forall\, x_0 \in M$, otherwise our theory is trivial.

Set

$$h := r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}, \tag{26}$$

where $r > 0$.

Initially assume that $h > 0$. Then

$$(\xi_2) \le \omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)$$

$$\left[\frac{\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}}{(n+1)!r} + \right. \tag{27}$$

$$\frac{\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2n!} + \tag{28}$$

$$\left.\frac{r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8(n-1)!}\right] =$$

$$\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}} \tag{29}$$

$$\left[\frac{1}{(n+1)!r} + \frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2n!r} + \frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8n!r}\right].$$

Consequently, we get

$$\left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{j=0}^{n}\frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!}\right\|_\beta\right)\right)(x_0) \le \tag{30}$$

$$\frac{\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{rn!}\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}$$

$$\left[\frac{1}{(n+1)} + \frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2} + \frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8}\right]$$

In case of

$$\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0) = 0, \tag{31}$$

the right hand side of (30) is zero. So we have by (23)

$$\int_M \|t - x_0\|_p^{n+1}\, d\mu_{Nx_0}(t) = 0, \tag{32}$$

which means $\|t - x_0\|_p = 0$, a.e. in $t \in M$.

That is $t = x_0$, a.e. in $t \in M$, i.e. $\mu_{Nx_0}(\{t \in M | t \ne x_0\}) = 0$. Hence μ_{Nx_0} concentrates at $\{x_0\}$, which means

$$\left(\widetilde{L}_N(F)\right)(x_0) = F(x_0)\,\mu_{Nx_0}(M) = F(x_0)\left(\widetilde{L}_N(1)\right)(x_0),$$

$\forall\, F \in C(M)$, $\forall\, N \in \mathbb{N}$.

Therefore, we have that

$$\left(\widetilde{L}_N\left(\left\|f(\cdot)-\sum_{j=0}^{n}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\beta\right)\right)(x_0)= \tag{33}$$

$$\left\|f(x_0)-\sum_{j=0}^{n}\frac{f^{(j)}(x_0)(x_0-x_0)^j}{j!}\right\|_\beta\left(\widetilde{L}_N(1)\right)(x_0)=\left\|f(x_0)-f(x_0)-\sum_{j=1}^{n}\frac{f^{(j)}(x_0)(0)^j}{j!}\right\|_\beta\left(\widetilde{L}_N(1)\right)(x_0)=0.$$

Consequently, inequality (30) is always true.

3 Main Results

We present our first main convergence result.

Theorem 3.1. *Let M be a nonempty compact convex subset of the open subset O of $\left(\mathbb{R}^k,\|\cdot\|_p\right)$, and let $\left(X,\|\cdot\|_\beta\right)$ be a general Banach space. For any $N\in\mathbb{N}$ let the linear operators $L_N: C(M,X)\to C(M,X)$ and the positive linear operators $\widetilde{L}_N: C(M)\to C(M)$, such that*

$$\|(L_N(f))(x_0)\|_\beta\leq\left(\widetilde{L}_N\left(\|f\|_\beta\right)\right)(x_0), \tag{34}$$

$\forall N\in\mathbb{N}, \forall f\in C(M,X), \forall x_0\in M$.

Furthermore assume that

$$L_N(cF)=c\widetilde{L}_N(F),\quad\forall F\in C(M),\forall c\in X. \tag{35}$$

Let $n\in\mathbb{N}$, here we deal with $f\in C^n(O,X)$, the space of n-times continuously Fréchet differentiable functions from O into X.

Here we study approximation to $f|_M$.

Let $x_0\in M$, $r>0$, and

$$\omega_1\left(f^{(n)},h\right):=\sup_{\substack{x,y\in M:\\ \|x-y\|_p\leq h}}\left\|f^{(n)}(x)-f^{(n)}(y)\right\|. \tag{36}$$

Then

1)

$$\left\|(L_N(f))(x_0)-\sum_{j=0}^{n}\frac{1}{j!}\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta\leq$$

$$\frac{\omega_1\left(f^{(n)},r\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{rn!}\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}$$

$$\left[\frac{1}{(n+1)}+\frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2}+\frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8}\right], \tag{37}$$

2) additionally if $f^{(j)}(x_0)=0$, $j=0,1,\ldots,n$, we have

$$\|(L_N(f))(x_0)\|_\beta\leq$$

$$\frac{\omega_1\left(f^{(n)},r\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{rn!}\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}$$

$$\left[\frac{1}{(n+1)}+\frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2}+\frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8}\right], \tag{38}$$

3)

$$\|(L_N(f))(x_0)-f(x_0)\|_\beta \leq \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0)-1\right| +$$

$$\sum_{j=1}^{n}\frac{1}{j!}\left\|\left(\widetilde{L}_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta +$$

$$\frac{\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{rn!}\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}$$

$$\left[\frac{1}{(n+1)}+\frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2}+\frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8}\right], \tag{39}$$

4)

$$\left\|\|L_N(f)-f\|_\beta\right\|_{\infty,M} \leq \left\|\|f\|_\beta\right\|_{\infty,M}\left\|\left(\widetilde{L}_N(1)\right)-1\right\|_{\infty,M}+$$

$$\sum_{j=1}^{n}\frac{1}{j!}\left\|\left\|\left(\widetilde{L}_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta\right\|_{\infty,x_0\in M}+\frac{\omega_1\left(f^{(n)}, r\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\frac{1}{n+1}}\right)}{rn!}$$

$$\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\frac{n}{n+1}}\left[\frac{1}{(n+1)}+\frac{r\left\|\widetilde{L}_N(1)\right\|_{\infty,M}^{\frac{1}{n+1}}}{2}+\frac{nr^2\left\|\widetilde{L}_N(1)\right\|_{\infty,M}^{\frac{2}{n+1}}}{8}\right]. \tag{40}$$

Proof. 1) Here L_N is a linear operator from $C(M,X)$ into $C(M,X)$ and $\widetilde{L}_N$ is a positive linear operator from $C(M)$ into $C(M)$ such that

$$\|(L_N(f))(x_0)\|_\beta \leq \left(\widetilde{L}_N\left(\|f\|_\beta\right)\right)(x_0), \tag{41}$$

$\forall\, N\in\mathbb{N}$, $\forall\, f\in C(M,X)$, $\forall\, x_0\in M$.

Therefore, we have

$$\left\|\left(L_N\left(f(\cdot)-\sum_{j=0}^{n}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right)\right)(x_0)\right\|_\beta \leq$$

$$\left(\widetilde{L}_N\left(\left\|f(\cdot)-\sum_{j=0}^{n}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\beta\right)\right)(x_0) \overset{(30)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{rn!}\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}} \tag{42}$$

$$\left[\frac{1}{(n+1)}+\frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2}+\frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8}\right].$$

2) It is obvious.

3) We have that

$$\|(L_N(f))(x_0)-f(x_0)\|_\beta =$$

$$\left\|(L_N(f))(x_0)-\sum_{j=0}^{n}\frac{1}{j!}\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)+\sum_{j=0}^{n}\frac{1}{j!}\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)-f(x_0)\right\|_\beta \leq \tag{43}$$

$$\left\|(L_N(f))(x_0)-\sum_{j=0}^{n}\frac{1}{j!}\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta +$$

$$\left\|\left(f(x_0)\left(\widetilde{L}_N(1)\right)(x_0)-f(x_0)\right)+\sum_{j=1}^{n}\frac{1}{j!}\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta \overset{(37)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)},r\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{rn!}\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{n}{n+1}}$$

$$\left[\frac{1}{(n+1)}+\frac{r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}}}{2}+\frac{nr^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}}{8}\right]+ \tag{44}$$

$$\|f(x_0)\|_\beta\left|\left(\widetilde{L}_N(1)\right)(x_0)-1\right|+\sum_{j=1}^{n}\frac{1}{j!}\left\|\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta,$$

proving (39).

4) Clearly here $\|f\|_\beta \in C(M)$, thus $\left\|\|f\|_\beta\right\|_{\infty,M}<\infty$. Also we notice that

$$\left\|\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta \leq \left(\widetilde{L}_N\left(\left\|f^{(j)}(x_0)(\cdot-x_0)^j\right\|_\beta\right)\right)(x_0)\leq$$

$$\left(\widetilde{L}_N\left(\left\|f^{(j)}(x_0)\right\|\|\cdot-x_0\|_p^j\right)\right)(x_0)=\left\|f^{(j)}(x_0)\right\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^j\right)\right)(x_0). \tag{45}$$

That is, it holds

$$\left\|\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta \leq \left\|f^{(j)}\right\|\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^j\right)\right)(x_0)\right\|_{\infty,x_0\in M}, \tag{46}$$

$\forall\, x_0\in M$, $j=1,\ldots,n$.

By Lemma 3.2, which follows, we get that $\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^j\right)\right)(x_0)\right\|_{\infty,x_0\in M}<\infty$, for all $j=1,\ldots,n+1$.

Therefore (46) it is obvious from (45), furthermore the right hand side of (40) is finite. □

We need

Lemma 3.2. *The function* $\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^m\right)\right)(x_0)$ *is continuous in* $x_0\in M$, $m\in\mathbb{N}$.

Proof. Let $x_n\to x_0$, $x_n,x_0\in M$, as $n\to\infty$. We observe that

$$\left\|\widetilde{L}_N\left(\|\cdot-x_n\|_p^m-\|\cdot-x_0\|_p^m\right)\right\|_\infty \leq \left\|\widetilde{L}_N\right\|\left\|\|\cdot-x_n\|_p^m-\|\cdot-x_0\|_p^m\right\|_{\infty,M}, \tag{47}$$

where $\left\|\widetilde{L}_N\right\|=\left\|\widetilde{L}_N(1)\right\|_{\infty,M}<\infty$, because $\widetilde{L}_N$ is a positive linear operator. We notice that $(t,x_n,x_0\in M)$

$$\left|\|t-x_n\|_p^m-\|t-x_0\|_p^m\right|=\left|\|t-x_n\|_p-\|t-x_0\|_p\right| \tag{48}$$

$$\left\{\|t-x_n\|_p^{m-1}+\|t-x_n\|_p^{m-2}\|t-x_0\|_p+\|t-x_n\|_p^{m-3}\|t-x_0\|_p^2+\ldots+\right.$$

$$\|t-x_n\|_p \|t-x_0\|_p^{m-2} + \|t-x_0\|_p^{m-1}\Big\} \leq$$

$$\left| \|t-x_n\|_p - \|t-x_0\|_p \right| m\,(diameter\,(M))^{m-1} \leq \|x_n - x_0\|_p\, m\,(diameter\,(M))^{m-1}.$$

Hence we have

$$\left| \|t-x_n\|_p^m - \|t-x_0\|_p^m \right| \leq \|x_n - x_0\|_p\, m\,(diam\,(M))^{m-1}, \quad \forall\, t \in M. \tag{49}$$

We further notice that in view of (47) and (49)

$$\left| \left(\widetilde{L}_N\left(\|\cdot - x_n\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0) \right| =$$

$$\Big| \left(\widetilde{L}_N\left(\|\cdot - x_n\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_n) +$$

$$\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0) \Big| = \tag{50}$$

$$\left| \left(\widetilde{L}_N\left(\|\cdot - x_n\|_p^m - \|\cdot - x_0\|_p^m\right)\right)(x_n) + \left[\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0) \right] \right| \leq$$

$$\left| \left(\widetilde{L}_N\left(\|\cdot - x_n\|_p^m - \|\cdot - x_0\|_p^m\right)\right)(x_n) \right| +$$

$$\left| \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0) \right| \leq \tag{51}$$

$$\left\|\widetilde{L}_N\right\| \left\| \|\cdot - x_n\|_p^m - \|\cdot - x_0\|_p^m \right\|_\infty + \left| \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0) \right| \leq$$

$$\left\|\widetilde{L}_N\right\| \|x_n - x_0\|_p\, m\,(diam\,(M))^{m-1} + \left| \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_n) - \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0) \right| \to 0,$$

proving the claim. ☐

Remark 3.3. From (24) we get

$$\left\| \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^k\right)\right)(x_0) \right\|_{\infty, x_0 \in M} \leq$$

$$\left\| \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0) \right\|_{\infty, x_0 \in M}^{\left(\frac{k}{n+1}\right)} \left\| \widetilde{L}_N(1) \right\|_{\infty, M}^{\frac{n+1-k}{n+1}}, \tag{52}$$

for all $k = 1, \ldots, n$.

Conclusion 3.4. Let $N \to \infty$ and $\widetilde{L}_N(1) \stackrel{u}{\to} 1$, uniformly, and $\left\| \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0) \right\|_{\infty, x_0 \in M} \to 0$. Then by (40) we get, as $N \to \infty$, that $L_n(f) \stackrel{u}{\to} f$, uniformly in $\|\cdot\|_\beta$ over M.

The last statement is also supported by (46) and (52). Here notice that $\left\| \widetilde{L}_N(1) \right\|_{\infty, M}$ turns out to be bounded, and $\omega_1\left(f^{(n)}, \cdot\right)$ is also bounded.

Conclusion 3.5. By (24), (39) and (45) we get the following: as $N \to \infty$, assume that $\left(\widetilde{L}_N(1)\right)(x_0) \to 1$, and $\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right) \to 0$. Then $(L_N(f))(x_0) \to f(x_0)$, as $N \to \infty$, pointwise. Notice that here $\left(\widetilde{L}_N(1)\right)(x_0)$ is bounded with respect to N, as well as $\omega_1\left(f^{(n)}, \cdot\right)$ is bounded in general.

Corollary 3.6. *(to Theorem 3.1, case $n = 1$) Additionally assume that $\left(\widetilde{L}_N(1)\right)(x_0) = 1, \forall\, x_0 \in M$. Then*

1)

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta \leq \left\| \left(L_N\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0) \right\|_\beta +$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}$$

$$\left[1+r\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{2}}+\frac{r^2\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)}{4}\right], \tag{53}$$

2)

$$\left\|\left\|(L_N(f))-f\right\|\right\|_{\infty,M}\leq$$
$$\left\|\left\|\left(\widetilde{L}_N\left(f^{(1)}(x_0)(\cdot-x_0)\right)\right)(x_0)\right\|_{\beta}\right\|_{\infty,x_0\in M}+$$
$$\frac{1}{2r}\omega_1\left(f^{(1)},r\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\frac{1}{2}}\right)\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\frac{1}{2}}$$
$$\left[1+r\left\|\widetilde{L}_N(1)\right\|_{\infty,M}^{\frac{1}{2}}+\frac{r^2\left\|\widetilde{L}_N(1)\right\|_{\infty,M}}{4}\right]. \tag{54}$$

We need

Definition 3.7. Let M be a convex and compact subset of $\left(\mathbb{R}^k,\|\cdot\|_p\right)$, $p\in[1,\infty]$, and $\left(X,\|\cdot\|_\beta\right)$ be a general Banach space. Let $f\in C(M,X)$. We define the first modulus of continuity of f as

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in M:\\ \|x-y\|_p\leq\delta}}\|f(x)-f(y)\|,\quad 0<\delta\leq diam(M). \tag{55}$$

If $\delta>diam(M)$, then
$$\omega_1(f,\delta)=\omega_1(f,diam(M)). \tag{56}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta>0$. For $f\in B(M,X)$ (bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 3.8. *We have $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$ iff $f\in C(M,X)$.*

Proof. ($\Rightarrow$) Let $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$. Then $\forall\,\varepsilon>0$, $\exists\,\delta>0$ with $\omega_1(f,\delta)\leq\varepsilon$, i.e. $\forall\,x,y\in M:\|x-y\|_p\leq\delta$ we get $\|f(x)-f(y)\|_\beta\leq\varepsilon$. That is $f\in C(M,X)$.

($\Leftarrow$) Let $f\in C(M,X)$. Then $\forall\,\varepsilon>0$, $\exists\,\delta>0$: whenever $\|x-y\|_p\leq\delta$, $x,y\in M$, it implies $\|f(x)-f(y)\|_\beta\leq\varepsilon$. Therefore $\forall\,\varepsilon>0$, $\exists\,\delta>0:\omega_1(f,\delta)\leq\varepsilon$. That is $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$. □

Lemma 3.9. *[1, p. 208] Let M be a convex compact subset of $\left(\mathbb{R}^k,\|\cdot\|_p\right)$, $p\in[1,\infty]$, and $\left(X,\|\cdot\|_\beta\right)$ be a general Banach space. Let $f\in B(M,X)$. Then*

$$\|f(x)-f(x_0)\|_\beta\leq\omega_1(f,h)\left\lceil\frac{\|x-x_0\|_p}{h}\right\rceil$$
$$\leq\omega_1(f,h)\left(1+\frac{\|x-x_0\|_p}{h}\right),\quad\forall\,x,x_0\in M, \tag{57}$$

where $\lceil\cdot\rceil$ is the ceiling of the number, $h>0$.

We present the basic pointwise convergence result.

Theorem 3.10. *Let $L_N:C(M,X)\to C(M,X)$ be linear operators, $\forall\,N\in\mathbb{N}$, where M is a convex compact subset of $\left(\mathbb{R}^k,\|\cdot\|_p\right)$, $p\in[1,\infty]$ and $\left(X,\|\cdot\|_\beta\right)$ is a general Banach space. Let $\widetilde{L}_N:C(M)\to C(M)$ be positive linear operators, such that*
$$\|(L_N(f))(x_0)\|_\beta\leq\left(\widetilde{L}_N\left(\|f\|_\beta\right)\right)(x_0), \tag{58}$$

$\forall\,N\in\mathbb{N}$, $\forall\,f\in C(M,X)$, $\forall\,x_0\in M$.

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\, g \in C(M), \forall\, c \in X. \tag{59}$$

Then

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta \le \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \tag{60}$$

$$\left[\left(\widetilde{L}_N(1)\right)(x_0) + 1\right] \omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right), \quad \forall\, f \in C(M, X).$$

By (60), as $\left(\widetilde{L}_N(1)\right)(x_0) \to 1$ *and* $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) \to 0$, *then* $(L_N(f))(x_0) \overset{\|\cdot\|_\beta}{\to} f(x_0)$, *as* $N \to \infty$, $\forall\, f \in C(M, X)$. *Here notice that* $\left(\widetilde{L}_N(1)\right)(x_0)$ *is bounded.*

Proof. We observe that

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta =$$

$$\|(L_N(f))(x_0) - (L_N(f(x_0)))(x_0) + (L_N(f(x_0)))(x_0) - f(x_0)\|_\beta \le \tag{61}$$

$$\|(L_N(f))(x_0) - (L_N(f(x_0)))(x_0)\|_\beta + \|(L_N(f(x_0)))(x_0) - f(x_0)\|_\beta =$$

$$\|(L_N(f - f(x_0)))(x_0)\|_\beta + \left\|f(x_0)\left(\widetilde{L}_N(1)\right)(x_0) - f(x_0)\right\|_\beta =$$

$$\|(L_N(f - f(x_0)))(x_0)\|_\beta + \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| \le \tag{62}$$

$$\left(\widetilde{L}_N\left(\|f - f(x_0)\|_\beta\right)\right)(x_0) + \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| \le$$

(let $h > 0$, and by Lemma 3.9)

$$\left(\widetilde{L}_N\left(\omega_1(f, h)\left(1 + \frac{\|\cdot - x_0\|_p}{h}\right)\right)(x_0)\right) + \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| = \tag{63}$$

$$\omega_1(f, h)\left[\left(\widetilde{L}_N(1)\right)(x_0) + \frac{1}{h}\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right] + \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| =$$

$$\omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right)\left[\left(\widetilde{L}_N(1)\right)(x_0) + 1\right] + \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right|, \tag{64}$$

by choosing

$$h := \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0), \tag{65}$$

if $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) > 0$.

Next we consider the case of

$$\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) = 0. \tag{66}$$

By Riesz representation theorem there exists a positive finite measure μ_{x_0} such that

$$\left(\widetilde{L}_N(g)\right)(x_0) = \int_M g(t)\, d\mu_{x_0}(t), \quad \forall\, g \in C(M). \tag{67}$$

That is

$$\int_M \|t - x_0\|_p\, d\mu_{x_0}(t) = 0, \tag{68}$$

which implies $\|t - x_0\|_p = 0$, a.e, hence $t - x_0 = 0$, a.e., and $t = x_0$, a.e. on M. Consequently $\mu_{x_0}(\{t \in M : t \neq x_0\}) = 0$. That is $\mu_{x_0} = \delta_{x_0}\overline{M}$ (where $0 < \overline{M} := \mu_{x_0}(M) = \left(\widetilde{L}_N(1)\right)(x_0)$). Hence, in that case $\left(\widetilde{L}_N(g)\right)(x_0) = g(x_0)\,\overline{M}$.

Consequently, by (66) it holds $\omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right) = 0$, and the right hand side of (60) equals $\|f(x_0)\|_\beta \left|\overline{M} - 1\right|$.

Also, it is $\left(\widetilde{L}_N\left(\|f - f(x_0)\|_\beta\right)\right)(x_0) = 0$, implying (see (58)) $\|(L_N(f - f(x_0)))(x_0)\|_\beta = 0$. Hence, $(L_N(f - f(x_0)))(x_0) = 0$, and

$$(L_N(f))(x_0) = f(x_0)\left(\widetilde{L}_N(1)\right)(x_0) = \overline{M}f(x_0). \tag{69}$$

Consequently, the left hand side of (60) becomes

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta = \left\|\overline{M}f(x_0) - f(x_0)\right\|_\beta = \|f(x_0)\|_\beta \left|\overline{M} - 1\right|. \tag{70}$$

So that (60) becomes an equality, both sides equal $\|f(x_0)\|_\beta \left|\overline{M} - 1\right|$.

In the extreme case of $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) = 0$. Thus inequality (60) is proved completely in all cases. ☐

Corollary 3.11. *All as in Theorem 3.10. Additionally assume that* $\left(\widetilde{L}_N(1)\right)(x_0) = 1, \forall\, x_0 \in M$. *Then*

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta \leq 2\omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right), \quad \forall\, f \in C(M, X). \tag{71}$$

Next we obtain the following uniform convergence result.

Theorem 3.12. *All as in Theorem 3.10. Then*

$$\left\|\|L_N(f) - f\|_\beta\right\|_{\infty,M} \leq \left\|\|f\|_\beta\right\|_{\infty,M} \left\|\widetilde{L}_N(1) - 1\right\|_{\infty,M} + \tag{72}$$

$$\left\|\widetilde{L}_N(1) + 1\right\|_{\infty,M} \omega_1\left(f, \left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right\|_{\infty, x_0\in M}\right), \quad \forall\, f \in C(M, X).$$

Proof. By (60). ☐

Conclusion 3.13. By Lemma 3.2 $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)$ is continuous in $x_0 \in M$. Thus $\left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right\|_{\infty, x_0\in M} < \infty$.

As $N \to \infty$, we assume that $\widetilde{L}_N(1) \overset{u}{\to} 1$, uniformly, and $\left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right\|_{\infty, x_0\in M} \to 0$, then $L_n(f) \overset{u}{\to} f$, uniformly on M, $\forall\, f \in C(M, X)$.

Here $\left\|\widetilde{L}_N(1)\right\|_{\infty,M}$ is bounded.

Under convexity we have the following sharp general pointwise convergence result.

Theorem 3.14. *All here as in Theorem 3.10. Additionally, assume that* $x_0 \in M^0$, *and the closed ball in* $\mathbb{R}^k$: $B\left(x_0, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right) \subset M$, *where* $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) \geq 0$, *and* $\|f(t) - f(x_0)\|_\beta$ *is convex in* $t \in M$. *Then*

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta \leq \|f(x_0)\|_\beta \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right), \quad \forall\, N \in \mathbb{N}. \tag{73}$$

Proof. Let $g(t) := \|f(t) - f(x_0)\|_\beta$, $\forall\, t \in M$; $g(x_0) = 0$. Then by [1, p. 243, Lemma 8.1.1], we obtain

$$g(t) \leq \frac{\omega_1(g, h)}{h}\|t - x_0\|_p, \quad \forall\, t \in M; \ h > 0 \text{ with } B(x_0, h) \subset M. \tag{74}$$

We notice the following

$$\|f(t_1) - f(x_0)\|_\beta = \|f(t_1) - f(t_2) + f(t_2) - f(x_0)\|_\beta \leq$$

$$\|f(t_1) - f(t_2)\|_\beta + \|f(t_2) - f(x_0)\|_\beta , \tag{75}$$

hence

$$\|f(t_1) - f(x_0)\|_\beta - \|f(t_2) - f(x_0)\|_\beta \le \|f(t_1) - f(t_2)\|_\beta . \tag{76}$$

Similarly, it holds

$$\|f(t_2) - f(x_0)\|_\beta - \|f(t_1) - f(x_0)\|_\beta \le \|f(t_1) - f(t_2)\|_\beta . \tag{77}$$

Therefore for any $t_1, t_2 \in M : \|t_1 - t_2\|_p \le h$ we get:

$$\left| \|f(t_1) - f(x_0)\|_\beta - \|f(t_2) - f(x_0)\|_\beta \right| \le \|f(t_1) - f(t_2)\|_\beta \le \omega_1(f, h) . \tag{78}$$

That is

$$\omega_1(g, h) \le \omega_1(f, h) . \tag{79}$$

The last implies

$$\|f(t) - f(x_0)\|_\beta \le \frac{\omega_1(f, h)}{h} \|t - x_0\|_p , \quad \forall\, t \in M. \tag{80}$$

As in the proof of Theorem 3.10 we have

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta \le \ldots \le$$

$$\|(L_N(f - f(x_0)))(x_0)\|_\beta + \|f(x_0)\|_\beta \left| \left(\widetilde{L}_N(1)\right)(x_0) - 1 \right| \le \tag{81}$$

$$\left(\widetilde{L}_N\left(\|f - f(x_0)\|_\beta\right)\right)(x_0) + \|f(x_0)\|_\beta \left| \left(\widetilde{L}_N(1)\right)(x_0) - 1 \right| \overset{(80)}{\le}$$

$$\frac{\omega_1(f, h)}{h} \left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_\beta\right)\right)(x_0)\right) + \|f(x_0)\|_\beta \left| \left(\widetilde{L}_N(1)\right)(x_0) - 1 \right| = \tag{82}$$

$$\omega_1\left(f, \left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_\beta\right)\right)(x_0)\right)\right) + \|f(x_0)\|_\beta \left| \left(\widetilde{L}_N(1)\right)(x_0) - 1 \right| ,$$

by choosing

$$h := \left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_\beta\right)\right)(x_0)\right) > 0, \tag{83}$$

if the last is positive. The case $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_\beta\right)\right)(x_0) = 0$ is treated the same way as in the proof of Theorem 3.10. The theorem is proved. □

Theorem 3.15. *All as in Theorem 3.14. Inequality (73) is sharp, in fact it is attained by* $f(t) = \overrightarrow{i} \|t - x_0\|_p$, $\overrightarrow{i}$ *is a unit vector of* $\left(X, \|\cdot\|_\beta\right)$, $t \in M$.

Proof. Indeed, this f here fulfills all the assumptions of the theorem.

We further notice that $f(x_0) = 0$, and $\|f(t) - f(x_0)\| = \|t - x_0\|_p$ is convex in $t \in M$.

The left hand side of (73) is

$$\|(L_N(f))(x_0) - f(x_0)\|_\beta = \left\| \left(L_N\left(\overrightarrow{i} \|\cdot - x_0\|_p\right)\right)(x_0) \right\|_\beta \overset{(59)}{=}$$

$$\left\| \overrightarrow{i} \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) \right\|_\beta = \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) . \tag{84}$$

The right hand side of (73) is

$$\omega_1\left(f, \left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right)\right) =$$

$$\omega_1\left(\overrightarrow{i} \|\cdot - x_0\|_p , \left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right)\right) =$$

$$\sup_{\substack{t_1,t_2\in M:\\ \|t_1-t_2\|_p\le\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p\right)\right)(x_0)}} \left\|\vec{i}\,\|t_1-x_0\|_p-\vec{i}\,\|t_2-x_0\|_p\right\|_\beta =$$

$$\sup_{\substack{t_1,t_2\in M:\\ \|t_1-t_2\|_p\le\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p\right)\right)(x_0)}} \left|\|t_1-x_0\|_p-\|t_2-x_0\|_p\right| \le \tag{85}$$

$$\sup_{\substack{t_1,t_2\in M:\\ \|t_1-t_2\|_p\le\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p\right)\right)(x_0)}} \|t_1-t_2\|_p = \left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p\right)\right)(x_0)\right).$$

Hence we have found that

$$\omega_1\left(f,\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p\right)\right)(x_0)\right)\right)\le\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p\right)\right)(x_0). \tag{86}$$

Clearly (73) is attained. The theorem is proved. □

4 Application

Let $[0,1]^2$ be a subset of O which is an open subset of $\left(\mathbb{R}^2,\|\cdot\|_{l_1}\right)$, and let $\left(X,\|\cdot\|_\beta\right)$ be a general Banach space.

For any $m,n\in\mathbb{N}$ define the Bernstein oeprators from $C\left([0,1]^2,X\right)$ into $C\left([0,1]^2,X\right)$, as follows:

$$B_{m,n}(f;x_1,x_2):=\sum_{k=0}^{m}\sum_{l=0}^{n}f\left(\frac{k}{m},\frac{l}{n}\right)\binom{m}{k}\binom{n}{l}x_1^k(1-x_1)^{m-k}x_2^l(1-x_2)^{n-l}, \tag{87}$$

$\forall(x_1,x_2)\in[0,1]^2$, $\forall f\in C(O,X)$.

For any $N\in\mathbb{N}$, we define the companion Bernstein operators from $C\left([0,1]^2\right)$ into $C\left([0,1]^2\right)$, as follows:

$$\widetilde{B}_{m,n}(g;x_1,x_2):=\sum_{k=0}^{m}\sum_{l=0}^{n}g\left(\frac{k}{m},\frac{l}{n}\right)\binom{m}{k}\binom{n}{l}x_1^k(1-x_1)^{m-k}x_2^l(1-x_2)^{n-l}, \tag{88}$$

$\forall(x_1,x_2)\in[0,1]^2$, $\forall g\in C(O)$, see also [1, p. 238].

We observe easily that

$$\|B_{m,n}(f;x_1,x_2)\|_\beta\le\sum_{k=0}^{m}\sum_{l=0}^{n}\left\|f\left(\frac{k}{m},\frac{l}{n}\right)\right\|_\beta \tag{89}$$

$$\binom{m}{k}\binom{n}{l}x_1^k(1-x_1)^{m-k}x_2^l(1-x_2)^{n-l}=\widetilde{B}_{m,n}\left(\|f\|_\beta;x_1,x_2\right).$$

Clearly we see also that

$$B_{m,n}(cg;x_1,x_2)=c\widetilde{B}_{m,n}(g;x_1,x_2), \tag{90}$$

$\forall g\in C(O)$, $\forall c\in X$.

We further observe that $\widetilde{B}_{m,n}(1;x_1,x_2)=1$, $\forall(x_1,x_2)\in[0,1]^2$.

1) Let $f\in C^1(O,X)$, the space of once continuously Fréchet differentiable functions from O into X. Here we study approximation to $f|_{[0,1]^2}$. We choose $r=\frac{1}{2}$.

By giving the obvious probabilistic interpretation to $\widetilde{B}_{m,n}$, as well as using Schwarz's inequality one obtains (see (39), $n=1$)

$$\|(B_{m,n}(f))(x_1,x_2)-f(x_1,x_2)\|_\beta\le$$

$$\left\|\left(B_{m,n}\left(f^{(1)}(x_1,x_2)(\cdot-x_1,\cdot-x_2)\right)\right)(x_1,x_2)\right\|_\beta+$$

$$\frac{25}{32}\omega_1\left(f^{(1)},\frac{1}{4}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right)\right)\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right). \tag{91}$$

But by (24) and (45) we have that

$$\left\|\left(B_{m,n}\left(f^{(1)}\left(x_1,x_2\right)\left(\cdot-x_1,\cdot-x_2\right)\right)\right)\left(x_1,x_2\right)\right\|_{\beta}\leq$$

$$\left\|f^{(1)}\right\|\left(\widetilde{B}_{m,n}\left(\left\|\left(\cdot-x_1,\cdot-x_2\right)\right\|_{l_1}\right)\right)\left(x_0\right)\leq\left\|f^{(1)}\right\|\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right). \tag{92}$$

Consequently, it holds

$$\left\|\left(B_{m,n}(f)\right)\left(x_1,x_2\right)-f\left(x_1,x_2\right)\right\|_{\beta}\leq$$

$$\left[\frac{\left\|f^{(1)}\right\|}{2}+\frac{25}{32}\omega_1\left(f^{(1)},\frac{1}{4}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right)\right)\right]\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right), \tag{93}$$

$\forall\left(x_1,x_2\right)\in[0,1]^2$.

Therefore,

$$\left\|\left\|\left(B_{m,n}(f)\right)-f\right\|_{\beta}\right\|_{\infty,[0,1]^2}\leq$$

$$\left[\frac{\left\|f^{(1)}\right\|}{2}+\frac{25}{32}\omega_1\left(f^{(1)},\frac{1}{4}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right)\right)\right]\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right), \tag{94}$$

$\forall\, m,n\in\mathbb{N}$.

2) If we consider any $f\in C\left([0,1]^2,X\right)$, now without any differentiation assumption, then, by Corollary 3.11, we get similarly that

$$\left\|\left\|\left(B_{m,n}(f)\right)-f\right\|_{\beta}\right\|_{\infty,[0,1]^2}\leq 2\omega_1\left(f,\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right)\right),\quad\forall m,n\in\mathbb{N}. \tag{95}$$

3) Next we consider $x_0\in(0,1)^2$, where $x_0=\left(x_1,x_2\right)$. We have that the closed disk

$$B\left(x_0,\left(\widetilde{B}_{m,n}\left(\left\|\cdot-x_0\right\|_{l_1}\right)\right)\left(x_0\right)\right)\subseteq B\left(x_0,\left(\left(\widetilde{B}_{m,n}\left(\left\|\cdot-x_0\right\|_{l_2}^2\right)\right)\left(x_0\right)\right)^{\frac{1}{2}}\right)$$

$$\subseteq B\left(x_0,\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right)\right)\subseteq[0,1]^2, \tag{96}$$

for sufficiently large $m,n\in\mathbb{N}$.

Let $f\in C\left([0,1]^2,X\right)$ such that $\left\|f(t)-f\left(x_0\right)\right\|_{\beta}$ is convex in $t\in[0,1]^2$.

Then, by (73), we have

$$\left\|\left(B_{m,n}(f)\right)\left(x_1,x_2\right)-f\left(x_1,x_2\right)\right\|_{\beta}\leq\omega_1\left(f,\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{n}}\right)\right), \tag{97}$$

$\forall\, m,n\in\mathbb{N}$.

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