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Commentary

Ravindra K. Bisht*

Note on strict contractive conditions and common fixed point theorems with application

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Abstract: In this note we point out errors in the proofs of first two theorems contained in Demonstratio Mathematica, Vol. XLVI, no 2, 2013, 405-413. We also rectify the erratic theorems by employing a proper setting.

Keywords: coincidence point, strict contractive condition, noncompatible mappings.

MSC: 47H09, 47H10.

1 Introduction

In order to avert repetition, we follow the same terminology and the notations employed in [1] rather than presenting the same again.

The following theorems are essentially contained in [1].

Theorem 1. *Let f and g be conditionally commuting noncompatible self-mappings of a metric space (X, d) such that*

(i) $fX \in gX$,

(ii) $d(fx, fy) < max\{d(gx, gy), \frac{k}{2}[d(fx, gx) + d(fy, gy)], \frac{k}{2}[d(fy, gx) + d(fx, gy)]\}, 1 \leq k < 2.$

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point.

Theorem 2. *Let f and g be conditionally commuting self-mappings of a metric space (X, d) satisfying (i) and (ii). If f and g satisfy the property (E. A.) and the range of f or g is a complete subspace of X, then f and g have a unique common fixed point.*

2 Main results

The first error occurs in Lines 13 – 15 on Page 408 which claim that the inequality

$$d(fu, ffu) < max\{d(gu, gfu), \frac{k}{2}[d(fu, gu) + d(ffu, gfu)], \frac{k}{2}[d(ffu, gu) + d(fu, gfu)]\} < kd(fu, ffu),$$

***Corresponding Author: Ravindra K. Bisht:** Department of Mathematics, National Defence Academy, Khadakwasla, Pune, 411023, India, E-mail: ravindra.bisht@yahoo.com

leads to a contradiction. However, this inequality does not lead to a contradiction unless condition (ii) of Theorem 1 (above) is slightly modified. To get over this problem we can replace the condition (ii) of Theorem 1 by

$$(ii')\ d(fx, fy) < m(x, y) = max\{d(gx, gy), \frac{k}{2}[d(fx, gx) + d(fy, gy)], \frac{1}{2}[d(fx, gy) + d(fy, gx)]\},$$
$$1 \leq k < 2, \text{ whenever the right – hand side is positive.}$$

With the above modification the Theorem 1 can be restated as:

Theorem 3. *Let f and g be noncompatible self-mappings of a metric space (X, d) satisfying $(i)'$ $fX \subset gX$ and $(ii)'$.*

Then

$(iii)'$ f and g have a coincidence point $u \in X$;

$(iv)'$ f and g have a unique common fixed point in X, provided that f and g commute at u, i.e., $fgu = gfu$.

Proof. Using the same argument as in proof of Theorem 1 [1], we get $fu = gu$. This proves $(iii)'$. In view of $(iv)'$ we get $fgu = gfu = ffu = ggu$. Again, if $fu \neq ffu$ then by virtue of (ii') we obtain $d(fu, ffu) < m(u, fu) = d(fu, ffu)$, a contradiction. Hence fu is a common fixed point of f and g. Uniqueness of the common fixed point follows from (ii'). This establishes the theorem. □

A similar error is involved in Lines 19 – 22 on Page 409 and can be corrected using the same arguments as used in Theorem 3.

We now furnish an example [2] to demonstrate the validity of the hypotheses of the above theorem.

Example 1 Let $X = [2, 20]$ equipped with the Euclidean metric d. Define $f, g : X \to X$ by

$$fx = \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } 2 < x \leq 5, \\ 2 & \text{if } x > 5, \end{cases}$$

and

$$gx = \begin{cases} 2 & \text{if } x = 2, \\ 12 & \text{if } 2 < x \leq 5, \\ \frac{(x+1)}{3} & \text{if } x > 5. \end{cases}$$

Then f and g satisfy all the conditions of above theorem and have a common fixed point at $x = 2$. It can be verified in this example that

$d(fx, fy) = 6 - 2 = 4$ and $4 < m(x, y) \leq 10$ when $x = 2, 2 < y \leq 5$,

$d(fx, fy) = 2 - 2 = 0$ and $0 < m(x, y) \leq 5$ when $x = 2, y > 5$,

$d(fx, fy) = 6 - 6 = 0$ and $0 < m(x, y) \leq 3$ when $2 < x, y \leq 5$,

$d(fx, fy) = 6 - 2 = 4$ and $6 < m(x, y) < 7$ when $2 < x \leq 5, y > 5$,

$d(fx, fy) = 2 - 2 = 0$ and $0 < m(x, y) \leq \frac{40}{3}$ when $x, y > 5$.

Also, it is easy to observe that

(i) $fX = \{2, 6\}, g(X) = [2, 7] \bigcup 12$ and $f(X) \subset g(X)$;

(ii) f and g satisfy $(ii)'$ for $\frac{4}{3} < k < 2$;

(iii) f and g are non-compatible weakly compatible mappings.

Remark 1. We also point out that Theorems 1 and 2 are not real generalizations of the results of Pant and Pant [3, 4]. Since for a pair of single valued mappings, contractive condition excludes the possibility of more than one coincidence point or fixed point. For example, suppose that a pair (f, g) of self-mappings of a metric space (X, d) satisfies the contractive condition (ii') and suppose u and v are distinct coincidence points of f and g, i.e., $fu = gu$ and $fv = gv$ for $u \neq v$. Then $d(fu, fv) < m(u, v) = d(fu, fv)$, a contradiction. Hence under contractive condition $(ii)'$, conditional commutativity reduces to weak compatibility or point-wise R-weakly commuting and no real generalization is obtained by assuming conditional commutativity.

References

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