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Research Article

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Multiplication operators on the space of functions of bounded variation

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Abstract: In this paper, we study the properties of the multiplication operator acting on the bounded variation space $BV[0,1]$. In particular, we show the existence of non-null compact multiplication operators on $BV[0,1]$ and non-invertible Fredholm multiplication operators on $BV[0,1]$.

Keywords: Multiplication operator, Functions of bounded variation

MSC: Primary 26B30, 47B38; Secondary 46E40

1 Introduction

A function f is called a BV-function on $[0,1]$ if its total variation $V_0^1(f) < \infty$, where

$$V_0^1(f) = \sup_{P\in\Pi} \sum_{k=1}^{m_P} |f(t_k) - f(t_{k-1})|,$$

Π is the set of all partitions of $[0,1]$ and a partition $P = \{t_0, t_1, \cdots, t_{m_P}\}$ of $[0,1]$ is a finite and ordered subset of $[0,1]$ such that its first element is 0 and its last element is 1. The space of all bounded variation functions is denoted by $BV[0,1]$ and was introduced by Jordan in 1881 (see [1]), while he studied convergence of Fourier series. The space $BV[0,1]$ has been extensively studied and many generalizations of this concept have appeared recently. We refer the interested reader to the work of Appell, Banas and Merentes [2] for an updated study of this subject. In particular, $BV[0,1]$ is a Banach space with the norm

$$\|f\|_{BV[0,1]} = \|f\|_\infty + V_0^1(f),$$

where

$$\|f\|_\infty = \sup_{t\in[0,1]} \left\{ |f(t)| \right\}.$$

It is known that $BV[0,1]$ is a subspace of $B[0,1]$, the set of all bounded functions on $[0,1]$ and that a function $f \in BV[0,1]$ if and only if $f = f_1 - f_2$, where f_1, f_2 are increasing functions on $[0,1]$. Hence $BV[0,1]$ is a subspace of $L_0([0,1])$, the set of all measurable functions on $[0,1]$. However, unlike other spaces of measurable functions such as $L_p([0,1])$ spaces, Orlicz spaces or Lorentz spaces, $BV[0,1]$ is not a Köthe space (we refer to [3, 4] for definition and properties of Köthe spaces). For instance, there exists a measurable subset A of $[0,1]$ such that the characteristic function $\mathbf{1}_A$ does not belong to $BV[0,1]$. Also, the fact that $f(t) \le g(t)$ for all $t \in [0,1]$ and $g \in BV[0,1]$ does not imply that $f \in BV[0,1]$ neither than $\|f\|_{BV[0,1]} \le \|g\|_{BV[0,1]}$. Furthermore,

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the fact that $f = g$ almost everywhere and $g \in BV[0,1]$ does not imply that $f \in BV[0,1]$. Therefore, in the context of BV-functions, $f = g$ means that $f(t) = g(t)$ for all $t \in [0,1]$.

An important property enjoyed by the space $BV[0,1]$, which the $L_p([0,1])$ spaces do not have, is the fact that the product of two BV-functions is also a BV-function, that is, if $f, g \in BV[0,1]$ then $f \cdot g \in BV[0,1]$ and

$$\|f \cdot g\|_{BV[0,1]} \leq \|f\|_{BV[0,1]} \|g\|_{BV[0,1]}.$$

Hence, $BV[0,1]$ is a Banach algebra. Thus, if $u \in BV[0,1]$ is fixed then the multiplication operator M_u induced by u and defined by $M_u(f) = u \cdot f$ maps the space $BV[0,1]$ into itself and conversely, if the multiplication operator maps $BV[0,1]$ into itself, then, since the constant function $\mathbf{1} \in BV[0,1]$, we conclude that the function symbol u belongs to $BV[0,1]$. We can summarize this observation in the following proposition:

Proposition 1. *The multiplication operator M_u maps $BV[0,1]$ into itself if and only if $u \in BV[0,1]$. In this case, $\|M_u\| = \|u\|_{BV[0,1]}$.*

It is remarkable that this last property of the multiplication operator does not hold in Köthe spaces, in which this operator is continuous if and only if the symbol is an essentially bounded function (see [5]). The properties of the multiplication operator on measurable function spaces have been studied by numerous mathematicians. It is worth referring to the outstanding works of Abrahamese [6], Halmos [7], Takagi and Yokouchi [8] and Castillo, Rafeiro and Ramos-Ferández [9]. Recently, Castillo, Ramos-Fernández and Salas-Brown [5] made a very comprehensive study about the properties of M_u acting on Köthe spaces.

The main goal of this article is to make an exhaustive study of the properties of multiplication operator M_u acting on the space $BV[0,1]$. In Section 2, we characterize the symbols $u \in BV[0,1]$ inducing bounded below and invertible multiplication operators. In Section 3 we characterize the symbols $u \in BV[0,1]$ which induce multiplication operators with closed range. In Section 4, we characterize the finiteness range and the compactness of M_u in terms of the finiteness of certain subsets of the support of u. In Section 5, we show that Fredholm multiplication operators are the same as lower or upper semi-Fredholm multiplication operators and they are induced by symbols $u \in BV[0,1]$ which have a finite number of zero and are away from zero on their support. Finally, in Section 6 we calculate the spectrum and the spectral radius of M_u when acting on $BV[0,1]$ space.

2 Bounded below multiplication operators on $BV[0,1]$

The objective of this section is to characterize bounded below and invertible multiplication operators on $BV[0,1]$. We characterize the symbols $u \in BV[0,1]$ for which M_u is injective. We show that if M_u is onto on $BV[0,1]$ then is injective and we show that M_u is bounded below on $BV[0,1]$ if and only if the symbol u is away from zero on $[0,1]$, that is if there exists a $\delta > 0$ such that $|u(t)| > \delta$ for all $t \in [0,1]$. We recall that the support of a function u, denoted by $\operatorname{supp}(u)$, is defined as the set

$$\operatorname{supp}(u) = \{t \in [0,1] : |u(t)| > 0\}.$$

With this notation, we have the following result.

Proposition 2. *The operator $M_u : BV[0,1] \to BV[0,1]$ is injective if and only if* $\operatorname{supp}(u) = [0,1]$.

Proof. Suppose first that $\operatorname{supp}(u) \neq [0,1]$, then there is a $t_0 \in [0,1]$ such that $u(t_0) = 0$. We set the function

$$f(t) = \begin{cases} 1 & , \quad t = t_0 \\ 0 & , \quad \text{other case.} \end{cases}$$

Clearly, $f \in BV[0,1]$ since $V_0^1(f) \leq 2$. f is not the null function and furthermore $u(t)f(t) = 0$ for all $t \in [0,1]$. Hence $f \in \ker(M_u) \neq \{0\}$ and M_u is not injective on $BV[0,1]$.

Conversely, if $\text{supp}(u) = [0,1]$ and $f \in \ker(M_u)$, then $u(t)\cdot f(t) = 0$ for all $t \in [0,1]$ and hence $f(t) = 0$ for all $t \in [0,1]$. This shows that $\ker(M_u) = \{0\}$ and M_u is injective on $BV[0,1]$. □

It is very interesting that an onto multiplication operator on $BV[0,1]$ is also injective and therefore bijective such as we show in the following result.

Proposition 3. *If $M_u : BV[0,1] \to BV[0,1]$ is onto then is injective.*

Proof. Indeed, let us suppose that $M_u : BV[0,1] \to BV[0,1]$ is not injective. Then by Proposition 2 there exists a $t_0 \in [0,1]$ such that $u(t_0) = 0$. Thus, the function

$$f(t) = \begin{cases} 1 & , \quad t = t_0, \\ 0 & , \quad \text{other case}, \end{cases}$$

is of bounded variation on $[0,1]$. If $f \in \text{Ran}(M_u)$, the range of M_u, then there exists a function $h \in BV[0,1]$ such that $f(t) = u(t)\cdot h(t)$ for all $t \in [0,1]$. In particular, for $t = t_0$ we have $1 = f(t_0) = u(t_0)\,h(t_0) = 0$. Which give us a contradiction. Therefore $M_u : BV[0,1] \to BV[0,1]$ is not onto. □

From the above proposition it is clear that $M_u : BV[0,1] \to BV[0,1]$ is bijective if and only if this operator is onto. But we are able to give a better conclusion.

Theorem 4. *Suppose that $u \in BV[0,1]$. $M_u : BV[0,1] \to BV[0,1]$ is bijective (with continuous inverse) if and only if there exists a $\delta > 0$ such that $|u(t)| > \delta$ for all $t \in [0,1]$.*

Proof. Let us suppose first that $M_u : BV[0,1] \to BV[0,1]$ is bijective, then there exists a linear operator $T : BV[0,1] \to BV[0,1]$ such that $M_u \circ T = T \circ M_u = I$, the identity operator on $BV[0,1]$. Thus, for each $f \in BV[0,1]$, we have $M_u \circ T(f) = M_u(Tf) = u \cdot Tf = f$ and since M_u is 1-1, $u(t) \neq 0$ for all $t \in [0,1]$. It follows that $Tf = \frac{f}{u} = M_{\frac{1}{u}}(f)$ for all $f \in BV[0,1]$. In particular, since the constant function $\mathbf{1} \in BV[0,1]$, we conclude that $\frac{1}{u} \in BV[0,1]$ and $T = M_{\frac{1}{u}}$ is continuous on $BV[0,1]$. Furthermore, from the fact that $BV[0,1] \subset B[0,1]$ we can see that there exists $M > 0$ such that $\left|\frac{1}{u(t)}\right| < M$ for all $t \in [0,1]$. Therefore, if we set $\delta = \frac{1}{M}$, then we obtain that $|u(t)| > \delta$ for all $t \in [0,1]$.

Conversely, if there exists a $\delta > 0$ such that $|u(t)| > \delta$ for all $t \in [0,1]$. Since $u \in BV[0,1]$, it is easy to see that $\frac{1}{u} \in BV[0,1]$. Thus the operator $M_{\frac{1}{u}}$ is continuous on $BV[0,1]$, $M_u \circ M_{\frac{1}{u}} = M_{\frac{1}{u}} \circ M_u = I$ and therefore the operator $M_u : BV[0,1] \to BV[0,1]$ is bijective with continuous inverse. □

For our next result, we recall that a linear operator $T : X \to X$, where X is a Banach space, is said *bounded below* if there exists a constant $L > 0$ such that $\|Tf\| \geq L\|f\|$ for all $f \in X$. It is well known that an operator $T : X \to X$ is bounded below if and only if $T : X \to X$ is 1-1 and it has closed range. In our case of multiplication operator acting on $BV[0,1]$ space, we have the following result:

Theorem 5. *The following statements are equivalent:*
(1) $M_u : BV[0,1] \to BV[0,1]$ *is bijective (with continuous inverse),*
(2) $\text{Ran}(M_u) = BV[0,1]$,
(3) $M_u : BV[0,1] \to BV[0,1]$ *is bounded below,*
(4) $\inf_{t\in[0,1]}(|u(t)|) > 0$.

Proof. According to the Propositions 3 and 4, and since all bijective operators with continuous inverse are bounded below, then it is enough to show that (3) implies (4). Indeed, if $\inf_{t\in[0,1]}(|u(t)|) = 0$, then for each $n \in \mathbb{N}$ we can find a $t_n \in [0,1]$ such that $0 \leq |u(t_n)| < \frac{1}{n}$. Thus, the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, defined by

$$f_n(t) = \begin{cases} 1 & , \quad t = t_n, \\ 0 & , \quad t \neq t_n \end{cases}$$

are of bounded variation on $[0, 1]$. Furthermore, we have $2 \le \|f_n\|_{BV[0,1]} \le 3$ for all $n \in \mathbb{N}$. Also, we can see that $V_0^1(u \cdot f_n) \le 2\left|u(t_n)\right|$ for all $n \in \mathbb{N}$ and therefore

$$\|u \cdot f_n\|_{BV[0,1]} \quad \le \quad 3\left|u(t_n)\right| \le \frac{3}{2n}\|f_n\|_{BV[0,1]},$$

for all $n \in \mathbb{N}$. This means that $M_u : BV[0, 1] \to BV[0, 1]$ is not bounded below. The proof is complete. □

3 Some M_u-invariants subspaces of $BV[0, 1]$ and multiplication operator with closed range on $BV[0, 1]$

The aim of this section is to characterize the symbols $u \in BV[0, 1]$ which induce multiplication operator M_u with closed range on $BV[0, 1]$. The key of our result lies in considering the following set

$$X_{\mathbf{Z}_u} \quad = \quad \{f \in BV[0,1] : f(t) = 0 \, \forall t \in \mathbf{Z}_u\},$$

where from now, $\mathbf{Z}_u$ denotes the set of all zeros of the function u, that is, $\mathbf{Z}_u = \left\{t \in [0, 1] : u(t) = 0\right\}$.

Proposition 6. *If $\mathbf{Z}_u \neq \emptyset$ then the set $X_{\mathbf{Z}_u}$ is a proper closed subspace of $BV[0, 1]$ which is absorbent ($f \in X_{\mathbf{Z}_u}$ and $g \in BV[0, 1]$ implies that $f \cdot g \in X_{\mathbf{Z}_u}$) and M_u-invariant, that is, $M_u\left(X_{\mathbf{Z}_u}\right) \subset X_{\mathbf{Z}_u}$. Furthermore* $\operatorname{Ran}(M_u) \subset X_{\mathbf{Z}_u}$.

Proof. Since $\mathbf{Z}_u \neq \emptyset$, the set $X_{\mathbf{Z}_u}$ is not empty because this set has the null function. Clearly, $X_{\mathbf{Z}_u}$ is an absorbent and proper subspace of $BV[0, 1]$, since the non-null constant functions belong to $BV[0, 1] \setminus X_{\mathbf{Z}_u}$. Also, if $f \in X_{\mathbf{Z}_u}$ then $h = M_u f = u \cdot f \in BV[0, 1]$, because $u \in BV[0, 1]$, and satisfies $h(t) = u(t) \cdot f(t) = 0$ for all $t \in \mathbf{Z}_u$. Thus, $\operatorname{Ran}(M_u) \subset X_{\mathbf{Z}_u}$ and $X_{\mathbf{Z}_u}$ is M_u-invariant. Finally, if $f \in \overline{X_{\mathbf{Z}_u}}$, then there exists a sequence $\{f_n\} \subset X_{\mathbf{Z}_u}$ such that $\|f_n - f\|_{BV[0,1]} \to 0$ as $n \to \infty$. Hence $\|f_n - f\|_\infty \to 0$ as $n \to \infty$ which implies that $\left|f_n(t) - f(t)\right| \to 0$ as $n \to \infty$ for all $t \in \mathbf{Z}_u$. That is, $f \in X_{\mathbf{Z}_u}$. □

In the next result we give a condition in order to $\operatorname{Ran}(M_u) = X_{\mathbf{Z}_u}$. This will be useful in the proof of the main result of this section (Theorem 8).

Lemma 7. *If there exists a $\delta > 0$ such that $\left|u(t)\right| \ge \delta$ for all $t \in \operatorname{supp}(u)$ then* $\operatorname{Ran}(M_u) = X_{\mathbf{Z}_u}$.

Proof. Indeed, it is enough to show that $X_{\mathbf{Z}_u} \subset \operatorname{Ran}(M_u)$. Suppose that $f \in X_{\mathbf{Z}_u}$ and we define the function

$$g(t) \quad = \quad \begin{cases} \dfrac{f(t)}{u(t)} & , \quad \text{si } t \notin \mathbf{Z}_u \\ \\ 0 & , \quad \text{other case.} \end{cases}$$

Clearly $f = u \cdot g$ and hence we only have to show that $g \in BV[0, 1]$. To see this last, let $P : 0 = t_0 < t_1 < \cdots < t_n = 1$ be any partition of $[0, 1]$, then we have the following cases:

Case I: If $t_k, t_{k-1} \in \mathbf{Z}_u$ then

$$|g(t_k) - g(t_{k-1})| \quad = \quad 0 = |f(t_k) - f(t_{k-1})|$$

Case II: If $t_k, t_{k-1} \notin \mathbf{Z}_u$ then

$$\begin{aligned} |g(t_k) - g(t_{k-1})| &= \left|\frac{f(t_k)}{u(t_k)} - \frac{f(t_{k-1})}{u(t_{k-1})}\right| \le \frac{|u(t_{k-1}) f(t_k) - u(t_k) f(t_{k-1})|}{\delta^2} \\ &\le \frac{\|u\|_\infty}{\delta^2}|f(t_k) - f(t_{k-1})| + \frac{\|f\|_\infty}{\delta^2}|u(t_k) - u(t_{k-1})| \end{aligned}$$

Case III: If $t_k \in \mathbf{Z}_u$ and $t_{k-1} \notin \mathbf{Z}_u$ then

$$|g(t_k) - g(t_{k-1})| \quad = \quad \left|\frac{f(t_{k-1})}{u(t_{k-1})}\right| \le \frac{|f(t_{k-1})|}{\delta} = \frac{|f(t_k) - f(t_{k-1})|}{\delta}$$

Case IV: If $t_k \notin \mathbf{Z}_u$ and $t_{k-1} \in \mathbf{Z}_u$ then

$$|g(t_k) - g(t_{k-1})| = \left|\frac{f(t_k)}{u(t_k)}\right| \le \frac{|f(t_k)|}{\delta} = \frac{|f(t_k) - f(t_{k-1})|}{\delta}$$

Thus, we have

$$\begin{aligned}\sum_{k=1}^{n} |g(t_k) - g(t_{k-1})| &\le \sum_{\mathbf{II}} \left(\frac{\|u\|_\infty}{\delta^2} |f(t_k) - f(t_{k-1})| + \frac{\|f\|_\infty}{\delta^2} |u(t_k) - u(t_{k-1})| \right) \\ &\quad + \sum_{\mathbf{III}} \frac{|f(t_k) - f(t_{k-1})|}{\delta} + \sum_{\mathbf{IV}} \frac{|f(t_k) - f(t_{k-1})|}{\delta} \\ &\le \left(\frac{1}{\delta} + \frac{\|u\|_\infty}{\delta^2} \right) \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| + \frac{\|f\|_\infty}{\delta^2} \sum_{k=1}^{n} |u(t_k) - u(t_{k-1})| \\ &\le \left(\frac{1}{\delta} + \frac{\|u\|_\infty}{\delta^2} \right) V_0^1(f) + \frac{\|f\|_\infty}{\delta^2} V_0^1(u) < \infty\end{aligned}$$

since $u, f \in BV[0,1]$. This shows that $g \in BV[0,1]$ and the proof of lemma is now completes. □

Now we can enunciate and show the main result of this section.

Theorem 8. *The operator $M_u : BV[0,1] \to BV[0,1]$ has closed range if and only if there exists a $\delta > 0$ such that $|u(t)| \ge \delta$ for all $t \in \operatorname{supp}(u)$.*

Proof. By Lemma 7, the condition that there exists a $\delta > 0$ such that $|u(t)| \ge \delta$ for all $t \in \operatorname{supp}(u)$ implies that $\operatorname{Ran}(M_u) = X_{\mathbf{Z}_u}$ which is a closed subspace of $BV[0,1]$ by Proposition 6.

Next, let us suppose that the operator $M_u : BV[0,1] \to BV[0,1]$ has closed range and that the conclusion is false, then for each $n \in \mathbb{N}$ we can find a $t_n \in \operatorname{supp}(u)$ such that

$$0 < |u(t_n)| < \frac{1}{n^2}.$$

In particular, we have that $u(t_n) \to 0$ as $n \to \infty$ and $\{t_n : n \in \mathbb{N}\}$ is an infinite set. For each $n \in \mathbb{N}$, we define the set $A_{2n+1} = \{t_1, t_3, t_5, \cdots, t_{2n+1}\}$, then the function

$$h_n(t) = \begin{cases} u(t) & , \quad t \in A_{2n+1}, \\ 0 & , \quad \text{other case} \end{cases}$$

is of bounded variation on $[0,1]$ since A_{2n+1} is a finite set. Furthermore, $h_n \in \operatorname{Ran}(M_u)$ since $h_n = u \cdot \mathbf{1}_{A_{2n+1}}$. We go to show that $\{h_n\}$ is a Cauchy sequence in $\operatorname{Ran}(M_u)$. Indeed, if $n, m \in \mathbb{N}$ and we suppose that $n > m$, then we have

$$(h_n - h_m)(t) = \begin{cases} u(t) & , \quad \text{if } t \in \{t_{2m+3}, \cdots, t_{2n+1}\}, \\ 0 & , \quad \text{other case.} \end{cases}$$

Hence,

$$\begin{aligned}\|h_n - h_m\|_{BV[0,1]} &\le 2 \sum_{k=2m+3}^{2n+1} |u(t_k)| \le 2 \sum_{k=2m+3}^{\infty} |u(t_k)| \\ &\le 2 \sum_{k=2m+3}^{\infty} \frac{1}{k^2} \to 0 \text{ as } m \to \infty\end{aligned}$$

which shows the affirmed. Now, since $\operatorname{Ran}(M_u)$ is a closed set of $BV[0,1]$, there exists a function $h \in \operatorname{Ran}(M_u)$ such that $\|h_n - h\|_{BV[0,1]} \to 0$ as $n \to \infty$. Furthermore, there exists a function $f \in BV[0,1]$ such that $h = u \cdot f$ and hence $\left\|u \cdot \left(\mathbf{1}_{A_{2n+1}} - f\right)\right\|_\infty \to 0$ as $n \to \infty$. In particular, for each $t \in A_{2n+1} \subset \operatorname{supp}(u)$, we have

$$\left|u(t)\left(\mathbf{1}_{A_{2n+1}} - f\right)(t)\right| \to 0 \text{ as } n \to \infty$$

and we conclude that the function f must satisfy

$$f(t) = \begin{cases} 1 & , \quad \text{if } t \in A_{2n+1}, \\ \\ 0 & , \quad \text{if } t \in \text{supp}(u) \setminus A_{2n+1}. \end{cases}$$

But, if we consider, for each $n \in \mathbb{N}$, the partition $P_n = \{0, t_1, t_2, \cdots, t_n, 1\}$, then we have

$$V_0^1(f) \geq \sum_{k=2}^{n} |f(t_k) - f(t_{k-1})| = \sum_{k=2}^{n} 1 = n - 1.$$

Which implies that the function f is not an element of $BV[0,1]$ and we get a contradiction. Therefore, we conclude that there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \text{supp}(u)$. □

4 Finite range and compactness

Recall that if X is a Banach space, an operator $T : X \to X$ is said to have finite range if $\dim(\text{Ran}(T)) < \infty$ and it is compact if $\{Tx_n\}$ has a convergent subsequence for all bounded sequence $\{x_n\} \subset X$. In this section, we characterize all the symbols $u \in BV[0,1]$ which induce multiplication operator M_u with finite range on $BV[0,1]$. Also we characterize the compactness of $M_u : BV[0,1] \to BV[0,1]$ in terms of the supp(u).

Theorem 9. *The operator $M_u : BV[0,1] \to BV[0,1]$ has finite range if and only if* supp(u) *is a finite set.*

Proof. Suppose first that supp(u) is an infinite set. There exists a sequence $\{t_n\} \subset \text{supp}(u)$ such that $t_i \neq t_j$ for all $i \neq j$. In particular, $u(t_n) \neq 0$ for all $n \in \mathbb{N}$. We set the functions

$$h_n(t) = \begin{cases} u(t) & , \quad t = t_n, \\ 0 & , \quad t \neq t_n. \end{cases}$$

Clearly, $h_n \in \text{Ran}(M_u)$ since $h_n = u \cdot f_n$, where

$$f_n(t) = \begin{cases} 1 & , \quad t = t_n, \\ 0 & , \quad t \neq t_n \end{cases}$$

and $V_0^1(f_n) \leq 2$. Furthermore, if $\alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_m}$ are scalars and we suppose that

$$\sum_{k=1}^{m} \alpha_{n_k} h_{n_k} = 0,$$

then, by evaluating at $t = t_{n_j}$, we have $\alpha_{n_j} h_{n_j}(t_{n_j}) = \alpha_{n_j} u(t_{n_j}) = 0$, which implies that $\alpha_{n_j} = 0$. This means that the infinite set $\{h_n\}_{n\in\mathbb{N}} \subset \text{Ran}(M_u)$ is linearly independent and $\dim(\text{Ran}(M_u)) = \infty$.

Conversely, if supp(u) is a finite set, then we can write $\text{supp}(u) = \{t_1, \cdots, t_m\}$, with $t_k \in [0,1]$ for all $k = 1, 2, \cdots, m$. For each $n \in \{1, \ldots, m\}$, we set the function

$$h_n(t) = \begin{cases} u(t) & , \quad t = t_n, \\ 0 & , \quad t \neq t_n. \end{cases}$$

We affirm that the set $H = \{h_1, h_2, \cdots, h_m\}$ is a basis for $\text{Ran}(M_u)$. Indeed, clearly $h_n \in BV[0,1]$ for all $n = 1, \cdots, m$ and that $\{h_1, h_2, \cdots, h_m\}$ is a linearly independent set, hence it is enough to show that each $f \in \text{Ran}(M_u)$ is a linear combination of H. We observe that there exists $g \in BV[0,1]$ such that $M_u g = u \cdot g = f$, thus for each $k \in \{1, 2, \ldots, m\}$ we can set the scalar $\alpha_k = g(t_k)$ and then for $t = t_j$, we have

$$f(t_j) = u(t_j) \cdot g(t_j) = u(t_j) \cdot \alpha_j = \alpha_j \cdot h_j(t) = \sum_{k=1}^{m} \alpha_k h_k(t).$$

While if $t \notin \operatorname{supp}(u)$ then $h_k(t) = 0$ for all $k = 1, 2, \ldots, m$ and we can write

$$f(t) = u(t) \cdot g(t) = 0 = \alpha_j \cdot 0 = \sum_{k=1}^{m} \alpha_k h_k(t).$$

Therefore, $f(t) = \sum_{k=1}^{m} \alpha_k h_k(t)$ for all $t \in [0, 1]$ and H is a basis for Ran (M_u). This shows that dim (Ran (M_u)) = $m < \infty$. □

Now, we are going to characterize the compactness of $M_u : BV[0, 1] \to BV[0, 1]$. Our characterization is given in terms of the finiteness of certain subsets of supp(u) which we define below. For $\epsilon > 0$ given we set

$$E_\epsilon = \left\{ t \in [0, 1] : |u(t)| \geq \epsilon \right\}.$$

Associated to the set E_ϵ we have the following subspace of $BV[0, 1]$:

$$X_{E_\epsilon} = \left\{ f \in BV[0, 1] : f(t) = 0 \ \ \forall t \in [0, 1] \setminus E_\epsilon \right\},$$

which is a closed and M_u-invariant subspace of $BV[0, 1]$. Our result now can be enunciated as follows:

Theorem 10. *Suppose that $u \in BV[0, 1]$. The following statements are equivalents:*
1. *The operator $M_u : BV[0, 1] \to BV[0, 1]$ is compact,*
2. dim $\left(X_{E_\epsilon}\right) < \infty$ *for all $\epsilon > 0$,*
3. *E_ϵ is a finite set for all $\epsilon > 0$.*

Proof. (1)⇒(2): Suppose that $M_u : BV[0, 1] \to BV[0, 1]$ is a compact operator and let $\epsilon > 0$ be given. Since X_{E_ϵ} is a closed subspace of $BV[0, 1]$, the inclusion operator $i_{E_\epsilon} : X_{E_\epsilon} \to BV[0, 1]$ given by $i_{E_\epsilon} f = f$ is continuous and hence the composition $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to BV[0, 1]$ is a compact operator. We affirm that Ran $\left(M_u \circ i_{E_\epsilon}\right) = X_{E_\epsilon}$. Indeed, clearly Ran $\left(M_u \circ i_{E_\epsilon}\right) \subset X_{E_\epsilon}$, while if $f \in X_{E_\epsilon}$ we can define the function $h : [0, 1] \to \mathbb{R}$ by

$$h(t) = \begin{cases} \dfrac{f(t)}{u(t)} & , \quad t \in E_\epsilon, \\ \\ 0 & , \quad \text{other case.} \end{cases}$$

Then if $P = \{t_0, t_1, \cdots, t_n\}$ is any partition of $[0, 1]$ and by considering the cases $t_k, t_{k-1} \in E_\epsilon$, $t_k \in E_\epsilon$ and $t_{k-1} \notin E_\epsilon$ and $t_k, t_{k-1} \notin E_\epsilon$ we obtain that

$$\sum_{k=1}^{n} |h(t_k) - h(t_{k-1})| \leq \left(\frac{1}{\epsilon} + \frac{\|u\|_\infty}{\epsilon^2} \right) V_0^1(f) + \frac{\|f\|_\infty}{\epsilon^2} V_0^1(u) < \infty$$

since $u, f \in BV[0, 1]$. Hence $h \in BV[0, 1]$ and h belongs to X_{E_ϵ}. Furthermore, we also have $(M_u \circ i_{E_\epsilon})h = M_u(i_{E_\epsilon} h) = M_u h = u \cdot h = f$ and $f \in$ Ran $\left(M_u \circ i_{E_\epsilon}\right)$. This shows the affirmed and $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to X_{E_\epsilon}$ is onto.

Now, we will show that $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to X_{E_\epsilon}$ is also injective. Indeed, if $f \in$ Ker $\left(M_u \circ i_{E_\epsilon}\right)$ then $u \cdot f = 0$ and hence $f(t) = 0$ for all $t \in [0, 1]$ since $f \in X_{E_\epsilon}$. Thus, the operator $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to X_{E_\epsilon}$ is bijective and compact which implies that dim $\left(X_{E_\epsilon}\right) < \infty$ since it is a known fact the identity operator $\mathbf{I} : X \to X$ is compact if and only if dim$(X) < \infty$.

(2)⇒(3): Suppose that E_ϵ is infinite for some $\epsilon > 0$. Then there exists a sequence $\{t_n\} \in E_\epsilon$ such that $t_i \neq t_j$ for $i \neq j$. Thus, for each $n \in \mathbb{N}$, we can define the function

$$f_n(t) = \begin{cases} 1 & , \quad t = t_n, \\ 0 & , \quad t \neq t_n. \end{cases}$$

Clearly, $f_n \in X_{E_\epsilon}$ for all $n \in \mathbb{N}$. Furthermore, if $\alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_m}$ are scalars and we suppose that

$$\sum_{k=1}^{m} \alpha_{n_k} f_{n_k} = 0,$$

then by evaluating at $t = t_{n_j}$ we conclude that $\alpha_{n_j} = 0$ and the set $\{f_n\}_{n\in\mathbb{N}} \subset X_{E_\epsilon}$ is linearly independent. This means that $\dim\left(X_{E_\epsilon}\right) = \infty$.

(3)⇒(1): Suppose now that E_ϵ is finite for all $\epsilon > 0$. Observe that

$$\operatorname{supp}(u) = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} = \bigcup_{n=1}^{\infty} \left\{ t \in [0, 1] : |u(t)| \geq \frac{1}{n} \right\}.$$

Since $E_{\frac{1}{n}}$ is finite for all $n \in \mathbb{N}$, we can deduce that $\operatorname{supp}(u)$ is a countable set. If $\operatorname{supp}(u)$ is finite then Theorem 9 implies that $M_u : BV[0, 1] \to BV[0, 1]$ has finite range and therefore is compact since it is a known fact that all operators with finite range are compact. In the case that $\operatorname{supp}(u)$ is an infinite set, then we can write

$$\operatorname{supp}(u) = \{t_1, t_2, \cdots, t_n, \cdots\} \subset [0, 1],$$

where $t_i \neq t_j$ for $i \neq j$. Hence, we have

$$\sum_{k=1}^{\infty} |u(t_k)| \leq V_0^1(u) \leq 2 \sum_{k=1}^{\infty} |u(t_k)|,$$

and since $V_0^1(u) < \infty$, we conclude that the numerical series $\sum_{k=1}^{\infty} |u(t_k)|$ converges absolutely. Thus, any rearrangement of $\{t_n\}_{n\in\mathbb{N}}$ does not affect the value of this series. Now, for each $n \in \mathbb{N}$, we consider the set $E_n = \{t_1, t_2, \cdots, t_n\}$ and we define the function

$$u_n(t) = u(t) \cdot \mathbf{1}_{E_n}(t) = \begin{cases} u(t) & , \quad t \in E_n, \\ 0 & , \quad t \in [0, 1] \setminus E_n. \end{cases}$$

Then $u_n \in BV[0, 1]$ for all $n \in \mathbb{N}$ since each E_n is finite and by Theorem 9, the operator $M_{u_n} : BV[0, 1] \to BV[0, 1]$ has finite range for each $n \in \mathbb{N}$ and, in particular, they are compact operators.

Observe that for each $n \in \mathbb{N}$, the operator $M_{u_n - u} : BV[0, 1] \to BV[0, 1]$ is continuous since $u_n, u \in BV[0, 1]$. Next, we will prove that

$$\|M_{u_n} - M_u\| = \|M_{u_n - u}\| = \|u_n - u\|_{BV[0,1]} \to 0$$

as $n \to \infty$. Indeed, for each $n \in \mathbb{N}$, we have

$$u_n(t) - u(t) = \begin{cases} -u(t) & , \quad \text{if } t \in \{t_{n+1}, t_{n+2}, \cdots\}, \\ 0 & , \quad \text{other case}. \end{cases}$$

Hence, we obtain

$$V_0^1\left(u_n - u\right) \leq 2 \sum_{k=n+1}^{\infty} |u(t_k)| \to 0$$

as $n \to \infty$, since the series $\sum_{k=1}^{\infty} |u(t_k)|$ is convergent. Thus, using the fact that

$$\|u_n - u\|_\infty \leq \left|(u_n - u)(t_1)\right| + V_0^1\left(u_n - u\right),$$

we conclude that

$$\|M_{u_n} - M_u\| = \|M_{u_n - u}\| = \|u_n - u\|_{BV[0,1]} \to 0$$

as $n \to \infty$. This means that $M_u : BV[0, 1] \to BV[0, 1]$ is the limit of operators having finite range and therefore it must be a compact operator. This shows the result. □

5 Fredholm multiplication operators on $BV[0, 1]$

Let X be a Banach space and let $T : X \to X$ be a continuous operator. The operator T is said *upper semi-Fredholm* if it has finite dimensional kernel and $\mathrm{Ran}(T)$ is a closed subspace of X. T is said *lower semi-Fredholm* if $\mathrm{codim}\,(\mathrm{Ran}\,(T)) = \dim\left(X/\mathrm{Ran}\,(T)\right) < \infty$. It is known that the condition $\mathrm{codim}\,(\mathrm{Ran}\,(T)) < \infty$ implies that $\mathrm{Ran}\,(T)$ is a closed subspace of X. An operator T is called *Fredholm* if it is lower and upper semi-Fredholm. Fredholm multiplication operators have been studied in L_p spaces by Jabbarzadeh and Pourreza [10], in Orlicz spaces by Komal and Gupta [11] and more generally, in Köthe spaces by Castillo, Ramos-Fernández and Salas-Brown [5]. It is remarkable that in those spaces, for non-atomic measures, Fredholm multiplication operators are the same as invertible multiplication operators. In the case of multiplication operators acting on $BV[0, 1]$ space we have the following result:

Theorem 11. *Suppose that $u \in BV[0, 1]$. The following statements are equivalents:*
1. *The operator $M_u : BV[0, 1] \to BV[0, 1]$ is Fredholm,*
2. *the operator $M_u : BV[0, 1] \to BV[0, 1]$ is upper semi-Fredholm,*
3. *the operator $M_u : BV[0, 1] \to BV[0, 1]$ is lower semi-Fredholm,*
4. *$\mathbf{Z}_u$ is a finite set and there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \mathrm{supp}(u)$.*

Proof. Is is enough to show that (2)⇒(4)⇒(3). Suppose first (2), that is, the operator $M_u : BV[0, 1] \to BV[0, 1]$ is upper semi-Fredholm, then by definition, $\dim\,(\mathrm{Ker}\,(M_u)) < \infty$ and $\mathrm{Ran}\,(M_u)$ is a closed subspace of $BV[0, 1]$. By Theorem 8 it follows that there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \mathrm{supp}(u)$, hence it is enough to show that $\mathbf{Z}_u$ is a finite set.

If $\mathbf{Z}_u$ is an infinite set, then we can find a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbf{Z}_u$ such that $t_i \neq t_j$ for $i \neq j$. Thus for each $n \in \mathbb{N}$, we can define the function

$$f_n(t) = \begin{cases} 1 & , \quad t = t_n, \\ 0 & , \quad \text{other case.} \end{cases}$$

Clearly $\{f_n : n \in \mathbb{N}\}$ is an infinite linearly independent set contained into $\mathrm{Ker}\,(M_u)$, which is a contradiction to the fact that $\dim\,(\mathrm{Ker}\,(M_u)) < \infty$. Therefore, $\mathbf{Z}_u$ is a finite set. This shows the implication (2)⇒(4).

(4)⇒(3): Suppose that $\mathbf{Z}_u$ is a finite set, we say $\mathbf{Z}_u = \{t_1, t_2, \cdots, t_m\}$, and that there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \mathrm{supp}(u)$. We are going to show that $\dim\left(BV[0, 1]/\mathrm{Ran}\,(M_u)\right) < \infty$. Recall that

$$BV[0, 1]/\mathrm{Ran}\,(M_u) = \left\{[f] := f + \mathrm{Ran}\,(M_u) : f \in BV[0, 1]\right\},$$

$[f] = [g]$ if and only if $f - g \in \mathrm{Ran}\,(M_u)$ and $h \in [f]$ if and only if $h - f \in \mathrm{Ran}\,(M_u)$.

For each $k \in \{1, 2, \ldots, m\}$ we set the function

$$f_k(t) = \begin{cases} 1 & , \quad t = t_k, \\ 0 & , \quad \text{other case.} \end{cases}$$

We are about to show that $B = \{[f_k] : k = 1, 2, \ldots, m\}$ is a basis for space $BV[0, 1]/\mathrm{Ran}\,(M_u)$. Indeed, if $\alpha_1, \alpha_2, \cdots, \alpha_m$ are scalars and

$$\alpha_1\,[f_1] + \alpha_2\,[f_2] + \cdots + \alpha_m\,[f_m] = [0],$$

then there exists a function $h \in BV[0, 1]$ such that $\alpha_1 f_1 + \alpha_2 f_2 + \cdots, \alpha_m f_m = u \cdot h$. In particular, evaluating at $t = t_k \in \mathbf{Z}_u$ we conclude that $\alpha_k = 0$ for all $k = 1, 2, \ldots, m$ and B is a linearly independent set of $BV[0, 1]/\mathrm{Ran}\,(M_u)$.

Next, we are about to show that the function $g \in BV[0, 1]/\mathrm{Ran}\,(M_u)$ is a linear combination of the vectors in B. Let g be any function in $BV[0, 1]$, then $[g] \in BV[0, 1]/\mathrm{Ran}\,(M_u)$. Thus if we consider the scalars $\alpha_k =$

$g(t_k)$, with $t_k \in \mathbf{Z}_u$ and we set the function

$$h(t) = \begin{cases} \frac{g(t)}{u(t)} & , \quad t \notin \mathbf{Z}_u, \\ 0 & , \quad \text{other case}, \end{cases}$$

then, the hypothesis that there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \text{supp}(u)$ implies that the function h belongs to $BV[0,1]$. Furthermore, for $t \in \text{supp}(u)$ we have

$$g(t) - \sum_{k=1}^{m} \alpha_k f_k(t) = g(t) = u(t) \cdot h(t),$$

while if $t \in \mathbf{Z}_u$, then $t = t_j$ for some $j = 1, 2, \dots, m$ then

$$g(t) - \sum_{k=1}^{m} \alpha_k f_k(t) = g(t_j) - \alpha_j f_j(t_j) = \alpha_j - \alpha_j = 0 = u(t) \cdot h(t).$$

Hence $g - \sum_{k=1}^{m} \alpha_k f_k \in \text{Ran}(M_u)$ and $[g] = \sum_{k=1}^{m} \alpha_k [f_k]$. Therefore, B is a basis for $BV[0,1]/\text{Ran}(M_u)$ and $\dim\left(BV[0,1]/\text{Ran}(M_u)\right) < \infty$. This shows the result. □

As an immediate consequence of the above result, we have:

Corollary 12. *Suppose that* $u \in BV[0,1]$.
1. *The operator* $M_u : BV[0,1] \to BV[0,1]$ *is Fredholm if and only if* $\mathbf{Z}_u$ *is a finite set and* Ran (M_u) *is a closed subspace of* $BV[0,1]$.
2. *The operator* $M_u : BV[0,1] \to BV[0,1]$ *is invertible (with continuous inverse) if and only if* codim (Ran $(M_u)) < \infty$ *and* supp$(u) = [0,1]$.

Proof. The statement (1) is a consequence of Theorem 11 and Theorem 8. It is enough to show (2). If the operator $M_u : BV[0,1] \to BV[0,1]$ is invertible, then by Theorem 4 we have that $\text{supp}(u) = [0,1]$ and $\text{codim}(\text{Ran}(M_u)) = \dim\left(BV[0,1]/BV[0,1]\right) = 1 < \infty$. While if $\text{codim}(\text{Ran}(M_u)) < \infty$ and $\text{supp}(u) = [0,1]$ then the operator $M_u : BV[0,1] \to BV[0,1]$ is lower semi-Fredholm and by Theorem 11, there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \text{supp}(u) = [0,1]$. Therefore, Theorem 4 tells us that the operator $M_u : BV[0,1] \to BV[0,1]$ is invertible (with continuous inverse). □

6 On the spectrum and the spectral radius of $M_u : BV[0,1] \to BV[0,1]$

The results obtained for us in the sections above give us a powerful tool to build examples of operators with prescribed properties. For example, if we wish to build an unbounded operator, we only must consider a multiplication operator on $BV[0,1]$ whose symbol does not belong to $BV[0,1]$. If we need a Fredholm no invertible operator, then we consider a symbol $u \in BV[0,1]$ with at least one zero and away from the zero on supp(u), for example, we can consider the function

$$u(t) = \begin{cases} 0 & , \quad t = \frac{1}{2}, \\ 1 & , \quad t \neq \frac{1}{2}. \end{cases}$$

Also we can use our results in the above sections to obtain other properties of $M_u : BV[0,1] \to BV[0,1]$ with $u \in BV[0,1]$. For instance, in this section, we utilize Theorem 4 to calculate the spectrum and the spectral radius of M_u. We recall that the spectrum of M_u, denoted by $\sigma(M_u)$, is defined by

$$\sigma(M_u) = \{\lambda \in \mathbb{C} : M_u - \lambda I \text{ is not invertible}\}.$$

The elements of $\sigma(M_u)$ are known as eigenvalues. The spectral radius of M_u, denoted by $r(M_u)$, is defined as

$$r(M_u) = \sup\{|\lambda| : \lambda \in \sigma(M_u)\}.$$

It is known that $r(M_u) \le \|M_u\| = \|u\|_{BV[0,1]}$, but in our case, we have the following result:

Theorem 13. *Suppose that $u \in BV[0,1]$, then $\sigma(M_u) = \overline{u([0,1])}$ and hence $r(M_u) = \|u\|_\infty$.*

Proof. Indeed, observe that for any $\lambda \in \mathbb{C}$, the operator $M_u - \lambda I = M_{u-\lambda}$, that is, it is a multiplication operator with symbol $u - \lambda$. By Theorem 4, the operator $M_{u-\lambda}$ is not invertible on $BV[0,1]$ if and only if for each $n \in \mathbb{N}$ we can find a $t_n \in [0,1]$ such that $|u(t_n) - \lambda| \le \frac{1}{n}$, which means that $\lambda \in \overline{u([0,1])}$. Hence, we also obtain that

$$r(M_u) = \sup\left\{|\lambda| : \lambda \in \overline{u([0,1])}\right\} = \|u\|_\infty.$$

This shows the result. □

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