Demonstr. Math. 2017; 50:100–104 **DE GRUYTER** OPEN

Demonstratio Mathematica **Open Access**

Research Article

Adam Nawrocki*

Diophantine approximations and almost periodic functions

DOI 10.1515/dema-2017-0011
Received November 17, 2015; accepted April 27, 2016

Abstract: In this paper we investigate the asymptotic behaviour of the classical continuous and unbounded almost periodic function in the Lebesgue measure. Using diophantine approximations we show that this function can be estimated by functions of polynomial type and we give the best polynomial estimation.

Keywords: Almost periodic function in the Lebesgue measure; continued fraction; Stepanov almost periodic function

MSC: 42A75, 11A55, 41A10

1 Introduction

The classical theory of almost periodic functions initiated by H. Bohr (see [1–3]) has been extended by many mathematicians e.g. by A. S. Besicovitch [4], S. Bochner [5], B. M. Levitan [6], J. von Neumann [7], V. V. Stepanov [8] and H. Weyl [9] (see also [10]).

One of essential generalizations of classical almost periodic functions connected with the Lebesgue measure, is the class of μ-almost periodic functions which was introduced by Stepanov (see [8]). In the literature these functions are sometimes also called measurably almost periodic (see e.g. [10, 11]). Let us emphasize that these functions are defined on the class of measurable functions which do not have to be locally integrable. Some interesting results on μ-almost periodic functions one can find for example in the papers [12] and [13]. In this paper we investigate the asymptotic behaviour of the classical continuous and unbounded μ-almost periodic function $f\colon \mathbb{R} \to \mathbb{R}$ defined by formula

$$f(x) = \frac{1}{2 + \cos(x) + \cos(\sqrt{2}x)} \quad \text{for } x \in \mathbb{R}.$$

It was proved in [14] that the function f can be estimated by functions of exponential type. To be precise, it was shown that for every $\lambda < 0$ the following equality holds

$$\lim_{x\to+\infty} \frac{e^{\lambda x}}{2 + \cos(x) + \cos(\sqrt{2}x)} = 0. \tag{1}$$

Let us add that the direct proof of the above equality is long and technically complex, and it is based on continuous fractions machinery. In this paper, using in particular the Liouville Theorem, we are going to establish that the exponential function which appears in the limit (1), can be replaced by a function of polynomial type. Moreover, among such polynomial estimations, we indicate the best one.

Now we recall basic definitions and facts which will be needed in the proof of the main results. By $L^0(\mathbb{R})$ we denote the family of all equivalence classes of real-valued Lebesgue measurable functions.

***Corresponding Author: Adam Nawrocki:** Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland, E-mail: adam.nawrocki@amu.edu.pl

Let X be an arbitrary set and let $f, g\colon X \to \mathbb{R}$. If there exists $C > 0$ such that for every $x \in X$ we have $|f(x)| \le Cg(x)$, then we write $f(x) \ll g(x)$. For $a \in \mathbb{R}$, we define $\lfloor a \rfloor := z$, where $z \in \mathbb{Z}$ is such that $z \le a < z+1$.

Now we recall the notion of a μ-almost periodic function.

Definition 1. A function $f \in L^0(\mathbb{R})$ is said to be *almost periodic in the Lebesgue measure* μ (or simply μ*-almost periodic*), if for arbitrary numbers $\varepsilon, \eta > 0$ the set

$$E\{\varepsilon, \eta; f\} := \left\{ \tau \in \mathbb{R} : \sup_{u \in \mathbb{R}} \mu\left(\left\{x \in [u, u+1] : |f(x+\tau) - f(x)| \ge \eta\right\}\right) \le \varepsilon \right\}$$

is relatively dense, that is, there exists a positive number ω, such that each open interval $(a, a+\omega)$, where $a \in \mathbb{R}$, contains at least one element of the set $E\{\varepsilon, \eta; f\}$.

Let us also recall the well-known Liouville's Theorem.

Theorem 1 ([15]). *If α is an irrational algebraic number of degree n, then there exists $c > 0$ such that for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, we have*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}.$$

We will need the following definition and results connected with continued fractions.

Definition 2 ([15]). For a sequence $a = (a_n)_{n=0}^{\infty}$, such that $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \in \mathbb{N}$, we define the sequence $(\frac{P_n}{Q_n})_{n=1}^{\infty}$ in the following way

$$P_{-1} := 1, \qquad Q_{-1} := 0, \qquad P_0 := a_0, \qquad Q_0 := 1$$

and

$$P_k := a_k P_{k-1} + P_{k-2} \quad \text{and} \quad Q_k := a_k Q_{k-1} + Q_{k-2} \qquad \text{for } k \in \mathbb{N}.$$

Theorem 2 ([15]). *The limit $\alpha := \lim_{n\to\infty} \frac{P_n}{Q_n}$ exists and, moreover,*

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n^2} \quad \text{for } n \in \mathbb{N}.$$

Remark 1. It is easy to establish that

$$\sqrt{2} = \lim_{n\to\infty} \frac{P_n}{Q_n},$$

when $(a_n)_{n=0}^{\infty} = (1, 2, 2, 2, \ldots)$.

2 Main results

At the beginning of this section we are going to prove

Theorem 3. *For $x > \frac{1}{2}\pi$ we have*

$$\frac{1}{2 + \cos(x) + \cos(\sqrt{2}x)} \ll x^2.$$

Proof. Given $x > \frac{1}{2}\pi$ let us put

$$P(x) := \left\lfloor \frac{x\sqrt{2}}{\pi} + \frac{1}{2} \right\rfloor \qquad \text{and} \qquad Q(x) := \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor$$

as well as $d(x) := \max\{d_1(x), d_2(x)\}$, where

$$d_1(x) := \left|\frac{x\sqrt{2}}{\pi} - P(x)\right| \qquad \text{and} \qquad d_2(x) := \left|\frac{x}{\pi} - Q(x)\right|.$$

It is easy to show that the integers $P(x)$, $Q(x)$ are positive and that $0 < d(x) \le \frac{1}{2}$. Then

$$\begin{aligned}|\sqrt{2}Q(x)\pi - P(x)\pi| &\le |\sqrt{2}Q(x)\pi - x\sqrt{2}| + |x\sqrt{2} - P(x)\pi| \\ &= d_2(x)\pi\sqrt{2} + d_1(x)\pi \ll d(x),\end{aligned}$$

which implies that

$$\left|\sqrt{2} - \frac{P(x)}{Q(x)}\right| \ll \frac{d(x)}{Q(x)}.$$

Moreover, since $\sqrt{2}$ is an algebraic number of degree 2, by the Liouville Theorem, we get

$$\frac{1}{Q(x)^2} \ll \left|\sqrt{2} - \frac{P(x)}{Q(x)}\right|.$$

Thus

$$\frac{1}{d(x)} \ll Q(x). \tag{2}$$

On the other hand, because $0 \le d_2(x) \le \frac{1}{2}$, we infer that

$$\frac{1}{2}Q(x) \le Q(x) - \frac{1}{2} \le \frac{x}{\pi},$$

and therefore we have

$$Q(x) \ll x. \tag{3}$$

Furthermore

$$1 + \cos(y + k\pi) \ge 1 - \cos(|y|) \qquad \text{for } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], k \in \mathbb{Z},$$

and

$$1 - \cos(y) \ge \frac{y^2}{2} - \frac{y^4}{24} \qquad \text{for } y \in \mathbb{R}.$$

By the definitions of d_1 and d_2, we get

$$\begin{aligned}2 + \cos(x) + \cos(\sqrt{2}x) &\ge 1 - \cos(d_2(x)\pi) + 1 - \cos(d_1(x)\pi) \\ &\ge 1 - \cos(d(x)\pi) \ge \frac{(d(x)\pi)^2}{2} - \frac{(d(x)\pi)^4}{24} \ge d(x)^2.\end{aligned}$$

Therefore, by (2), (3) and the above inequalities we obtain

$$\frac{1}{2 + \cos(x) + \cos(\sqrt{2}x)} \ll \frac{1}{d(x)^2} \ll Q(x)^2 \ll x^2 \quad \text{for } x > \frac{1}{2}\pi.$$

□

Corollary 1. For every $\varepsilon > 0$ we have

$$\lim_{x\to+\infty} \frac{x^{-2-\varepsilon}}{2 + \cos(x) + \cos(\sqrt{2}x)} = 0.$$

The following theorem shows that the exponent 2 in the above equality cannot be improved.

Theorem 4. *The limit*

$$\lim_{x\to+\infty} \frac{x^{-2}}{2 + \cos(x) + \cos(\sqrt{2}x)} \tag{4}$$

does not exist.

Proof. Let P, Q, d_1, d_2, d be the functions defined at the beginning of the proof of Theorem 3. Furthermore, let $x_n := Q_{2n}\pi$ for $n \in \mathbb{N}$, where the numbers Q_{2n} correspond to the sequence $(1, 2, 2, 2, \ldots)$ (cf. Definition 2). Then, it is easy to see that the numbers Q_{2n} are odd and that

$$Q(x_n) = \left\lfloor \frac{Q_{2n}\pi}{\pi} + \frac{1}{2}\right\rfloor = Q_{2n} \qquad \text{for } n \in \mathbb{N}.$$

By Theorem 2 and Remark 1 we get

$$\left|\sqrt{2}-\frac{P_{2n}}{Q_{2n}}\right| < \frac{1}{Q_{2n}^2} < \frac{1}{2Q_{2n}}, \tag{5}$$

and therefore

$$\left|\frac{x_n\sqrt{2}}{\pi}-P_{2n}\right| = |\sqrt{2}Q_{2n}-P_{2n}| < \frac{1}{2}$$

as well as

$$P_{2n} < \frac{x_n\sqrt{2}}{\pi}+\frac{1}{2} < P_{2n}+1.$$

Thus

$$P(x_n) = \left\lfloor \frac{x_n\sqrt{2}}{\pi}+\frac{1}{2}\right\rfloor = P_{2n},$$

which means that

$$d_1(x_n) = |\sqrt{2}Q_{2n}-P_{2n}| \qquad \text{and} \qquad d_2(x_n) = \left|\frac{x_n}{\pi}-Q_{2n}\right| = 0.$$

Consequently $d(x_n) = d_1(x_n)$, and hence

$$d(x_n) = |\sqrt{2}Q_{2n}-P_{2n}| < \frac{1}{Q_{2n}}.$$

Furthermore, for $y \in \mathbb{R}$ we have $1-\cos(y) \le \frac{1}{2}y^2$, and so

$$2+\cos(x_n)+\cos(\sqrt{2}x_n) = 1-\cos(d(x_n)\pi) \le \frac{1}{2}(d(x_n)\pi)^2 < \frac{\pi^2}{2Q_{2n}^2} = \frac{\pi^4}{2x_n^2}.$$

Hence

$$\frac{2x_n^2}{\pi^4} \le \frac{1}{2+\cos(x_n)+\cos(\sqrt{2}x_n)},$$

and therefore

$$\limsup_{x\to+\infty}\frac{x^{-2}}{2+\cos(x)+\cos(\sqrt{2}x)} \ge \frac{2}{\pi^4} > 0.$$

On the other hand, for the sequence $y_n = 2n\pi$, $n \in \mathbb{N}$ we have

$$\frac{y_n^{-2}}{2+\cos(y_n)+\cos(\sqrt{2}y_n)} = \frac{y_n^{-2}}{3+\cos(\sqrt{2}y_n)} \le \frac{1}{2y_n^2},$$

which shows that

$$\liminf_{x\to+\infty}\frac{x^{-2}}{2+\cos(x)+\cos(\sqrt{2}x)} = 0,$$

and thus the limit (4) cannot exist. □

Acknowledgement: I am very grateful to Professor D. Bugajewski for his valuable remarks.
I would like to thank the anonymous referee for his/her valuable comments and suggestions.

References

[1] Bohr H., Zur Theorie der fastperiodischen Funktionen, I. Teil Eine Verallgemeinerung der Theorie der Fourierreihen, Acta Math., 1925, 45, 29–127

[2] Bohr H., Zur Theorie der fastperiodischen Funktionen, II. Teil Zusammenhang der fastperiodischen Functionen mit Funktionen von unendkich vielen Variablen; gleichmassige Approximation durch trigonometrische Summen, Acta Math., 1925, 46, 101–214

[3] Bohr H., Zur Theorie der fastperiodischen Funktionen, III. Teil Dirichletentwicklung analytischer Functionen, Acta Math., 1926, 47, 237–281

[4] Besicovitch A.S., On generalized almost periodic functions, Proc. London Math. Soc., 1926, 25, no. 2, 495–512

[5] Bochner S., Properties of Fourier series of almost periodic functions, Proc. London Math. Soc., 1926, 26, no. 2, 433–452
[6] Levitan B. M., Pochti-periodicheskie Funkcii, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953, (in Russian)
[7] Neumann J., Almost periodic functions in a group, Trans. Amer. Math. Soc., 1934, 36, 445–492
[8] Stepanov V. V., Uber einige Verallgemeinerungen der fastperiodischen Funktionen, Ann. Math., 1926, 95, 437–498
[9] Weyl H., Integralgleichungen und fastperiodische Funktionen, Ann. Math., 1926, 97, 338–356
[10] Andres J., Bersani A. M., Grande R. F., Hierarchy of almost-periodic function spaces, Rend. Mat. Appl.(7), 2006, 26, 121–188
[11] Franklin P., Approximation theorems for generalized almost periodic functions, Math. Z., 1929, 29, no. 1, 70–86
[12] Stoiński S., Almost periodic functions in the Lebesgue measure, Comment. Math. Prace Mat., 1994, 34, 189–198
[13] Stoiński S., On compactness of almost periodic functions in the Lebesgue measure, Fasc. Math., 1999, 30, 171–175
[14] Bugajewski D., Nawrocki A., Some remarks on almost periodic functions in view of the Lebesgue measure with applications to linear differential equations, Ann. Acad. Sci. Fenn. Math., (in press)
[15] Hardy G. H., Wright E. M., An introduction to the number theory, 4th ed., Clarendon Press, 1971